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Harvey Rosen, Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487, e-mail: hrosen@@gp.as.ua.edu

EVERY REAL FUNCTION IS THE SUM OF TWO EXTENDABLE CONNECTIVITY FUNCTIONS

Abstract

It is shown that an arbitrary fucntion $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of two extendable connectivity functions.

Let K be any one of the following three classes of functions from $\mathbb{R}$ into $\mathbb{R}$: Darboux functions, connectivity functions, or almost continuous functions. It is known that an arbitrary function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two functions belonging to K [8], [3], [10], [2] and [7]. We show that this result is also true for the class K of extendable connectivity functions. This answers a question of Gibson in [4]. Consequently, just like for the other classes, summation does not preserve for extendable functions any topological properties.

For a Darboux function $g : \mathbb{R} \to \mathbb{R}$, $g(C)$ is connected whenever C is connected. Let $X = \mathbb{R}$ or $\mathbb{R}^2$. A function $G : X \to \mathbb{R}$ is called connectivity if the graph of the restriction $G|C$ is connected for each connected subset C of X. According to [6], [14], [13], when $X = \mathbb{R}^2$, this last concept is equivalent to the notion of peripheral continuity, which means that for each $x \in X$ and each open neighborhood U of x and V of $G(x)$, there exists an open neighborhood W of x in U such that $G(\mathrm{bd}(W)) \subset V$, where $\mathrm{bd}(W)$ denotes the set-theoretic boundary of W in X. We say $g : \mathbb{R} \to \mathbb{R}$ is an extendable connectivity function if there exists a connectivity function $G : \mathbb{R}^2 \to \mathbb{R}$ such that $G(x, 0) = g(x)$ for all $x \in \mathbb{R}$, and we say a set $A \subset \mathbb{R}$ is g-negligible if every function from $\mathbb{R}$ into $\mathbb{R}$ obtained by arbitrarily redefining g on A is still an extendable connectivity function. Every open neighborhood of the graph of an almost continuous function $g : \mathbb{R} \to \mathbb{R}$ contains the graph of some continuous function from $\mathbb{R}$

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into $\mathbb{R}$. Let $I = [0,1]$. There is an extendable connectivity function $g : I \to I$ whose graph is dense in I^2 [1], [5] and [12]. Natkaniec remarks in [9] that it is unknown whether there exists an extendable connectivity function $g : I \to \mathbb{R}$ which is dense in $I \times \mathbb{R}$. We give such an example and use it and some results of Natkaniec [9] to verify the title.

Example 1 *There exists an extendable connectivity function $g : \mathbb{R} \to \mathbb{R}$ whose graph is dense in $\mathbb{R}^2$.*

PROOF. We outline how to show this. Let $Q = \{r_1, r_2, r_3, \ldots\}$ be the set of rational numbers in $\mathbb{R} \times \{0\}$, and let

$$\{d_1, d_2, d_3, d_4, d_5, d_6, \ldots\} = \{1, -1, 2, -2, 3, -3, \ldots\}.$$

In what follows, a "triangle" t will consist of the points interior to the three sides along with the points on its open base b. First we want to define partially a function $G : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ which is peripherally continuous. For $n = 1, 2, 3, \ldots$ we let T_n denote a countable collection of triangles t_i in $\mathbb{R} \times [0,\infty)$ whose bases b_i form a locally finite countable collection B_n of open intervals of $\mathbb{R} \times \{0\}$ with irrational endpoints. Furthermore, let G be a function such that

(1) $\mathrm{diam}(t_i) < \frac{1}{n}$,

(2) T_{n+1} is a refinement of T_n and B_{n+1} is a refinement of B_n,

(3) each element r_j of $\{r_1, r_2, \ldots, r_n\}$ belongs to exactly two members b'_j, b''_j of B_n which are bases of triangles t'_j, t''_j in T_n with $\mathrm{cl}(t'_j) \subset t''_j$,

(4) $T^*_n = T_n \setminus \{t''_j : 1 \le j \le n\}$ is a "sawtooth" countable collection of disjoint triangles, $B^*_n = B_n \setminus \{b''_j : 1 \le j \le n\}$ is a countable collection of disjoint open intervals, and $\mathbb{R} \times \{0\} = \cup\{\mathrm{cl}(b_i) : b_i \in B^*_n\}$,

(5) for $1 \le j \le n$, $G(\mathrm{bd}(t'_j) \setminus b'_j) = d_n$ and $G(\mathrm{bd}(t''_j) \setminus b''_j) = 0$,

(6) for each $t_i \in T_n$ the variation of $G(x)$ on $\mathrm{bd}(t_i) \setminus b_i$ is $< \frac{1}{n}$, and

(7) G maps the closed set $(\mathbb{R} \times [0,\infty)) \setminus \cup T^*_n$ continuously onto $[-d_n + 1, d_n]$ if n is odd or onto $[-|d_n|, |d_n|]$ if n is even.

Here is how to attain condition (6) for $n > 1$. Let E denote the set of endpoints of all intervals belonging to B_{n-1} along with the endpoints of each b'_j, $b''_j \in B_n$ described in (3) for each $r_j \in \{r_1, r_2, \ldots, r_n\}$. Suppose c and d are consecutive points of E with $r_j \notin (c, d)$ for $1 \le j \le n$. Even if $|G(d) - G(c)|$

is a large value, a sufficient number of consecutive small triangles of diameter $< \frac{1}{n}$ which are to belong to T_n can be constructed forming sawteeth from c to d so that the variation of $G(x)$ on the slanted sides of each of these triangles will be less than $\frac{1}{n}$. We may suppose $G(x)$ varies monotonically from $G(c)$ to $G(d)$ along the slanted edges of the sawteeth from c to d.

We now define G on the rest of $\mathbb{R} \times \{0\}$. Let $\varepsilon > 0$.

Case (i): x is a rational number $r_j \in \mathbb{R} \times \{0\}$. Then define $G(x) = 0$. For each $n \geq j$, there exist by (2), (3), and (5), open intervals b'_j, $b''_j \in B_n$ such that $G(\text{bd}(t'_j) \setminus b'_j) = d_n$ and $G(\text{bd}(t''_j) \setminus b''_j) = 0$. So $\text{diam}(\{G(x)\} \cup G(\text{bd}(t''_j) \setminus b''_j) = \text{diam}(\{0\}) = 0 < \varepsilon$.

Case (ii): x is an irrational number in $\mathbb{R} \times \{0\}$ that is not an endpoint of any b_i in any B_n. Suppose there exists an integer N such that for all $n > N$, x does not belong to any $b''_j \in B_n,\ 1 \leq j \leq n$. Then by (3), (5) and (6), for each $n > N$, there exists in T_n a triangle t_i whose base b_i contains x and on whose slanted sides the value of G lies in $[-|d_N|, |d_N|]$. For each $n > N$, choose a point x_n belonging to a slanted side of t_i. Then there exists a cluster point y of the sequence $G(x_1), G(x_2), G(x_3), \ldots$, and so define $G(x) = y$. By (1) and (6), $\text{diam}(\{G(x)\} \cup G(\text{bd}(t_i) \setminus b_i)) < \epsilon$ for infinitely many n. On the other hand, if we suppose there does not exist such an integer N, then there are infinitely many n such that $x \in b''_j \in B_n$ for some j with $1 \leq j \leq n$. Because of (5), $G(\text{bd}(t''_j) \setminus b''_j) = 0$, and so define $G(x) = 0$. Therefore $\text{diam}(\{G(x)\} \cup (G(\text{bd}(t''_j) \setminus b''_j)) = 0 < \epsilon$.

Case (iii): x is an irrational number in $\mathbb{R} \times \{0\}$ that is an endpoint of some interval belonging to some B_m. Then $G(x)$ is already defined. By (2) and (4), for each $n \geq m$, x is an endpoint of adjacent intervals b_i and b_k in B_n. Because of (1) and (6), there exists an $n \geq m$ such that $\frac{2}{n} < \epsilon$, $t_i \cup t_k$ has diameter $< \frac{2}{n}$, and the variation of G on $\text{bd}(t_i \cup t_k) \setminus (b_i \cup b_k)$ is $< \frac{2}{n}$. Since, by (7), G restricted to $(\mathbb{R} \times [0, \infty)) \setminus \cup T_n^*$ is continuous at x, there exists an open disk D centered at x and not containing the other vertices of t_i and t_k such that the diameter of the open neighborhood $U = t_i \cup t_k \cup (D \cap (\mathbb{R} \times [0, \infty)))$ of x in $\mathbb{R} \times [0, \infty)$ is $< \frac{2}{n}$ and $\text{diam}(\{G(x)\} \cup G(\text{bd}(U))) < \frac{2}{n} < \epsilon$.

Case (iv): $x \in \mathbb{R} \times (0, \infty)$. According to (7), G is already defined and continuous at x.

For each case, we have G is peripherally continuous at x. We can extend G to a peripherally continuous function $G : \mathbb{R}^2 \to \mathbb{R}$ by defining $G(s, t) = G(s, -t)$ whenever $(s, t) \in \mathbb{R} \times (-\infty, 0)$. On account of (5), the extendable connectivity function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = G(x, 0)$ for $x \in \mathbb{R}$ has its graph dense in $\mathbb{R}^2$. □

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then $f = g_1 + g_2$ for functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ which are extendable connectivity functions.*

PROOF. First let $g : \mathbb{R} \to \mathbb{R}$ be the above example of an extendable connectivity function whose graph is dense in $\mathbb{R} \times \mathbb{R}$. It follows from Theorem 1 in [11] that there exists a dense G_δ subset A of $\mathbb{R}$ that is g-negligible. Since $\mathbb{R} \setminus A$ is of the first category, it follows from Lemma 3 in [9] that there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $(\mathbb{R} \setminus A \cap h(\mathbb{R} \setminus A) = \emptyset$. Therefore $h(\mathbb{R} \setminus A) \subset A$; i.e., $\mathbb{R} \setminus A \subset h^{-1}(A)$. According to Corollary 1 and Lemma 2 (which still hold when $\mathbb{R}$ replaces I and J there) in [9], $g \circ h$ is an extendable connectivity function and $h^{-1}(A)$ is $g \circ h$-negligible. So $\mathbb{R} \setminus A$ is $g \circ h$-negligible. Define extendable connectivity functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ by

$$g_1 = \begin{cases} g \circ h & \text{on } A \\ f - g & \text{on } \mathbb{R} \setminus A \end{cases} \qquad \text{and} \qquad g_2 = \begin{cases} f - (g \circ h) & \text{on } A \\ g & \text{on } \mathbb{R} \setminus A. \end{cases}$$

Then $f = g_1 + g_2$. □

Question 1 *If $f : I \to I$ is an arbitrary bounded function, does $f = g_1 + g_2$, where g_1 and g_2 are bounded extendable connectivity functions? Natkaniec has shown that f is the sum of three such functions g_1, g_2, g_3 [9].*

Analogous results have been obtained by Ciesielski and Reclaw in their paper, Cardinal invariants concerning extendable and peripherally continuous functions, and according to the referee, a negative answer to the above question follows from latest results of Ciesielski and Maliszewski.

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