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CONVERGENCE AND KOLMOGOROV DIMENSION

Abstract

It is shown that under simple restrictions a series converges provided it's set of terms has Kolmogorov dimension strictly smaller than $\frac{1}{2}$.

1 Introduction

This paper is concerned with the relation between the convergence of an infinite series and the Kolmogorov dimension of its set of terms. For positive series (under a simple assumption), the cutoff occurs at dimension $\frac{1}{2}$. As might be expected from the situation with the ratio and root tests, at dimension $\frac{1}{2}$ a series may converge or diverge.

The restriction to nonnegative real series is required to get something definitive. Of course, terms must (eventually) be distinct as well, if one is to say much from knowledge of the set of terms.

The definition of Kolmogorov dimension that will be used is the diameter form.

Let X be a totally bounded metric space, and let $N(\varepsilon)$ be the smallest N for which X can be covered by N sets of diameter $\leq \varepsilon$. Let $I = \{d \geq 0 : \exists C_d > 0$ with $N(\varepsilon) \leq C_d\varepsilon^{-d}$ for ε sufficiently small$\}$. Then I is either empty (Kolmogorov dimension ∞), or I is a semi-infinite interval whose left endpoint ($\inf I$) is the Kolmogorov dimension. I may be open or closed. It suffices to check the inequality $N(\varepsilon) \leq C_d\varepsilon^{-d}$ on a sequence $\varepsilon_n \searrow 0$, provided $\varepsilon_n/\varepsilon_{n+1}$ is bounded above. Suppose $K \geq \varepsilon_n/\varepsilon_{n+1}$, for all n. Then for $\varepsilon \leq \varepsilon_1$ one can choose n with $\varepsilon_{n-1} \geq \varepsilon > \varepsilon_n$, so that $\varepsilon_n \geq \varepsilon/K$. Hence $N(\varepsilon) \leq N(\varepsilon_n) \leq C_d\varepsilon_n^{-d} \leq C_d K^d \varepsilon^{-d}$. The Kolmogorov dimension of X is denoted by $\dim_K(X)$.

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It should be noted that the Kolmogorov dimension can also be characterized in terms of ε-nets. Suppose $N'(\varepsilon)$ is the minimal size of an ε-net; i.e. a set such that each point is $< \varepsilon$ away from a point in the ε-net. Then X is a union of $N'(\varepsilon)$ balls of radius ε, so $N'(\varepsilon) \geq N(2\varepsilon)$. On the other hand, $N'(\varepsilon) \leq N(.9\varepsilon)$ by taking a point in each covering set.

The above, with an argument like that involving the ε_n's, shows that $N(\varepsilon)$ and $N'(\varepsilon)$ have the same asymptotic behavior as $\varepsilon \searrow 0$. The definition involving $N'(\varepsilon)$ goes by a number of names: Kolmogorov dimension, Kolmogorov capacity, limit capacity, and entropy to name four. We prefer the first, as it actually is a dimension. (It is also the form really used to find fractal dimensions in many applied situations, e.g. coastlines and mountains [1].)

2 The Theorem

Let $\langle a_n \rangle$ be a sequence of real numbers. We assume that $a_n > 0$ and $a_n > a_{n+1}$, for n sufficiently large. Also assume that $a_n \to 0$. Let $A = \{a_1, a_2, a_3, \dots\}$. We are concerned with the relation between $\dim_K(A)$ and the convergence of Σa_n. To this end an additional condition is imposed.

Definition 1 *Let $\langle a_n \rangle$ be a sequence satisfying the conditions above. Then $a_n \to 0$ regularly provided $a_n - a_{n+1} \geq a_{n+1} - a_{n+2}$ for n sufficiently large.*

The main theorem is as follows.

Theorem 1 *Let $\langle a_n \rangle$ be as above and let $A = \{a_1, a_2, a_3, \dots\}$.*

a) If Σa_n converges, then $\dim_K(A) \leq \frac{1}{2}$.

b) If $\dim_K(A) < \frac{1}{2}$ and $a_n \to 0$ regularly, then Σa_n converges.

Discussion: In [2], it is shown that, with $a_n = n^{-p}$ and $A = \{a_1, a_2, a_3, \dots\}$, then $\dim_K(A) = \frac{1}{p+1}$. This paper basically arose from investigating the fact that the cutoff for convergence occurs at dimension $\frac{1}{2}$. In the next section, some examples will be given. In particular, it will be shown with examples that for $\dim_K(A) = \frac{1}{2}$ the series may converge or may diverge. Also it will be shown in Section 4 that for $\dim_K(A) < \frac{1}{2}$ an extra condition, like $a_n \to 0$ regularly is required. The problem is that convergence of Σa_n measures how much the terms accumulate at 0, while $\dim_K(A)$ measures how much the terms accumulate around each other.

PROOF. By deleting a finite set, we may assume without loss of generality that $a_n > a_{n+1}$ for all n (and, in case (b), that $a_n - a_{n+1} \geq a_{n+1} - a_{n+2}$). In particular, $a_n > 0$ also.

(a) Suppose $s = \Sigma a_n < \infty$, and suppose $\varepsilon > 0$ is given. Let $m = [1 + \frac{1}{\sqrt{\varepsilon}}]$ Set $S = \{n : a_n \geq m\varepsilon\}$. Then $|S| \cdot m\varepsilon \leq \sum_{n=1}^{|S|} a_n \leq s$; so $|S| \leq \frac{s}{m\varepsilon}$. Now A can be covered with $|S|$ sets of the form $\{a_n\}$, $n \leq m$, together with m sets of the form $[(k-1)\varepsilon, k\varepsilon]$ for $k = 1, \ldots, m$. Hence $N(\varepsilon) \leq m + s/m\varepsilon$. Then $\frac{1}{\sqrt{\varepsilon}} < m \leq \frac{1}{\sqrt{\varepsilon}} + 1$; so

$$N(\varepsilon) \leq m + \frac{s}{m\varepsilon} \leq \frac{1}{\sqrt{\varepsilon}} + 1 + \frac{s}{\varepsilon^{-1/2} \cdot \varepsilon} = 1 + (1+s)\varepsilon^{-1/2}.$$

(b) Fix p with $\dim_K(A) < p < \frac{1}{2}$. For $k \in \mathbb{N}$ the set A can be covered by $\leq C \cdot 2^{kp}$ sets of diameter $\leq 2^{-k}$ for a number C independent of k. Set $n_k = \max\{n : a_n - a_{n+1} > 2^{-k}\}$. Then $a_n - a_{n+1} > 2^{-k}$ for $n \leq n_k$, and $a_n - a_{n+1} \leq 2^{-k}$ for $n > n_k$. Further, $n_1 \leq n_2 \leq n_3 \cdots$ and $n_k \to \infty$.

Now, if $n \leq n_k$, then no set of diameter $\leq 2^{-k}$ which contains a_n can contain any other a_m. So any cover of A by sets of diameter $\leq 2^{-k}$ must have at least n_k members. Hence $n_k \leq C \cdot 2^{kp}$.

Further, the set $\{a_n : n > n_k\}$ has no gaps wider than 2^{-k}. Thus if this set is minimally covered by N sets of diameter $\leq 2^{-k}$, then

i) these sets may be replaced without loss of generality by closed intervals $[b_j, b_j + 2^{-k}]$, with $b_j < b_{j+1}$ for all $j, j = 1, \ldots, N$;

ii) $b_1 = 0$ since $a_n \to 0$; and

iii) $b_{j+1} \leq (b_j + 2^{-k}) + 2^{-k}$ due to the lack of gaps.

Hence by induction, $b_j \leq (j-1) \cdot 2^{1-k}$. Thus $b_N + 2^{-k} \leq (2N-1)2^{-k} < 2N2^{-k}$. But $N \leq C \cdot 2^{kp}$; so for $n > n_k$,

$$a_n \leq b_N + 2^{-k} \leq 2C \cdot 2^{kp} \cdot 2^{-k} = 2C \cdot 2^{k(p-1)}.$$

Combine the above to get

$$\begin{aligned} \sum_{n=n_k+1}^{n_{k+1}} a_n &\leq n_{k+1} \cdot 2C \cdot 2^{k(p-1)} \\ &\leq C \cdot 2^{(k+1)p} \cdot 2C \cdot 2^{kp-k} \\ &= 2^{1+p} C^2 \cdot 2^{k(2p-1)}. \end{aligned}$$

Hence $\sum_{n=n_1}^{\infty} a_n \leq 2^{1+p} C^2 \sum_{k=1}^{\infty} 2^{k(2p-1)} < \infty$, since $2p - 1 < 0$. $\square$

3 Some Examples

First, a theorem that computes Kolmogorov dimension.

Theorem 2 *Let $a_n \searrow 0$, and let $A = \{a_n : n \in \mathbb{N}\}$. Suppose c, p, and q are positive.*

a) If $a_n \leq c/n^p$ for n sufficiently large, then $\dim_K(A) \leq \frac{1}{p+1}$.

b) If $a_n - a_{n+1} \geq c/(n+1)^q$ for n sufficiently large, then $\dim_K(A) \geq \frac{1}{q}$.

PROOF. Without loss of generality assume the inequalities hold for all n.

a) Without loss of generality suppose c is a positive integer. Set $\varepsilon_n = 1/n^{p+1}$. Then $\varepsilon_n/\varepsilon_{n+1} \to 1$. So it suffices to consider these ε's. Now $\{a_n, a_{n+1}, \dots\} \subset \cup_{k=1}^{cn} \left[\frac{k-1}{n^{p+1}}, \frac{k}{n^{p+1}}\right]$ while $\{a_1, \dots, a_{n-1}\}$ is covered by the sets $\{a_j\}$ for $j = 1, 2, \dots, n-1$. Hence
$$N(\varepsilon_n) < (c+1)n = (c+1)\varepsilon_n^{-1/(p+1)}.$$

b) Set $\varepsilon_m = c/m^q$. $n+1 < m \Rightarrow a_n - a_{n+1} \geq c/(n+1)^q > c/m^q$. Hence any two points in $\{a_1, \dots, a_{m-1}\}$ are more than ε_m units apart. So given any cover of A (and hence of $\{a_1, \dots, a_{m-1}\}$) by sets of diameter $\leq \varepsilon_m$, each set can contain at most one a_n with $n < m$. Hence $N(\varepsilon_m)$ is at least $m-1$, i.e., $N(\varepsilon_m) \geq m-1$. Now $N(\varepsilon_m) \leq k\varepsilon_m^{-d}$, m sufficiently large $\Rightarrow m - 1 \leq k\varepsilon_m^{-d} = kc^{-d} \cdot m^{qd}$ all m sufficiently large $\Rightarrow qd \geq 1$, i.e., $d \geq 1/q$. □

Corollary 1 *Under the same hypothesis as in Theorem 1, if $a_n = f(n)$, $f : [1, \infty) \to \mathbb{R}$, and if $-f'(x) \geq cx^{-q}$, then $\dim_K(A) \geq 1/q$.*

PROOF. By the Mean Value Theorem, there is a ζ_n between n and $n+1$ so that $a_n - a_{n+1} = f(n) - f(n+1) = -f'(\zeta_n) \geq c\zeta_n^{-q} \geq c(n+1)^{-q}$. □

Example 1 Let $a_n = \frac{1}{n(1+\log n)^2}$ and let $A = \{a_n : n \in \mathbb{N}\}$. (It will be shown in the next section that $a_n \to 0$ regularly.) Since $a_n \leq \frac{1}{n}, \dim_K(A) \leq \frac{1}{2}$. But if $q > 2$, then $f(x) = \dfrac{1}{x(1+\log x)^2}$ gives
$$\begin{aligned} -x^q f'(x) &= \frac{x^q}{x^2(1+\log x)^4} \cdot \left((1+\log x)^2 + x \cdot 2(1+\log x)\frac{1}{x}\right) \\ &= x^{q-2} \cdot \frac{3+\log x}{(1+\log x)^3} \to \infty. \end{aligned}$$

Hence $\exists\ c > 0$ with $-x^q f'(x) \geq c$. Thus $q > 2 \Rightarrow$ Kolmogorov dimension is $\geq \frac{1}{q}$. Letting $q \searrow 2$ gives Kolmogorov dimension $\frac{1}{2}$.

Example 2 Let $a_n = \frac{1}{1+\log n}$ and let $A = \{a_n : n \in \mathbb{N}\}$. If $q > 1$, $f(x) = (1+\log x)^{-1}$, then $-x^q f'(x) = x^{q-1}(1+\log x)^{-2} \to \infty$, so again $\exists\ c > 0$ with $-x^q f'(x) \geq c$. Hence $\dim_K(A) \geq \frac{1}{q}$. Letting $q \searrow 1$ gives $\dim_K(A) = 1$. It *can* get that large.

4 Regular Decrease

In both examples above it is assumed that the sequences tend to 0 regularly; that is, $a_{n+1} - a_n \geq a_{n+2} - a_{n+1}$. That they do tend to 0 regularly can be established using that there is a twice differentiable function, $f : (1-\delta, \infty) \to \mathbb{R}$ with $a_n = f(n)$ such that $f(x) > 0$, $f'(x) < 0$ and $\lim_{x\to\infty} f(x) = 0$. These properties imply that $a_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Further, if f' has only finitely many roots, then $f(x) > 0$ and $f(x) \to 0$ implies $f'(x) < 0$ for large x, since $f'(x) > 0$ is untenable for large x. Similarly, $a_n \to 0$ regularly provided $-f'(x)$ decreases, i.e., $f''(x) > 0$. Finally, if f' and f'' have only finitely many roots, then $f(x) > 0$ and $f(x) \to 0$ implies $f''(x) > 0$ for large x, since $f''(x) < 0$ is untenable for large x. ($f''(x) < 0$ would imply $-f'(x)$ increases, i.e., $a_n - a_{n+1}$ increases [and is positive; see above on why $f'(x) < 0$ for large x], contradicting $a_n - a_{n+1} \to 0$.)

So the following theorem results.

Theorem 3 *Let $\delta > 0$ and let $f : (1-\delta, \infty) \to \mathbb{R}$ be twice differentiable. Suppose for each $x \in (1-\delta, \infty)$ we have $f(x) > 0$, $f(x) \to 0$, and f' and f'' have only finitely many roots. Then $f(n) \to 0$ regularly.*

For example, generalized rational functions (with any nonnegative real exponent allowed) of x and $\log(x)$ satisfy the conditions of Theorem 3, since eventually one term will dominate in the numerator (and denominator), giving constant sign. Consequently, the sequences in Examples 1 and 2 do go to zero regularly.

Example 3 There is a sequence $a_n \searrow 0$ with $A = \{a_n : n \in \mathbb{N}\}$ having Kolmogorov dimension zero, while Σa_n diverges. Needless to say, a_n does not go to zero regularly.

For $n = 2^k + j$, $0 \leq j < 2^k$ set $a_n = 2^{-k} + (2^k - j - 1) \cdot 2^{-k} \cdot 2^{-2^k}$. $a_1 = 1$, $a_2 = \frac{1}{2} + \frac{1}{8}$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{4} + \frac{3}{64}$, $a_5 = \frac{1}{4} + \frac{2}{64}$, $a_6 = \frac{1}{4} + \frac{1}{64}$, $a_7 = \frac{1}{4}$, $a_8 = \frac{1}{8} + \frac{7}{2048}, \ldots$.

Note that

i) $a_n \geq 2^{-k}$. Thus $\sum_{n=2^k}^{2^{k+1}-1} a_n \geq 1$. Hence $\sum_{n=1}^{\infty} a_n$ diverges.

ii) a_n decreases for $2^k \leq n \leq 2^{k+1}-1$ and

$$\begin{aligned} a_{2^{k+1}} &= 2^{-(k+1)} + (2^{k+1}-1)2^{-(k+1)} \cdot 2^{-2^{k+1}} \\ &< 2^{-k} \cdot \frac{1}{2} + 2^{-2^{k+1}} = 2^{-k} \cdot (\frac{1}{2} + 2^{k-2^{k+1}}) \\ &< 2^{-k} = a_{2^{k+1}-1}, \end{aligned}$$

since $\frac{1}{2} + 2^{k-2^{k+1}} < 1$; i.e., $2^{k-2^{k+1}} < \frac{1}{2}$; i.e., $k - 2^{k+1} < -1$; i.e., $k+1 < 2^{k+1}$.

Claim: $A = \{a_n : n \in \mathbb{N}\}$ has Kolmogorov dimension zero.

Proof of Claim. Set $\varepsilon_l = 2^{-l}$. Divide A into three parts:

i) If $n = 2^k + j$ as above, and $k > l$, then $a_n < a_{2^{l+1}-1} = 2^{-l} = \varepsilon_l$. Hence $a_n \in [0, 2^{-l}]$. That's one set.

ii) If $\log_2 l \leq k \leq l$, then by the formula, $a_n \in [2^{-k}, 2^{-k} + 2^{-2^k}]$, which has length $2^{-2^k} \leq 2^{-2^{\log_2 l}} = 2^{-l} = \varepsilon_l$. That's at most l more sets.

iii) If $k < \log_2 l$, then $a_n \in \{a_n\}$. Now $k < \log_2 l \Rightarrow 2^k < l \Rightarrow n < 2l - 1$. That's at most $2l - 2$ more sets.

Combining, $N(\varepsilon_l) \leq 3l - 1$. Hence, for any $d > 0$, there is a C_d with $N(\varepsilon_l) \leq C_d \varepsilon_l^{-d}$.

References

[1] B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, San Francisco, 1982.

[2] C. Essex and M. Nerenberg, *Fractal dimension: Limit capacity or Hausdorff dimension?*, Am. J. Phys., **58** (10), (1990), 986–8.