RESEARCH

Real Analysis Exchange
Vol. 21(1), 1995–96, pp. 203–215

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ON $\mathcal{E}$-CONTINUOUS FUNCTIONS

Abstract

Some properties of $\mathcal{E}$-continuous functions are investigated. In particular, the maximal family with respect to outer and inner compositions for the family of all $\mathcal{E}$-continuous functions are described. Moreover, under some assumptions on $\mathcal{E}$ it is proved that every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as the composition of two $\mathcal{E}$-continuous function. Similarly, every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as the limit of a transfinite sequence of $\mathcal{E}$-continuous functions.

1 Introduction

This paper is a supplement to the article *Algebraic properties of $\mathcal{E}$-continuous functions* [1]. One can find there the following definitions.

Let $x \in \mathbb{R}$. A *path leading to* x is a set $E_x \subset \mathbb{R}$ such that $x \in E_x$ and x is a point of bilateral accumulation of E_x. For $x \in \mathbb{R}$ let $\mathcal{E}(x)$ be a family of paths leading to x. A *system of paths* is a collection $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ such that each $E_x \in \mathcal{E}(x)$ for every $x \in \mathbb{R}$ (compare with [2]). Sometimes we shall simply refer to E_x as a "path".

We say that L_x (R_x) is a *left (right) path leading to* x if $L_x = E_x \cap (-\infty, x]$ ($R_x = E_x \cap [x, \infty)$) for some path $E_x \in \mathcal{E}(x)$.

For a system of paths $\mathcal{E}$ we define its σ-closure $\sigma\mathcal{E}$ as the least σ-system of paths containing $\mathcal{E}$. We shall only consider systems of paths $\mathcal{E}$ having the property that if L_x is a left path leading to x and R_x is a right path leading to x, then $L_x \cup R_x$ is an element of $\mathcal{E}(x)$ and we shall assume that $\mathbb{R} \in \mathcal{E}(x)$ for each $x \in \mathbb{R}$. We shall classify systems of paths according to the following scheme: a system of paths $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ will be said to be

Key Words: system of paths, $\mathcal{E}$-continuous functions, $\mathcal{PR}$ functions, $\mathcal{PC}$ functions, bilaterally quasi-continuous functions, $\mathcal{C}(m)$ functions, $\mathcal{E}IVP$, $CIVP$, the maximal family with respect to outer and inner compositions, transfinite limit
Mathematical Reviews subject classification: Primary: 26A15
Received by the editors November 28, 1994

- of δ-*type*, if $E_x \cap [x-\delta, x+\delta]$ contains a path in $\mathcal{E}(x)$ for every $E_x \in \mathcal{E}(x)$ and for every $\delta > 0$.
- of Δ-*type*, if $\mathcal{E}$ is a δ-type system of paths, and there exists a path $E_y \in \mathcal{E}$ such that $E_y \subset E_x \setminus \{x\}$ for each a path $E_x \in \mathcal{E}$.
- of σ-*type*, if $\mathcal{E}$ is a δ-type system of paths, and for each triple of sequences of numbers $(a_n)_{n=1}^{\infty}$, $(x_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $b_{n+1} < a_n < x_n < b_n$, $(a_n < x_n < b_n < a_{n+1})$ $b_n \searrow x$ $(a_n \nearrow x)$ and for each left or right or bilateral paths $E_{x_n} \subset [a_n, b_n]$ leading to x_n for $n \in \mathbb{N}$, the set $\bigcup_{n=1}^{\infty} E_{x_n} \cup \{x\}$ contains a right path R_x (left path L_x) derived from an $E_x \in \mathcal{E}(x)$.
- of c-*type*, if $\mathcal{E}$ is a σ-system of paths and every Cantor set C_x such that x is a bilateral point of accumulation of C_x, belongs to $\mathcal{E}(x)$.

Such systems will be called simply δ-*systems*, σ-*systems* and c-*systems*, respectively. We consider real functions of a real variable, unless otherwise explicitly stated.

Let $f : \mathbb{R} \to \mathbb{R}$ and let $\mathcal{E} = \{\mathcal{E}(x) : x \in \mathbb{R}\}$ be a system of paths. We say that a function f is $\mathcal{E}$-*continuous at* x (f *has a path at* x) if there exists a path $E_x \in \mathcal{E}(x)$ such that $f|E_x$ is continuous at x. If f is $\mathcal{E}$-continuous at every point x, then we say that f is $\mathcal{E}-$*continuous.*

We say that a function f *has a left* (*right*) *path at* x if there exists a left (right) path E_x leading to x such that $f|E_x$ is continuous at x.

Let us set out some of the notation to be used in the article:

$\mathcal{C}$ – the class of all continuous functions,

$\mathcal{PR}$ – the class of all functions having perfect road at each point of the domain [5], (cf. [2] and [1]),

$\mathcal{PC}$ – the class of peripherally continuous functions [9, 2, 1],

$\mathcal{Q}_0$ – the class of bilaterally quasi-continuous functions [1],

$\mathcal{C}(m)$ – the class of functions which possess the *cardinality* m *property*, i.e. $\forall_{x\in\mathbb{R}}\forall_{\delta>0}\exists_{P\subset\mathbb{R}}$ $\operatorname{card}(P \cap (x, x+\delta)) \geq m$, $\operatorname{card}(P \cap (x-\delta, x)) \geq m$ and $f|P$ is continuous at x, where m is a fixed infinite cardinal number less than or equal to the continuum [1],

$\mathcal{E}const$ – the class of $\mathcal{E}$-*constant* functions, i.e. functions having the property: for each $x \in \mathbb{R}$ there exists a path E_x leading to x such that $f|E_x$ is constant,

$\mathcal{C}_{\mathcal{E}}$ – if $\mathcal{E}$ is a system of paths, then $\mathcal{C}_{\mathcal{E}}$ denote the class of all $\mathcal{E}$-continuous functions,

$\mathcal{E}IVP$ – the class of functions f having the $\mathcal{E}$-*intermediate value property*, i.e. functions for which the following condition is satisfied: for every $x, y \in \mathbb{R}$ and for each path $K \in \mathcal{E}$ between $f(x)$ and $f(y)$, there is a path $C \in \mathcal{E}$ between x and y such that $f(C) \subset K$ (cf. [3]).

Let $\mathcal{X}$ be a class of real functions. The family of functions $\mathcal{M}_{out}(\mathcal{X}) = \{f \in \mathcal{X}; \forall_{g\in\mathcal{X}} f \circ g \in \mathcal{X}\}$ is called the maximal family of $\mathcal{X}$ with respect to the outer component of the composition of functions. Similarly we define $\mathcal{M}_{in}(\mathcal{X})$, the maximal family of $\mathcal{X}$ with respect to the inner component of the composition of functions (cf. [6]).

Throughout this paper the symbols $K^-(f,x)$, $K^+(f,x)$ denote the cluster sets from the left and from the right of the function f at the point x, respectively and $K(f,x) = K^-(f,x) \cap K^+(f,x)$. By $Pr_x(A)$ we denote the x-projection of a set $A \subset \mathbb{R}^2$. Set $-A = \{-x : x \in A\}$.

2 Some Basic Lemmas

Remark 2.1 *If $\mathcal{E}$ is a σ-system of paths, then every bilaterally quasi continuous function is an $\mathcal{E}$-continuous function and each $\mathcal{E}$-continuous function is a peripherally continuous function, i.e. $\mathcal{Q}_0 \subset \mathcal{C}_\mathcal{E} \subset \mathcal{PC}$.*

Remark 2.2 *Let $\mathcal{E}$ be a σ-system of paths and let f be an $\mathcal{E}$-continuous function. Let $x \in \mathbb{R}$ and $x_n \searrow x$ ($y_n \nearrow x$). Then there exists a right path $R_x \in \mathcal{E}$ (left path $L_x \in \mathcal{E}$) such that $f|R_x$ ($f|L_x$) is continuous at x and the sets $(x_{n+1}, x_n) \cap R_x$ ($(y_n, y_{n+1}) \cap L_x$) contains a path E_n for infinite $n \in \mathbb{N}$.*

Lemma 2.1 *Let $\mathcal{E}$ be an σ-system of paths, $\{x_n\}_{n=1}^\infty$ be a sequence and $c \in \overline{\{x_n; n \in \mathbb{N}\}}$. Then there exists an $\mathcal{E}$-continuous function f such that $f(\mathbb{R} \setminus \{0\}) = \{x_n : n \in \mathbb{N}\}$ and $f(\{0\}) = \{c\}$.*

PROOF. Let C be the Cantor ternary set. For each $n \in \mathbb{N}$ let $I_{n,1}$, $I_{n,2}$, ..., $I_{n,2^{n-1}}$ be the components of $[0,1] \setminus C$ of length 3^{-n}. Let $c \in \overline{\{x_n; n \in \mathbb{N}\}}$ and let $\varphi = (\varphi_1, \varphi_2)$ be a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x_{\varphi_1(n)} & \text{if } |x| \in \overline{I}_{n,k},\ n \in \mathbb{N},\ k = 1,2,\ldots,2^{n-1} \\ c & \text{if } x = 0 \\ x_1 & \text{otherwise.} \end{cases}$$

Then f is bilaterally quasi-continuous function and, by Remark 2.1, it is an $\mathcal{E}$-continuous function. □

Theorem 2.1 *If $\mathcal{E}$ is an arbitrary system of paths and $f : \mathbb{R} \to \mathbb{R}$ is an $\mathcal{E}$-continuous function having closed graph, then f is continuous.*

PROOF. Suppose that f is not continuous at x_0 from the right. Notice that if there exists $y \in K^+(f, x_0) \setminus \{f(x_0), \pm\infty\}$, then f is not closed. Thus we have $K^+(f, x_0) \subset \{f(x_0), \pm\infty\}$. Therefore there exists $\delta > 0$ such that if

$x \in [x_0, x_0+\delta]$, then $|f(x)-f(x_0| < 1$ or $f(x) > f(x_0)+2$ or $f(x) < f(x_0)-2$. Put $A = \{(x, f(x)); x \in [x_0, x_0+\delta], |f(x)-f(x_0)| \leq 1\}$. Since f is closed and the set A is bounded, the set A is compact and therefore $Pr_x(A)$ is closed. Moreover $[x_0, x_0+\delta] \setminus Pr_x(A) \neq \emptyset$. Let (a, b) be a component of $[x_0, x_0+\delta] \setminus Pr_x(A)$. Then $a \in A$ and a function f is not $\mathcal{E}$-continuous at a from the right. This is impossible. □

3 The $\mathcal{E}$-intermediate Value Property

Remark 3.1 *Let $\mathcal{E}$ be a σ-system of paths. If $f \in \mathcal{E}IVP$, then for each $x \in \mathbb{R}$ there exists sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \searrow x$ ($x_n \nearrow x$) and $f(x_n) \to f(x)$.*

PROOF. If for some $\delta > 0$ $f|(x, x-\delta)$ is constant, then the sequence $(x_n)_{n=1}^{\infty}$ exists. Assume that $y_n \searrow x$ and $f(y_n) \neq f(x)$ for $n \in \mathbb{N}$. Let $(K_n)_{n=1}^{\infty}$ be a sequence of paths such that $K_n \subset (f(x), \min(f(y_n), f(x)+1/n))$ if $f(y) > f(x)$ and $K_n \subset (\max(f(y_n), f(x)-1/n, f(x))$ otherwise. Because $f \in \mathcal{E}IVP$, for each $n \in \mathbb{N}$ there exists a path $C_n \subset (x, y_n)$ such that $f(C_n) \subset K_n$. For each $n \in \mathbb{N}$ choose a point $x_n \in C_n$. Then $x_n \searrow x$ and $f(x_n) \to f(x)$. □

Lemma 3.1 *If $\mathcal{E}$ is a σ-system of paths, then $\mathcal{E}IVP \subset \mathcal{C}_{\mathcal{E}}$ and the opposite inclusion does not hold.*

PROOF. Let f be an arbitrary function satisfying $\mathcal{E}IVP$ and $x \in \mathbb{R}$. We shall construct a right path E_x leading to x such that $f|E_x$ is continuous at x.

Notice that if for some $\delta > 0$, $f(y) = f(x)$ for each $y \in [x, x+\delta]$, then for arbitrary right path R_x leading to x the function $f|R_x$ is continuous at x. Otherwise by Remark 3.1 there exists a sequence $(x_n)_{n=1}^{\infty}$ of reals such that $x_n \searrow x$, and $f(x_n)$ is monotonically convergent to $f(x)$. Suppose that $f(x_{n+1}) < f(x_n)$ for each n. Then for each path P_n between $f(x_{n+1})$ and $f(x_n)$ there exists a path E_n between x_{n+1} and x_n such that $f(E_n) \subset P_n$ for all $n \in \mathbb{N}$. Since $\mathcal{E}$ is a σ-system of paths, $\bigcup_{n=1}^{\infty} E_n \cup \{x\}$ is a right path leading to x and $f|E_x$ is continuous at x. In the same way we can prove that f has a left path at x.

By Lemma 2.1 there exists an $\mathcal{E}$-continuous function f such that $f(\mathbb{R}) = \{0, 1\}$. This function is not $\mathcal{E}IVP$. Thus $\mathcal{C}_{\mathcal{E}} \not\subset \mathcal{E}IVP$. □

Remark 3.2 *Note that if $\mathcal{E}$ is a collection of open intervals and $\mathcal{C}_{\mathcal{E}}$ is the class of all $\mathcal{E}$-continuous functions, then the first assertion of Lemma* 3.1 *is not true. Thus the assumption that $\mathcal{E}$ is σ-system is important.*

Theorem 3.1 *If $\mathcal{E}$ is a δ-system of paths, then $\mathcal{E}IVP = (\sigma\mathcal{E})IVP$.*

PROOF. Suppose that f is $\mathcal{E}IVP$. Choose $x, y \in \mathbb{R}$ such that $f(x) \neq f(y)$ and let $K \in \sigma\mathcal{E}$ be a path between $f(x)$ and $f(y)$. Then there exists $K_0 \in \mathcal{E}$ such that $K_0 \subset K$ and $C_0 \in \mathcal{E}$ for which $f(C_0) \subset K_0 \subset K$. Thus f is $(\sigma\mathcal{E})IVP$. Choose a function g having $(\sigma\mathcal{E})IVP$. Let x, y be such that $f(x) \neq f(y)$ and let $K \in \mathcal{E}$ be between $f(x)$ and $f(y)$. Then there exists $C \in \sigma\mathcal{E}$ such that $f(C) \subset K$. Each path $C \in \sigma\mathcal{E}$ contains a paths $C_0 \in \mathcal{E}$; so $f(C_0) \subset f(C) \subset K$, which completes the proof. □

Example 3.1 *There exists a σ-system of paths $\mathcal{E}$ for which $\mathcal{C} \not\subset \mathcal{E}IVP$ and $\mathcal{E}IVP \not\subset \mathcal{C}$.*

PROOF. Let W be the set of all algebraic numbers, $W = \{x \in \mathbb{R}; w(x) = 0$ for some $w \in \mathbb{Q}[x]\}$. Define $F_x^\varepsilon = \mathbb{Q} \cap (x-\varepsilon, x+\varepsilon)$ and $\mathcal{F}(x) = \{F_x^\varepsilon; \varepsilon > 0\}$ if $x \in \mathbb{Q}$, and $\mathcal{F}(x) = \{A \in 2^{\mathbb{R}\setminus W}$; x is a point of bilateral accumulation of A$\}$ otherwise. Put $\mathcal{E} = \sigma\mathcal{F}$. Define a continuous function f by $f(x) = \sqrt{|x|}$. We shall prove that f is not $\mathcal{E}IVP$. Let $x = 0$, $y = 1$ and $K = (0, 1) \cap \mathbb{Q}$. Then $K \in \mathcal{E}$ and $K \subset (f(x), f(y))$. Choose a path $C \subset (x, y)$. Since $\mathcal{E}IVP = \mathcal{F}IVP$, we can assume that $C \in \mathcal{F}$. If C is a path leading to $x \in \mathbb{R}\setminus\mathbb{Q}$, then $f(C) \cap (\mathbb{R}\setminus\mathbb{Q}) \neq \emptyset$, and thus $f(C) \not\subset K$. If C is a path leading to some $x \in \mathbb{Q}$, then it contains a rational number y such that $\sqrt{y} \notin \mathbb{Q}$. Therefore $f(C) \not\subset K$, too. Put

$$g(x) = \begin{cases} 4n(2n-1)x - 4n + 1 & \text{if } x \in [\frac{1}{2n}, \frac{1}{2n-1}],\ n \in \mathbb{N} \\ -4n(2n+1)x + 4n + 1 & \text{if } x \in (\frac{1}{2n+1}, \frac{1}{2n}),\ n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then g is a discontinuous function having $\mathcal{E}IVP$. □

Theorem 3.2 *$\mathcal{E}IVP = \mathcal{C}$ holds for no system of paths $\mathcal{E}$.*

PROOF. Assume that $\mathcal{C} \subset \mathcal{E}IVP$. Let $f(x) = \sin 1/x$ if $x \neq 0$ and $f(x) = 0$ for $x = 0$. Choose x, y such that $x < y$ and $f(x) \neq f(y)$. Let K be an arbitrary path from $\mathcal{E}$ which is between $f(x)$ and $f(y)$. Assume that $y > 0$. (The proof in the other case is similar.) Then there exists a point x_1 such that $0 < x_1 < y$ and $f(x_1) = f(x)$. Since $f|[x_1, y]$ is continuous and K is between $f(x_1)$ and $f(y)$, there exists a path $C \subset (x_1, y)$ from $\mathcal{E}$ for which $f(C) \subset K$. Therefore $f \in \mathcal{E}IVP \setminus \mathcal{C}$. □

Theorem 3.3 *If $\mathcal{E}$ is δ-system of paths and if $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{E}IVP$ closed function, then f is continuous.*

PROOF. If $\mathcal{E}$ is δ-system of paths, then by Theorem 3.1, $\mathcal{E}IVP = (\sigma\mathcal{E})IVP$. Then by Lemma 3.1 each function f having $(\sigma\mathcal{E})IVP$ is $\sigma\mathcal{E}$-continuous. Thus by Theorem 2.1, f is continuous. □

Remark 3.3 *Let $E_x = \mathbb{R}$ and $\mathcal{E} = \{\mathbb{R}\}$ be a system of paths. Then each function $f : \mathbb{R} \to \mathbb{R}$ has $\mathcal{E}IVP$ and claim of Theorem 3.3 is not true. Thus the assumption that $\mathcal{E}$ is a δ-system is important.*

Theorem 3.4 *If $\mathcal{E}$ is a system of paths, then $g \circ f \in \mathcal{E}IVP$ for all functions $f, g \in \mathcal{E}IVP$.*

PROOF. Choose $x, y \in \mathbb{R}$ such that $x < y$ and $g(f(x)) \neq g(f(y))$. Let K be arbitrary path between $g(f(x))$ and $g(f(y))$. Then $f(x) \neq f(y)$ and there exists a path P between $f(x)$ and $f(y)$ such that $g(P) \subset K$. Notice that there exists a path $C \subset (x, y)$ for which $f(C) \subset P$. Consequently, $g(f(C)) \subset K$. □

Lemma 3.2 *Let $\mathcal{E}$ be a δ-system of paths, $f \in \mathcal{E}IVP$ and $x_0 \in \mathbb{R}$. If $m = \inf K^+(f, x_0)$ ($m = \inf K^-(f, x_0)$) and if $M = \sup K^+(f, x_0)$ ($M = \sup K^-(f, x_0)$), then $K^+(f, x_0)$ ($K^-(f, x_0)$) is equal to the interval $[m, M]$.*

PROOF. If f is a continuous function from the right at x_0, then $m = M$. Suppose that f is discontinuous from the right at x_0 and there exists an open bounded interval $(a, b) \subset (m, M)$ such that $(a, b) \cap K^+(f, x_0) = \emptyset$ and $m, M, f(x_0) \notin [a, b]$. Then there exists a point $x_1 > x_0$ such that

$$[a, b] \subset (\min\{f(x_0), f(x_1)\}, \max\{f(x_0), f(x_1)\}) \text{ and } f(x) \notin (a, b)$$

for $x \in [x_0, x_1]$. Choose a path $K \subset (a, b)$. Then K is between $f(x_0)$ and $f(x_1)$ and $f(C) \not\subset K$ for each path $C \subset (x_0, x_1)$. This is impossible; so $K^+(f, x_0)$ is dense in $[m, M]$. Since $K^+(f, x_0)$ is closed, $K^+(f, x_0) = [m, M]$. □

Theorem 3.5 *Let $\mathcal{E}$ be a δ-system of paths, $f \in \mathcal{E}IVP$, $x_0 \in \mathbb{R}$ and $z \in K(f, x_0) \setminus \{\pm\infty\}$. Then the function*

$$g(x) = \begin{cases} f(x) & \text{if } x \neq x_0 \\ z & \text{otherwise} \end{cases}$$

has the $\mathcal{E}$-intermediate value property.

PROOF. Choose $x, y \in \mathbb{R}$ such that $x < y$ and $g(x) \neq g(y)$. We can assume that $g(x) < g(y)$. Choose an arbitrary path $K_s \subset (g(x), g(y))$ leading to some $s \in (g(x), g(y))$. We shall consider two cases.

1. Assume that $x = x_0$. (If $y = x_0$, the proof is analogous.)
   Set $c = \min\{|s - g(x)|, |s - g(y)|\}$. Let K_1 be a path leading to s such that $K_s \cap (s - \frac{c}{2}, s + \frac{c}{2})$. Because $(\min\{z, f(x_0)\}, \max\{z, f(x_0)\}) \subset K(f, x_0)$, we can choose a point x_1 such that $x_0 < x_1 < y$ and

   (a) $f(x_1) \in (f(x_0), z)$ if $f(x_0) < z$,

(b) $f(x_1) \in (z, s - \frac{c}{2})$ if $f(x_0) \in (z, g(y))$,

(c) $f(x_1) \in (s + \frac{c}{2}, g(y))$ if $f(x_0) \geq g(y)$.

Then $K_1 \subset (g(x_1), g(y))$ and there exists a path C such that $C \subset (x_1, y)$ if (a) or (b) holds, $C \subset (x_0, x_1)$ if (c) holds, and $g(C) \subset K_1$.

2. Suppose that $x \neq x_0 \neq y$.
If $x_0 \notin (x, y)$, then there exists a path C between x and y such that $g(C) = f(C) \subset K_s$. Let $x_0 \in (x, y)$. Then there exists a path C_r leading to some $r \in (x, y)$ such that $f(C_r) \subset K_s$. If $x_0 \neq r$, then there exists a positive number δ and path C leading to r such that $x_0 \notin C \subset C_r \cap (r - \delta, r + \delta)$. Thus $g(C) = f(C) \subset K_s$. Assume that $x_0 = r$ and $f(r) < g(r)$. (If $f(r) > g(r)$, then the proof is similar.) We shall consider two cases.

   (i) If $f(r) < s$, then $s \in (f(r), f(y))$ and $K_s \cap (f(r), f(y)) \subset (g(x), g(y))$ contains a path K. But then there exists a path $C \subset (r, y) \subset (x, y)$ such that $g(C) = f(C) \subset K \subset K_s$.

   (ii) If $f(r) > s$, then $s \in (f(x), f(r))$. Because $f \in \mathcal{E}IVP$, there exists a path $C \subset (x, r)$ such that $g(C) = f(C) \subset K \subset K_s$.

This completes the proof. □

Lemma 3.3 *Let $f : (-\infty, a) \to \mathbb{R}$, $g : (a, \infty) \to \mathbb{R}$, $f, g \in \mathcal{E}IVP$ and $c \in [K^-(f, a) \cap K^+(g, a)] \setminus \{\pm\infty\}$. If $\mathcal{E}$ is a Δ-system of paths, then the function*

$$h(x) = \begin{cases} f(x) & \text{if } x < a \\ c & \text{if } x = a \\ g(x) & \text{if } x > a \end{cases}$$

has $\mathcal{E}IVP$.

Proof. Choose x, y such that $h(x) < h(y)$ and a path $K \in \mathcal{E}$ such that $K \subset (h(x), h(y))$. Suppose that $x < y$. It is enough to prove that there exists a path $C \in \mathcal{E}$ between x and y such that $h(C) \subset K$. If $x, y \in (-\infty, a)$ or $x, y \in (a, \infty)$, then such a path C exists, because $f, g \in \mathcal{E}IVP$. If $x = a$ or $y = a$, then by Theorem 3.5, there exists a path $C \in \mathcal{E}$ such that $C \subset (x, y)$ and $f(C) \subset (h(x), h(y))$.
Suppose that $x < a < y$. We shall consider two cases.

1. $c \notin [h(x), h(y)]$. Assume that $c < h(x)$. Then there exists a point $s \in (a, y)$ such that $h(s) < h(x)$. Since $g \in \mathcal{E}IVP$ and $K \subset (h(s), h(y))$, there exists a path $C \subset (s, y) \subset (x, y)$ with $h(C) = g(C) \subset K$.

2. $c \in [h(x), h(y)]$. Since $\mathcal{E}$ is a Δ-system of path, $(h(x), c) \cap K$ or $(c, h(y)) \cap K$ contains a path $K_1 \in \mathcal{E}$. We can assume that $K_1 \subset (c, h(y)) \cap K$. (Otherwise the proof is analogous.) Then $h(y) = g(y)$. Let s be a point such that $a < s < y$, $g(x) < g(s) < g(y)$ and $(g(s), g(y)) \cap K_1$ contains a path K_2. Because $g \in \mathcal{E}IVP$, there exists a path $C \subset (s, y) \subset (x, y)$ such that $g(C) = h(C) \subset K_2 \subset K$. □

Remark 3.4 *There exists a δ-system of paths $\mathcal{E}$ and functions $f, g \in \mathcal{E}IVP$, $f : (-\infty, 0) \to \mathbb{R}$, $g : (0, \infty) \to \mathbb{R}$ such that $0 \in K^-(f, 0) \cap K^+(g, 0)$ and the function $h : \mathbb{R} \to \mathbb{R}$ defined by*

$$h(x) = \begin{cases} f(x) & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ g(x) & \text{if } x > 0 \end{cases}$$

does not have $\mathcal{E}IVP$.

PROOF. Let $\mathcal{E}$ be a δ-system of paths containing all sets having a point of bilateral accumulation. Then $E_0 = \{(-1)^n,\ n \in \mathbb{N}\} \cup \{0\} \in \mathcal{E}$ and $E_0 \setminus \{0\}$ contains no path; so $\mathcal{E}$ is not a Δ-system of paths. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of all open intervals having rational endpoints such that $I_n \subset (-\infty, 0)$ for $n \in \mathbb{N}$. Let $\{C_{n,\alpha}\}_{n \in \mathbb{N}, \alpha < c}$ be a family of pairwise disjoint Cantor sets such that $C_{n,\alpha} \subset I_n$ for $n \in \mathbb{N}$, $\alpha < c$ where c means the cardinality of the reals (cf. Lemma 2 [8] and [4]). Let $\{x_\alpha\}_{\alpha < c}$ be the net of $(-\infty, 0) \setminus E_0$. Put

$$f(x) = \begin{cases} x_\alpha & \text{if } x \in \bigcup_{n=1}^{\infty} C_{n,\alpha} \text{ and } \alpha < c \\ -1 & \text{otherwise,} \end{cases}$$

$$g(x) = \begin{cases} -x_\alpha & \text{if } x \in -\bigcup_{n=1}^{\infty} C_{n,\alpha} \text{ and } \alpha < c \\ 1 & \text{otherwise.} \end{cases}$$

Then $f, g \in \mathcal{E}IVP$.
Choose a x, y such that $x < 0 < y$, $h(x) < -1$ and $1 < h(y)$. Then $h(C) \not\subset E_0$ for each $C \in \mathcal{E}$ and $h \notin \mathcal{E}IVP$. □

Remark 3.5 *$\mathcal{E}IVP \not\subset \mathcal{E}const$ and $\mathcal{E}const \not\subset \mathcal{E}IVP$.*

PROOF. The function $f(x) = x$ has $\mathcal{E}IVP$ and $f \notin \mathcal{E}const$. Let $x_n = (-1)^n$ for $n \in \mathbb{N}$. By Lemma 2.1 there is an $\mathcal{E}$-continuous function g such that $g(\mathbb{R}) = \{-1, 1\}$. Note that $g \in \mathcal{E}const$ and $g \notin \mathcal{E}IVP$. □

4 Compositions with $\mathcal{E}$-continuous Functions

For the remainder of this paper $\mathcal{E}$ denotes a σ-system of paths.

Theorem 4.1 $\mathcal{M}_{out}(\mathcal{C}_{\mathcal{E}}) = \mathcal{C}$.

PROOF. The inclusion $\mathcal{C} \subseteq \mathcal{M}_{out}(\mathcal{C}_{\mathcal{E}})$ is obvious. Now we shall prove the opposite inclusion. Let g be an $\mathcal{E}$-continuous function and suppose that g is not continuous at y_0 from the right. Choose $y \in K^+(g, y_0) \setminus \{g(y_0)\}$. Let $c = |y - g(y_0)|$ if $|y| \neq \infty$ and $c = 1$ otherwise. Then there exists a sequence $(y_n)_{n=1}^{\infty}$ such that $y_n \searrow y_0$, $\lim_{n\to\infty} g(y_n) = y$ and $|g(y_n) - g(y_0)| > c/2$ for each $n \in \mathbb{N}$. By Lemma 2.1 there exists an $\mathcal{E}$-continuous function f for which $f(\mathbb{R} \setminus \{0\}) = \{y_n; n \in \mathbb{N}\}$ and $f(\{0\}) = \{y_0\}$. Then for each $x \neq 0$ there exists an $n \in \mathbb{N}$ such that $|g \circ f(x) - g \circ f(0)| = |g(y_n) - g(y_0)| > c/2 > 0$. Consequently, $g \circ f(0) \notin K^+(g \circ f, 0)$ and $g \circ f \notin \mathcal{C}_{\mathcal{E}}$, which completes the proof. $\square$

Corollary 4.1 $\mathcal{M}_{out}(\mathcal{PC}) = \mathcal{M}_{out}(\mathcal{PR}) = \mathcal{M}_{out}(\mathbb{Q}_0) = \mathcal{M}_{out}(\mathcal{C}(m)) = \mathcal{C}$.

Theorem 4.2 $\mathcal{E}IVP \subset \mathcal{M}_{in}(\mathcal{C}_{\mathcal{E}})$.

PROOF. Choose an $x \in \mathbb{R}$. We shall prove that $g \circ f$ has a right path leading to x. If there exists a right path R_x leading to x such that $f|R_x \equiv f(x)$, then $g \circ f|R_x \equiv g(f(x))$; so $g \circ f$ is continuous at x. Otherwise there exists a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \searrow x$ and $f(x_n)$ is monotonically convergent to $f(x)$. Assume that $f(x_{n+1}) < f(x_n)$. By Remark 2.2 there exists a path $E_{f(x)}$ leading to $f(x)$ such that $g|E_{f(x)}$ is continuous at $f(x)$ and $E_{f(x)} \cap (f(x_{n+1}), f(x_n))$ contains a path E_n for infinitely many $n \in \mathbb{N}$. Since $f \in \mathcal{E}IVP$ and $E_n \subseteq (f(x_{n+1}), f(x_n))$, there exists a path $F_n \subseteq (x_{n+1}, x_n)$ such that $f(F_n) \subseteq E_n$. Note that $E_x = \bigcup_{n=1}^{\infty} F_n \cup \{x\}$ is a right path leading to x and $g \circ f|E_x$ is continuous at x. $\square$

Corollary 4.2 *Note that if $f \in \mathcal{E}const$, then $g \circ f$ is an $\mathcal{E}$-continuous function for every $\mathcal{E}$-continuous function g. By Remark* 3.5, *$\mathcal{E}IVP \not\subset \mathcal{E}const$. Thus $\mathcal{M}_{in}(\mathcal{C}_{\mathcal{E}}) \not\subset \mathcal{E}IVP$.*

Question 4.1 *Characterize the class $\mathcal{M}_{in}(\mathcal{C}_{\mathcal{E}})$.*

Lemma 4.1 *If $\mathcal{E}$ is a c-system of paths, then there exists a one-to-one, $\mathcal{E}$-continuous function $f : \mathbb{R} \to \mathbb{R}$ of the second class of Baire, such that $f(\mathbb{R})$ is an F_σ, uncountable, first category, measure zero set.*

PROOF. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of all open intervals with rational endpoints. In each I_k choose a sequence $\{C_{k,n}\}_{n=1}^{\infty}$ of Cantor measure zero sets such that $C_{k,n} \cap C_{m,p} = \emptyset$ for $(k,n) \neq (m,p)$. Such a sequence $\{C_{k,n}\}_{k,n=1}^{\infty}$ exists since for $k \in \mathbb{N}$ the set $I_k \setminus \bigcup_{l=1}^{k-1} \bigcup_{m=1}^{\infty} C_{l,m} \setminus \bigcup_{p=1}^{n-1} C_{k,p}$ is a G_δ, uncountable set [4, p. 387]. Let $f_{k,n} : C_{k,n} \to C_{n,k}$ be a homeomorphism of the

Cantor sets $C_{k,n}$ and $C_{n,k}$ for $k,n \in \mathbb{N}$. Denote by C an arbitrary Cantor set contained in $\mathbb{R} \setminus \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} C_{k,n}$. By [7] there exists a bijection of the first class of Baire $\varphi : \mathbb{R} \setminus \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} C_{k,n} \to C$. Put

$$f(x) = \begin{cases} f_{k,n}(x) & \text{if } x \in C_{k,n},\ k,n \in \mathbb{N} \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Then $f(\mathbb{R}) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n,k} \cup C$ is a first category measure zero set and f is an injection of the second class of Baire. Note that the set $f|H$ where $H = \{x \in C_{k,n}; x \text{ is a point of bilateral accumulation of } C_{k,n} \text{ for } k,n \in \mathbb{N}\}$ is bilaterally dense in the graph of f and $f|C_{k,n}$ is continuous at x for any $x \in H$ and $x \in C_{k,n}$ for $k,n \in \mathbb{N}$. Then by Lemma 2.2 [1], f is $\mathcal{E}$-continuous. □

Theorem 4.3 *Let $\mathcal{E}$ be an arbitrary c-system. There exists a one-to-one $\mathcal{E}$-continuous function of the second class of Baire $f_0 : \mathbb{R} \to \mathbb{R}$ such that every $f : \mathbb{R} \to \mathbb{R}$ can be represented as a composition of f_0 with some measurable $\mathcal{E}$-continuous function $f_1 : \mathbb{R} \to \mathbb{R}$ which has the Baire property. Thus every function f is a composition of two $\mathcal{E}$-continuous functions.*

PROOF. Let $\{I_k\}_{k=1}^{\infty}$ be a sequence of all open intervals with rational endpoints and let f_0 be a function from Lemma 4.1. Since $f(\mathbb{R})$ is an F_σ, uncountable, first category, measure zero set, in each interval I_k we can choose a sequence of Cantor sets $\{K_{k,n}\}_{n=1}^{\infty}$ such that $K_{k,n} \cap f(\mathbb{R}) = \emptyset$ and $K_{k,n} \cap K_{m,p} = \emptyset$ for $(k,n) \neq (m,p)$. Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of rationals. Define

$$f_1(y) = \begin{cases} f(f_0^{-1}(y)) & \text{if } y \in f_0(\mathbb{R}) \\ q_k & \text{if } y \in K_{k,n} \\ 0 & \text{otherwise.} \end{cases}$$

If x is a point of bilateral accumulation of $K_{k,n}$, then $f_1|K_{k,n}$ is continuous at x and the union of the set of all points of bilateral accumulation of $K_{k,n}$ is bilaterally dense in the graph of f_1. Thus f_1 is $\mathcal{E}$-continuous. Choose an $x \in \mathbb{R}$. Then $f_1 \circ f_0(x) = f(f_0^{-1}(f_0(x))) = f(x)$. □

Lemma 4.2 *Assume that a function $f : \mathbb{R} \to \mathbb{R}$ fulfills the condition:*

(i) for each interval $J \subset \mathbb{R}$ and for each first category set $F \subset \mathbb{R}$ there exist Cantor sets $C_1,\ C_2 \subset J \setminus F$ such that $f(C_1)$ and $f^{-1}(C_2)$ are of first category.

Then there exist families of sets $\{A_y;\ y \in \mathbb{R}\}$ and $\{B_y;\ y \in \mathbb{R}\}$ such that:

(1) *Let $\{I_n\}_{n \in \mathbb{N}}$ be a set of all intervals having rational endpoints. For each $y \in \mathbb{R}$ there exist families of Cantor sets $\{C_{n,y}\}_{n \in \mathbb{N}}$, $\{K_{n,y}\}_{n \in \mathbb{N}}$ such that $\bigcup_{n=1}^{\infty} C_{n,y} \subset A_y$, $\bigcup_{n=1}^{\infty} K_{n,y} \subset B_y$, $C_{n,y} \cap C_{m,y} = \emptyset = K_{n,y} \cap K_{m,y}$ for $m \neq n$ and $C_{n,y}$, $K_{n,y} \subset I_n$, interval $J \subset \mathbb{R}$, $A_y \cap J$ and contain a Cantor set,*

(2) *if* $y \neq y_1$, *then* $A_y \cap A_{y_1} = \emptyset = B_y \cap B_{y_1}$,
(3) $\bigcup_{y\in\mathbb{R}} A_y$, $\bigcup_{y\in\mathbb{R}} B_y$ *are of first category,*
(4) $f(\bigcup_{y\in\mathbb{R}} A_y) \cap \bigcup_{y\in\mathbb{R}} B_y = \emptyset$.

Proof. Let $\{I_k\}_{n=1}^{\infty}$ be a sequence of all open intervals with rational endpoints. Let $\{C_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ be sequences of Cantor sets such that $f(C_n)$, $f^{-1}(K_n)$ are first category sets and

$$\begin{array}{lll} K_1 & \subset & I_1, \\ C_1 & \subset & I_1 \setminus f^{-1}(K_1), \\ \vdots & & \vdots \\ K_n & \subset & I_n \setminus [\bigcup_{k=1}^{n-1} K_k \cup f(\bigcup_{k=1}^{n-1} C_k)], \\ C_n & \subset & I_n \setminus [\bigcup_{k=1}^{n-1} C_k \cup f^{-1}(\bigcup_{k=1}^{n} K_k)], \\ \vdots & & \vdots \end{array}$$

Represent all sets C_n and K_n as unions $C_n = \bigcup_{\alpha<c} C_{n,\alpha}$ and $K_n = \bigcup_{\alpha<c} K_{n,\alpha}$ of pairwise disjoint perfect sets (cf. [4]). Let $(y_\alpha)_{\alpha<c}$ be a transfinite sequence of all reals. Put $A_y = \bigcup_{n=1}^{\infty} C_{n,\alpha}$, $B_y = \bigcup_{n=1}^{\infty} K_{n,\alpha}$ where $y = y_\alpha$ and $y \in \mathbb{R}$. Obviously, the families of sets $\{A_y;\ y \in \mathbb{R}\}$, $\{B_y;\ y \in \mathbb{R}\}$ fulfill the conditions (1)–(4). □

Theorem 4.4 *If a function* $f : \mathbb{R} \to \mathbb{R}$ *fulfills condition (i) and* $g : \mathbb{R} \to \mathbb{R}$ *is a surjection, then there exist* $\mathcal{E}$*-continuous surjections* $h_1,\ h_2 : \mathbb{R} \to \mathbb{R}$ *such that* $h_1 \circ f = g \circ h_2$.

Proof. Let a function f fulfill condition (i), g be a surjection and $\{A_y; y \in \mathbb{R}\}$, $\{B_y; y \in \mathbb{R}\}$ be families of sets from Lemma 4.2. We shall construct functions $h_2 : \mathbb{R} \to \mathbb{R}$ and $h_1 : \mathbb{R} \to \mathbb{R}$.

Put $h_2|A_y \equiv h_1|B_y \equiv y$ for $y \in \mathbb{R}$. If $x \in f^{-1}(B_y)$, then let $h_2(x) = z$, where z is an arbitrary point from a set $g^{-1}(\{y\})$. For $x \in f(A_y)$ define $h_1(x) = g(y)$. Now we define a value of functions h_2 and h_1 in a set $S_2 = \bigcup_{y\in\mathbb{R}}(A_y \cup f^{-1}(B_y))$ and $S_1 = \bigcup_{y\in\mathbb{R}}(B_y \cup f(A_y))$, respectively. For $x \in \mathbb{R} \setminus S_1$ let $h_1(x) = 0$. Fix an $x \in \mathbb{R} \setminus S_2$. If $f(x) \in \mathbb{R} \setminus S_1$, then define $h_2(x) = t$ where t is an arbitrary point for which $g(t) = 0$. Suppose that $f(x) \in S_1$. If $f(x) \in B_y$ for some $y \in \mathbb{R}$, then put $h_2(x) = z$, where z is an arbitrary point belonging to the set $g^{-1}(\{y\})$. If $f(x) \in f(A_y)$ for $y \in \mathbb{R}$ set $h_2(x) = y$.

We shall prove that $h_1 \circ f = g \circ h_2$.

a) If $x \in A_y$ for some $y \in \mathbb{R}$, then $h_2(x) = y$, $f(x) \in f(A_y)$ and we have $h_1 \circ f(x) = g(y) = g \circ h_2(x)$.

b) If $x \in f^{-1}(B_y)$ for some $y \in \mathbb{R}$, then $h_2(y) \in g^{-1}(\{y\})$ and therefore $g(h_2(x)) = y$. Notice that $f(x) \in B_y$ and $h_1(f(x)) = y$, also.

c) If $x \in \mathbb{R} \setminus S_2$ and $f(x) \in \mathbb{R} \setminus S_1$, then $g(h_2(x)) = 0 = h_1(f(x))$.

d) If $x \in \mathbb{R} \setminus S_2$ and $f(x) \in S_1$, then either $g(h_2(x)) = y = h_1(f(x))$ if $f(x) \in B_y$ or $g(h_2(x)) = g(y) = h_1(f(x))$ if $f(x) \in f(A_y)$.

By Lemma 4.2 there exists a sequence of pairwise disjoint Cantor sets $\{C_{n,y}\}_{n\in\mathbb{N},y\in\mathbb{R}}$ such that $C_{n,y} \subset I_n$ and $C_{n,y} \subset A_y$. Let P_2 be the set of all points $z \in \bigcup_{y\in\mathbb{R}} A_y$ such that z is a point of bilateral accumulation of a $C_{n,y}$ for some $n \in \mathbb{N}$, $y \in \mathbb{R}$. Then $h_2|P_2$ is bilaterally dense in the graph of the function h_2 and because each $z \in P_2$ is a point at which the function h_2 is $\mathcal{E}$-continuous, h_2 is $\mathcal{E}$-continuous everywhere. In the same way we can prove that h_1 is $\mathcal{E}$-continuous. □

5 Transfinite Limits

Recall that a function f is a limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of functions iff for each positive $\varepsilon > 0$ and $x \in \mathbb{R}$ there exists an $\alpha < \omega_1$ such that $|f(x) - f_\beta(x)| < \varepsilon$ for all $\beta > \alpha$.

Theorem 5.1 *Let $\mathcal{E}$ be a c-system. Then every function $f : \mathbb{R} \to \mathbb{R}$ is a limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of $\mathcal{E}$-continuous functions. Moreover, if f is measurable or $f$is Baire γ ($\gamma \geq 2$), then f_α can be taken from the same class for $\alpha < \omega_1$.*

PROOF. Let $(I_k)_{k=1}^\infty$ be a sequence of all open intervals with rational endpoints. We shall use the fact that in each interval I_k we can choose a sequence $(C_{k,n})_{n=1}^\infty$ of Cantor sets such that $C_{k,n} \cap C_{m,p} = \emptyset$ for $(k,n) \neq (m,p)$ (cf. Theorem 4.1). Because there exists a homeomorphism between $C_{k,n}$ and $C_{k,n} \times C_{k,n}$; so we can represent each $C_{k,n}$ as a union $\bigcup_{\alpha<\omega_1} C_{k,n,\alpha}$ of pairwise disjoint perfect sets. Let $(q_n)_{n=1}^\infty$ be a sequence of all rationals. Put

$$D_{n,\alpha} = \bigcup_{k=1}^{\infty} C_{k,n,\alpha} \quad \text{and} \quad f_\alpha(x) = \begin{cases} q_n & \text{if } x \in D_{n,\alpha},\ n \in \mathbb{N} \\ f(x) & \text{otherwise} \end{cases}$$

for $\alpha < \omega_1$. Then each function f_α is $\mathcal{E}$-continuous ($\alpha < \omega_1$). We shall show that

$$f(x) = \lim_{\alpha\to\omega_1} f_\alpha(x). \tag{1}$$

Choose an $x \in \mathbb{R}$. Then either $x \notin \bigcup_{n=1}^\infty \bigcup_{\alpha<\omega_1} D_{n,\alpha}$; so $f_\alpha(x) = f(x)$ for each $\alpha < \omega_1$ and (1) holds, or $x \in D_{n,\beta}$ for some $\beta < \omega_1$ and $n \in \mathbb{N}$. Then $x \notin D_{k,\alpha}$ for $\alpha > \beta$ and $k \in \mathbb{N}$; so $f_\alpha(x) = f(x)$ for $\alpha > \beta$. If f is measurable or if f belongs to Baire class γ ($\gamma \geq 2$), then by the definition it is easy see that f_α belongs to the same class. □

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