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Jadwiga Wolnicka, Institute of Mathematics, Technical University of Łódź, al. Politechniki 11, 93-590, Łódź, Poland.

QUASI-UNIFORM CONVERGENCE OF SEQUENCES OF 1-IMPROVABLE DISCONTINUOUS FUNCTIONS

Abstract

In the paper it is shown that the strongly quasi-uniform limit of a sequence of 1-improvable discontinuous functions on a complete space X is a 1-improvable discontinuous function or a continuous function. Automatically the same result will be valid for uniform convergence.

Definition 1 ([3]) *Let $f : X \to Y$ (X, Y metric spaces). The function f has at some point x_0 an improvable discontinuity if $\lim_{x\to x_0} f(x)$ exists and $\lim_{x\to x_0} f(x) \neq f(x_0)$.*

In this paper we consider the class A_1 of real-valued functions f on some metric space X such that the function

$$f^{(1)}(x) = \begin{cases} \lim_{t\to x} f(t) & \text{if the limit exists} \\ f(x) & \text{if } \lim_{t\to x} f(t) \text{ doesn't exist} \end{cases} \tag{1}$$

are continuous.

We shall prove that the strongly quasi-uniform limit of a sequence of functions of the class A_1 belongs to A_1. The problem was suggested by T. Świątkowski.

Let A_0 denote the class of continuous functions. If a function $f \in A_1 \setminus A_0$, then it is called a 1-improvable discontinuous function ([1] and [2]). First we shall consider a subclass $\widetilde{A_1}$ of the class A_1; namely $f \in \widetilde{A_1}$ if and only if

$$\forall_{x\in X} f(x) \geq 0 \text{ and } \forall_{x\in X} f^{(1)}(x) = 0 \tag{2}$$

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Lemma 1 *Let $f : X \to \mathbb{R}$ and $f(x) = 0$ if x is an isolated point. Then the function f belongs to $\widetilde{A_1}$ if and only if*

$$\forall_{x\in X} f(x) \geq 0 \text{ and } \forall_{\sigma>0}(\{x \in X : f(x) \geq \sigma\}^d \cap A(f) = \emptyset), \quad (3)$$

where $A(f) = \{x \in X : f(x) > 0\}$. (The notation $\{\cdot\}^d$ means the set of limit points of a set).

PROOF. First we assume that $f \in \widetilde{A_1}$. If $f \in A_0 \cap \widetilde{A_1}$, then $f \equiv 0$ and the above condition is obvious. Assume that $f \in \widetilde{A_1} \setminus A_0$. Suppose that there exist a real number $\sigma_0 > 0$ and a point $x_0 \in X$ such that $f(x_0) > 0$ and $x_0 \in \{x \in X : f(x) \geq \sigma_0\}^d$. Hence there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x_0$ and $\liminf_{n\to\infty} f(x_n) \geq \sigma_0 > 0$. Simultaneously $f(x_0) > 0$ and $f^{(1)}(x_0) = 0$; so $\lim_{x\to x_0} f(x) = 0$. Thus we have a contradiction. Hence condition (3) holds.

Now, we assume that condition (3) holds. If $A(f) = \emptyset$, then $f \equiv 0$ and $f \in A_0 \cap \widetilde{A_1}$. Assume that $A(f) \neq \emptyset$. Let us take an arbitrary $x_0 \in X$. We will consider two cases. First, we assume that $x_0 \in A(f)$. Then $x_0 \notin \{x \in X : f(x) \geq \sigma\}^d$ for each $\sigma > 0$. Thus

$$\forall_{\sigma>0}\exists_{r>0}(K(x_0, r) \setminus \{x_0\}) \cap \{x : f(x) \geq \sigma\} = \emptyset,$$

where $K(x_0, r) = \{x \in X : \rho(x, x_0) < r\}$. So

$$\forall_{\sigma>0}\exists_{r>0}\forall_{x\in K(x_0,r)\setminus\{x_0\}} (0 \leq f(x) < \sigma),$$

which means that $\lim_{x\to x_0} f(x) = 0$ and then $f^{(1)}(x_0) = 0$.

Now, we assume that $x_0 \notin A(f)$. If $\lim_{x\to x_0} f(x)$ does not exist, then $f^{(1)}(x_0) = f(x_0) = 0$. Assume that $\lim_{x\to x_0} f(x)$ exists and equals y with $y > 0$. Then there exists a real number $r > 0$ such that $f(x) > y/2$ for $x \in K(x_0, r) \setminus \{x_0\}$. Let $x' \in K(x_0, r) \setminus \{x_0\}$ be arbitrary. There exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x'$ and, for each $n \in \mathbb{N}$, $x_n \in K(x_0, r) \setminus \{x_0\}$. Then $x' \in \{x : f(x) > y/2\}^d \cap A(f)$, a contradiction. Hence $\lim_{x\to x_0} f(x) = 0$ and then $f^{(1)}(x_0) = 0$. Thus $f^{(1)}(x) = 0$ for each $x \in X$. Hence $f \in \widetilde{A_1}$ and the proof is complete. □

Definition 2 ([3]) *The sequence $\{f_n\}$ is quasi-uniformly convergent on X to f if f_n approaches f on X and*

$$\forall_{\epsilon>0}\forall_{n\in\mathbb{N}}\exists_{p\in\mathbb{N}}\forall_{x\in X}\exists_{0\leq l\leq p} \mid f_{n+l}(x) - f(x) \mid < \epsilon.$$

Theorem 1 *If functions f_n $(n = 1, 2, \ldots)$ belong to $\widetilde{A_1}$ and $\{f_n\}$ is quasi-uniformly convergent on X to f, then f belongs to $\widetilde{A_1}$.*

PROOF. In view of the above assumption and Lemma 1 we confirm that

$$\forall_{n\in\mathbb{N}}\forall_{\sigma>0}(\{x \in X : f_n(x) \geq \sigma\}^d \cap A(f_n) = \emptyset), \tag{4}$$

where $A(f_n) = \{x \in X : f_n(x) > 0\}$.

Suppose that f does not belong to $\widetilde{A_1}$. Then there exists a real number $\sigma_0 > 0$ such that $\{x \in X : f(x) \geq \sigma_0\}^d \cap A(f) \neq \emptyset$. Now let $x_0 \in A(f)$ and $\{x_k\}$ be a sequence such that $\lim_{k\to\infty} x_k = x_0$ and

$$\forall_{k\in\mathbb{N}}(f(x_k) \geq \sigma_0). \tag{5}$$

Put $\epsilon_0 = 1/2\min(\sigma_0, f(x_0))$. Since the sequence $\{f_n(x_0)\}$ converges to $f(x_0)$, there exists $n' \in \mathbb{N}$ such that

$$\forall_{n>n'}(f_n(x_0) > f(x_0) - \epsilon_0). \tag{6}$$

This follows from the assumption that the sequence $\{f_n\}$ is quasi-uniformly convergent to f on X, and thus also on $\{x_k\}$. Therefore there exists a number $p_0 \in \mathbb{N}$ such that

$$\forall_{k\in\mathbb{N}}\exists_{0\leq l\leq p_0}\,(f_{n'+l}(x_k) > f(x_k) - \epsilon_0).$$

Hence, by (5) and by the selection of ϵ_0, we have

$$\forall_{k\in\mathbb{N}}\exists_{0\leq l\leq p_0}\,(f_{n'+l}(x_k) > \sigma_0/2).$$

Thus there exist $l' \in \{0, 1, \ldots, p_0\}$ and a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that $f_{n'+l'}(x_{k_m}) > \sigma_0/2$ for each $m \in \mathbb{N}$. Simultaneously, by (6) and by the selection of ϵ_0, $f_{n'+l'}(x_0) > 0$. Then $x_0 \in \{x \in X : f_{n'+l'}(x) \geq \sigma_0/2\}^d \cap A(f_{n'+l'})$. This contradicts (4). □

Definition 3 ([4]) *The sequence $\{f_n\}$ of functions is said to be strongly quasi-uniformly convergent to f on X if every subsequence $\{f_{n_k}\}$ converges quasi-uniformly to f.*

Theorem I ([5]) *For $n \in \mathbb{N}$ let f_n be continuous on X and let $\{f_n\}$ converge strongly quasi-uniformly on some dense subset Z of X to a function $f : Z \to \mathbb{R}$. Then $\{f_n\}$ is strongly quasi-uniformly convergent on X to a function φ, obviously continuous, whose restriction to Z coincides with f.*

Theorem 2 *For $n \in \mathbb{N}$ let f_n belong to the class A_1 and let $\{f_n\}$ converge strongly quasi-uniformly on a complete space X. Then the sequence $\{f_n^{(1)}\}$ (where each $f_n^{(1)}$ is defined by formula (1)) is also strongly quasi-uniformly convergent on X to a continuous function φ.*

PROOF. We denote by E_n the set of all points in which the function f_n has an improvable discontinuity ($n = 1, 2, \ldots$). E_n is a set of the first category ([3]). The restriction of f_n to $X \setminus E_n$ is continuous. Put $E = \bigcup_{n=1}^{\infty} E_n$. The set E is of the first category; so $X \setminus E$ is a residual subset of X and since X is complete, $X \setminus E$ is a dense subset of X. The sequence $\{f_n\}$ converges strongly quasi-uniformly on $X \setminus E$, and $f_n^{(1)} \mid (X \setminus E) = f_n \mid (X \setminus E)$. Then the sequence $\{f_n^{(1)}\}$ converges strongly quasi-uniformly on the dense subset $X \setminus E$ of X. Since each $f_n^{(1)}$ is continuous on X, we conclude that $\{f_n^{(1)}\}$ converges strongly quasi-uniformly on X to a continuous function φ, by Theorem 1. □

Lemma 2 *If a function $\mid f \mid$ belongs to $\widetilde{A_1}$, then f belongs to A_1 and $f^{(1)}(x) = 0$ for each $x \in X$.*

PROOF. Let E denote the set of points in which the function $\mid f \mid$ has an improvable discontinuity. Then by assumption, the function $\mid f \mid$ has positive values on the set E and is zero on $X \setminus E$. Thus $f(x) = 0$ for $x \in X \setminus E$. We shall prove that $f^{(1)}(x) = 0$ for each $x \in X$. It suffices to show that $\lim_{t \to x} f(t) = 0$ for each $x \in E$. Obviously, since $x \in E$, we have $\lim_{t \to x} \mid f(t) \mid = 0$. So $\lim_{t \to x} f(t) = 0$. □

Lemma 3 *If a function f belongs to A_1, then $\mid f - f^{(1)} \mid$ belongs to $\widetilde{A_1}$.*

We omit the easy proof.

Theorem 3 *If for each $n \in \mathbb{N}$ the function f_n belongs to A_1 and $\{f_n\}$ is strongly quasi-uniformly convergent on a complete space X to f, then f belongs to A_1 and $f^{(1)}$ is the strongly quasi-uniform limit of the sequence $\{f_n^{(1)}\}$.*

PROOF. The sequence $\{f_n\}$ is strongly quasi-uniformly convergent on X to f. Then, by Theorem 2, the sequence $\{f_n^{(1)}\}$ is strongly quasi-uniformly convergent on X to a continuous function φ. Therefore the sequence $\{f_n - f_n^{(1)}\}$ is strongly quasi-uniformly convergent on X to $f - \varphi$ ([4]) and consequently the sequence $\{\mid f_n - f_n^{(1)} \mid\}$ is strongly quasi-uniformly convergent on X to $\mid f - \varphi \mid$. Note that each function $\mid f_n - f_n^{(1)} \mid$ belongs to $\widetilde{A_1}$ (Lemma 3). By Theorem 1, we have $\mid f - \varphi \mid \in \widetilde{A_1}$. Then, the function $f - \varphi$ belongs to A_1 (Lemma 2). Therefore the function f ($f = (f - \varphi) + \varphi$) belongs to A_1.

It remains to prove that $f^{(1)}$ is the strongly quasi-uniform limit of the sequence $\{f_n^{(1)}\}$ or $f^{(1)} = \varphi$. Let $f - \varphi = g$. The function $\mid g \mid \in \widetilde{A_1}$; so by Lemma 2, $g^{(1)}(x) = 0$ for each $x \in X$. Let $x \in X$ be arbitrary. First, we assume that $\lim_{t \to x} g(t)$ exists. Then $\lim_{t \to x} f(t)$ exists and $f^{(1)}(x) = \lim_{t \to x} f(t) =$

$\lim_{t\to x} g(t)+\lim_{t\to x}\varphi(t) = 0+\varphi(x) = \varphi(x)$. Now, we assume that $\lim_{t\to x} g(t)$ doesn't exist. Then $\lim_{t\to x} f(t)$ doesn't exist. Hence $f^{(1)}(x) = f(x) = g(x) + \varphi(x) = 0 + \varphi(x) = \varphi(x)$. Thus $f^{(1)}(x) = \varphi(x)$ for each $x \in X$. □

Definition 4 ([1]) *Let $f : X \to \mathbb{R}$ and let $U(f) = \{x \in X : \lim_{t\to x} f(t) \neq f(x)\}$. For every ordinal α we define a function $f^{(\alpha)}$ by*

$$f^{(\alpha)}(x) = \begin{cases} f(x) & \text{if } \{\gamma < \alpha : x \in U(f^{(\gamma)})\} = \emptyset \\ \lim_{t\to x} f^{(\gamma_0)}(t) & \text{where } \gamma_0 = \min\{\gamma < \alpha : x \in U(f^{(\gamma)})\} \end{cases}$$

($f^{(0)}(x) = f(x)$ for each $x \in X$). We denote by A_α the class of functions f such that the function $f^{(\alpha)}$ is continuous. If a function $f \in A_\alpha \setminus \bigcup_{0\leq\beta<\alpha} A_\beta$, then it is called an α-improvable discontinuous function.

Problem 1 Does Theorem 3 remain valid for sequences of functions of the class A_α?

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