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MORE ON THE METRIC SPACE OF METRICS

Abstract

In this note we look at a subset of the metric space of metrics for an arbitrary set X and show that in terms of cardinality this can be very large while being extremely small in the porosity sense.

1 Introduction

We begin with some definitions.

Let (X, d) be a metric space. For $x \in X$ we denote by $B(x, \varepsilon)$ the open ball about x of radius r; that is

$$B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}.$$

Now let $x \in A \subseteq X$, and fix $R > 0$, a real number. We denote by $\gamma(x, R, A)$ the supremum of all real numbers $r > 0$ for which there exists a $z \in X$ such that $B(z, r) \subseteq B(x, R)$ and $B(z, r) \cap A = \emptyset$. The porosity of A at x is then given by

$$p(A, x) = \limsup_{R \to 0^+} \frac{2\gamma(x, R, A)}{R}.$$

Porosity values range between zero (not porous) and one (strongly porous). The set A is porous at x if $p(A, x) > 0$. A set is called porous if it is porous at each of its points. In a general metric space the class of porous sets is a

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subset of the nowhere dense sets. They are not equal; there exists nowhere dense sets which are not porous (see [4]).

If the "lim sup" in the definition of porous is replaced by a "lim inf" we have the definition of a set being *very porous* at a point, very strongly porous sets follow similarly. For a more detailed essay on porosities see [5].

For X a nonempty set we denote by $card\,(X)$ the cardinality of X and write $\aleph_0$ for the cardinality of the natural numbers. Define $\mathcal{M} = \mathcal{M}\,(X)$ to be the set of all metrics on X. In [2] T. Šalát, J. Tóth, and L. Zsilinszky placed a metric on $\mathcal{M}$ defined by

$$d^*\,(d, d') = \min\left\{1, \sup_{x,y\in X} \left(d\,(x,y) - d'\,(x,y)\right)\right\}.$$

In [2] subspaces $\mathcal{H}_\alpha$ and $\mathcal{H}$ were defined by $\mathcal{H}_\alpha = \{d: \ x \neq y \Rightarrow d\,(x,y) \geq \alpha\}$ and $\mathcal{H} = \bigcup_{\alpha>0} \mathcal{H}_\alpha = \bigcup_{k\in\mathbb{N}} \mathcal{H}_{1/k}$.

Results shown in [2] include $\mathcal{M}$ is a non-complete Baire space and $\mathcal{M}\backslash\mathcal{H}$ is nowhere dense in $\mathcal{M}$. Other results on the metric space of metrics may be found in [1] and [3].

2 Results

To gain our results we work with the mostly unexplored set $\mathcal{M}\backslash\mathcal{H}$ and a simple, but seemingly overlooked question. Can we tell, from the set X, whether or not $\mathcal{M}\backslash\mathcal{H}$ is nonempty and, if nonempty, how big is it? Two answers are given in the following theorems.

Theorem 1 *We have*

$$card\,(\mathcal{M}\backslash\mathcal{H}) = \begin{cases} 0 & \text{if } card\,(X) < \aleph_0 \\ 2^{card(X)} & \text{if } card\,(X) \geq \aleph_0 \end{cases}$$

PROOF. If $card\,(X) = n$, then for each $d \in \mathcal{M}$ there is a natural number k such that $\min_{x\neq y} \{d\,(x,y)\} > 1/k$. Thus each metric is in $\mathcal{H}$ and $card\,(\mathcal{M}\backslash\mathcal{H}) = 0$.

If $card\,(X) \geq \aleph_0$, let $A = \{a_n\}$ be a countable, infinite subset of X with infinite complement and let B be a subset of $X\backslash A$ with $card\,(B) = card\,(X)$. Place the following metric on X :

$$\rho_B\,(x,y) = \begin{cases} 0 & x = y \\ 1 & x \text{ or } y \text{ is in } X\backslash A,\ x, y \notin B,\ x \neq y \\ 2 & x \text{ or } y \text{ is in } B,\ x \neq y \\ \max\{1/n, 1/m\} & x = a_n,\ y = a_m. \end{cases}$$

It is easy to verify that ρ_B is a metric and that $\rho_B \neq \rho_{B'}$ if $B \neq B'$. Now, pick $\varepsilon > 0$. For this ε there is an $N = N(\varepsilon)$ such that for all $n \geq N$, $1/n < \varepsilon$. Thus for $n, m \geq N$, $\rho_B(a_n, a_m) < \varepsilon$. This says there are points in X arbitrarily close to one another so $\rho_B \notin \mathcal{H}$. There are $2^{card(X)}$ many choices for B so we can see

$$2^{card(X)} \leq card(\mathcal{M} \backslash \mathcal{H}) \leq card(\mathcal{M}) \leq 2^{card(X)}.$$

□

We now contrast this "largeness" of $\mathcal{M} \backslash \mathcal{H}$ in cardinality terms by showing that $\mathcal{M} \backslash \mathcal{H}$ is in fact as very porous as possible.

Theorem 2 *The set of metrics on the space X with values arbitrarily close to zero is a very strongly porous subset in the space of all metrics on X.*

Proof. We only need concern ourselves with the case that $card(X) \geq \aleph_0$ so assume this is so. Let ρ be an arbitrary metric in $\mathcal{M} \backslash \mathcal{H}$ and let ε_n be a sequence of positive values which converge to zero as n approaches infinity. Fix n. Let ρ_n be the metric defined by

$$\rho_n(x, y) = \begin{cases} 0 & \text{if } x = y \\ \rho(x, y) + \dfrac{\varepsilon_n}{2} & \text{if } x \neq y \end{cases}$$

Note two things: First that $\rho_n \in \mathcal{H}$ and second that $d^*(\rho, \rho_n) = \frac{1}{2}\varepsilon_n$, so that $\rho_n \in B(\rho, \varepsilon_n)$. It is easy to see that since $\rho_n \in \mathcal{H}_{\varepsilon_n/2}$, we have

$$B(\rho_n, (\varepsilon_n/2)) \subset \mathcal{H}.$$

So in terms of porosity

$$p(\mathcal{M} \backslash \mathcal{H}, \rho) \geq \lim_{n \to \infty} 2 \frac{(\varepsilon_n/2)}{\varepsilon_n} = 1.$$

Since ρ and $\{\varepsilon_n\}$ were arbitrary, this actually shows $\mathcal{M} \backslash \mathcal{H}$ is a very strongly porous subset of $\mathcal{M}$. □

What we have here is a metric space and a very porous subset which have equal cardinality. The next question to ask is if this is always the case. First note that in (X, d) with $d \in \mathcal{H}$ the only porous set is the empty set. This is the case also for some metrics in $\mathcal{M} \backslash \mathcal{H}$ when X is infinite (see Theorem 2.1).

There are however metric spaces where the cardinality of the nonempty very porous sets is neither zero nor the cardinality of the space, e.g. if

$$X = \left\{2^{-n} : n \in \mathbb{N}\right\} \cup \{0\}$$

with the usual topology, then the only very porous nonempty set in X is $\{0\}$.

Question 1 Can one, for any give space X, describe the set of metrics on X such that the cardinality of the very porous sets equals the cardinality of the space?

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