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SOME TYPICAL PROPERTIES OF SYMMETRICALLY CONTINUOUS FUNCTIONS, SYMMETRIC FUNCTIONS AND CONTINUOUS FUNCTIONS

Abstract

In this paper we show that the typical symmetrically continuous function and the typical symmetric function have c-dense sets of points of discontinuity. Also we show the existence of a nowhere symmetrically differentiable function and a nowhere quasi-smooth function by showing directly such functions are typical in the space of all real continuous functions.

1 Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetrically continuous at $x \in \mathbb{R}$ if

$$\lim_{h\to 0}[f(x+h) - f(x-h)] = 0.$$

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetric at $x \in \mathbb{R}$ if

$$\lim_{h\to 0}[f(x+h) + f(x-h) - 2f(x)] = 0.$$

In 1964 Stein and Zygmund [1, p. 25] showed that if $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and is symmetrically continuous on a Lebesgue measurable set E, then f is continuous a.e. on E. Also they obtained the same conclusion for symmetric functions [1, p. 27]. In 1971 Preiss [1, p. 52] constructed a

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bounded measurable, 2π-periodic function that is symmetrically continuous everywhere and whose set of points of discontinuity is of power c. In 1989 Tran in [2] constructed a bounded measurable symmetric function whose set of points of discontinuity is of power c, and also showed that the absolute value function of this function is symmetrically continuous and its set of points of discontinuity is of power c.

In 1964 Neugebauer first studied typical properties of symmetric functions. He showed that the typical function of the set of all bounded, measurable symmetric functions equipped with supremum metric has a dense set of discontinuity. His methods would also give a typical result for symmetrically continuous functions.

In Section 2 by using the Preiss and Tran constructions we give an elementary proof to show that the typical symmetrically continuous function and the typical symmetric function have c-dense sets of points of discontinuity. This answers two questions posed in [1, p. 422].

Let us use the following expressions,

$$D^1 f(x,h) = [f(x+h) - f(x-h)]/h,$$
$$D^2 f(x,h) = [f(x+h) + f(x-h) - 2f(x)]/h.$$

In 1969 Filipczak in [3] constructed a continuous function f defined on [0,1] which satisfies for each $x \in (0,1)$, $\limsup_{h\to 0} D^1 f(x,h) = +\infty$. In 1972 Kostyrko in [4] used this example to show that the typical function $f \in C[0,1]$, the set of all real continuous functions with the supremum metric, satisfies for each $x \in (0,1)$,

$$\limsup_{h\to 0} D^1 f(x,h) = +\infty \text{ and } \liminf_{h\to 0} D^1 f(x,h) = -\infty.$$

In 1987 Evans [5, Theorem 1] constructed a function $f \in C[0,1]$ which satisfies that for each $x \in (0,1)$,

$$\operatorname{ap}\limsup_{h\to 0^+} D^1 f(x,h) = +\infty, \qquad \operatorname{ap}\liminf_{h\to 0^+} D^1 f(x,h) = -\infty,$$
$$\text{and } \operatorname{ap}\limsup_{h\to 0^+} |D^2 f(x,h)| = +\infty.$$

He used this example to show that such functions are typical in $C[0,1]$.

In Section 3 we directly show that the typical function $f \in C[0,1]$ satisfies for each $x \in (0,1)$,

$$(1) \quad \limsup_{h\to 0} |D^1 f(x,h)| = +\infty, \qquad (2) \quad \limsup_{h\to 0} |D^2 f(x,h)| = +\infty$$

without using the constructions of Filipczak and Evans.

Throughout this paper, $BSC[a,b]$ denotes the set of all bounded measurable, symmetrically continuous functions defined on the interval $[a,b]$ and equipped with the supremum metric ρ, and $BS[a,b]$ denotes the set of all bounded measurable, symmetric functions defined on $[a,b]$ and equipped with the supremum metric ρ. $D(f)$ denotes the set of points of discontinuity of function f. A^c denotes the complement of a set A.

2 Typical Properties of Symmetrically Continuous Functions and Symmetric Functions

Lemma 1 (Tran [2 *)]There are functions $g_1 \in BSC[a,b]$ and $g_2 \in BS[a,b]$ both of which have continuum points of discontinuity in every subinterval of $[a,b]$.*

PROOF. Tran gave a construction of a function $g \in BS[a,b]$ for which $D(g)$ is of power c and constructed g_1 and g_2 from g. We can also use the Preiss result [1, p. 52] to construct a function g_1 as in the lemma. Let $\{(a_n,b_n)\}$ be an enumeration of the set of all subintervals of $[a,b]$ with rational endpoints. For every n there are a set E_n that is of power c and contained in (a_n,b_n) and a symmetrically continuous function f_n such that $0 \le f_n \le 1$, $f_n(x) > 0$ for $x \in E_n$ and $f_n(x) = 0$ outside of a set of measure zero. Note that such a function is discontinuous at a point if and only if it is positive there. Set

$$g_1 = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n$$

Then g_1 too is symmetrically continuous everywhere and is discontinuous precisely on the set $\{x \in [a,b] : g_1(x) > 0\}$. Clearly this latter set is c-dense in $[a,b]$. □

Theorem 2 *Given $(c,d) \subseteq [a,b]$, let*

$$A((c,d)) = \{f \in BSC[a,b] : D(f) \bigcap (c,d) \text{ is of power } c\}$$

Then $A((c,d))$ is a dense open set in $BSC[a,b]$.

PROOF. Let $\{f_n\} \subseteq A((c,d))^c$ be a convergent sequence. Then there is a function $f \in BSC[a,b]$ such that $f_n \longrightarrow f$ uniformly. Let e_n denote the set $D(f_n) \bigcap (c,d)$. Then e_n is at most countable and so the union $\bigcup_{n=1}^{\infty} e_n$ is at most countable. We know that f is continuous at each point $x \in (c,d) \setminus \bigcup_{n=1}^{\infty} e_n$, so $f \in A((c,d))^c$. Hence $A((c,d))^c$ is closed and $A((c,d))$ is open.

Now we show that $A((c,d))$ is dense in $BSC[a,b]$. For every ball $B(f,\epsilon) \subseteq BSC[a,b]$, if $f \in A((c,d))$ there is nothing to prove. We assume $f \in A((c,d))^c$. Then f has at most countably many points of discontinuity in (c,d). From Lemma 1 there is a function $g \in BSC[a,b]$ such that g has a c-dense set of points of discontinuity on (c,d). Let M be a constant such that $|g(x)| \le M$ for all $x \in [a,b]$ and set $h = f + \frac{\epsilon}{2M} g$. Then $h \in BSC[a,b]$ is discontinuous in continuum many points of (c,d) and

$$\rho(h,f) = \rho(f + \frac{\epsilon}{2M} g, f) = \rho(\frac{\epsilon}{2M} g, 0) < \epsilon$$

where ρ is the supremum metric on $BSC[a,b]$. Thus $h \in A((c,d)) \bigcap B(f,\epsilon)$ and hence $A((c,d))$ is dense. □

Theorem 3 *The typical function $f \in BSC[a,b]$ has a c-dense set of points of discontinuity.*

PROOF. From Theorem 2 $A(I)$ is a dense open set for each open subinterval I. The result follows by taking the intersection $\bigcap_I A(I)$ for all rational open subintervals $I \subseteq [a,b]$. □

The same methods can be used to prove the following theorem.

Theorem 4 *The typical function $f \in BS[a,b]$ has a c-dense set of points of discontinuity.*

3 An Application of the Baire Category Theorem to the Space of Continuous Functions

Lemma 5 *Let $f \in C[0,1]$, n be a positive integer, m and ϵ be two given positive constants. Then there exists a finite piecewise linear function $g \in C[0,1]$ such that for each $x \in [0,1]$, $|f(x) - g(x)| < \epsilon$ and for each $x \in [1/n, 1 - 1/n]$, $|D^2 g(x,h)| > m$ for some h with $0 < |h| < 1/n$.*

PROOF. The function f is uniformly continuous on $[0,1]$. For $\epsilon > 0$ there exists $\delta_1 > 0$ such that $|f(x_1) - f(x_2)| < \epsilon/16$ whenever $x_1, x_2 \in [0,1]$, $|x_1 - x_2| < \delta_1$. Take $\delta = \min\{\frac{\epsilon}{6m}, \frac{\delta_1}{10}, \frac{1}{10n}\}$ and partition [0,1] as $0 = x_0 < x_1 < \cdots < x_k = 1$. Here $x_i - x_{i-1} = \delta$ if i is not a number of the form $4l+2$ where l is a nonnegative integer. If i is a number of the form $4l+2$, $x_i - x_{i-1} = 3\delta$ except $k = 4l+2$.

If k is a number of form $4l+2$, $x_k - x_{k-1} = \delta$ or 2δ or 3δ depending on how many subintervals we get if we partition [0,1] into subintervals with length δ.

Let g be a finite piecewise linear function which connects the following points $a_0, a_1, a_2, \ldots, a_k$. Here $a_0 = (x_0, f(x_0) + (3/8)\epsilon), a_1 = (x_1, f(x_1) - (3/8)\epsilon)$. The point a_2 is the intersection point of the line $x = x_2$ with the half line starting from the point a_1 and parallel to the x-axis, $a_3 = (x_3, f(x_3) + (3/8)\epsilon), a_4$ is the intersection point of the line $x = x_4$ with the half line starting from the point a_3 and parallel to the x-axis, $a_5 = (x_5, f(x_5) - (3/8)\epsilon)$. Similarly as for a_2 we can define a_6, and continue in this way to get $a_0, a_1, a_2, \ldots, a_k$. See the figure (ii) where $r = \epsilon$.

We now verify that the function g satisfies our requirements. Obviously g is a finite piecewise linear, continuous function and for each $x \in [0,1]$, $|f(x) - g(x)| < \epsilon$. For the remainder we need to verify that for each $x \in [x_{i-2}, x_{i+2}]$ as indicated in the figure (ii), $|D^2 g(x,h)| > m$ for some h with $0 < |h| < 1/n$. We can assume $3 < i < k-3$ since $x \in [1/n, 1-1/n]$ and $\delta \le \frac{1}{10n}$. For $x \in [x_{i-1}, x_i]$, choose $h = \min\{x - x_{i-2}, x_{i+1} - x\}$ and note $\delta \le \frac{\epsilon}{6m}$,

$$\begin{aligned} |D^2 g(x,h)| &= \left|\frac{g(x+h) - g(x)}{h}\right| + \left|\frac{g(x) - g(x-h)}{h}\right| \\ &\ge \frac{(3/4)\epsilon - (1/16)\epsilon}{(5/2)\delta} = \frac{11\epsilon}{40\delta} > m. \end{aligned}$$

Partition $[x_i, x_{i+1}]$ into three subintervals of equal length $[x_i, x^1]$, $[x^1, x^2]$ and $[x^2, x_{i+1}]$. For $x \in [x_i, x^1]$, choose $h = x_{i+1} - x$. Then

$$\begin{aligned} |D^2 g(x,h)| &= \left|\frac{g(x+h) - g(x)}{h}\right| - \left|\frac{g(x) - g(x-h)}{h}\right| \\ &\ge (1 - 1/3)\frac{(3/4)\epsilon - (1/16)\epsilon}{\delta} = \frac{11\epsilon}{24\delta} > m. \end{aligned}$$

For $x \in [x^1, x^2]$, choose $h = x_{i+3} - x$. Then

$$\begin{aligned} |D^2 g(x,h)| &= \left|\frac{g(x+h) - g(x)}{h}\right| + \left|\frac{g(x) - g(x-h)}{h}\right| \\ &\ge 2[\frac{(1/3)((3/4)\epsilon - (1/16)\epsilon)}{(2 + (2/3))\delta}] = \frac{11\epsilon}{64\delta} > m. \end{aligned}$$

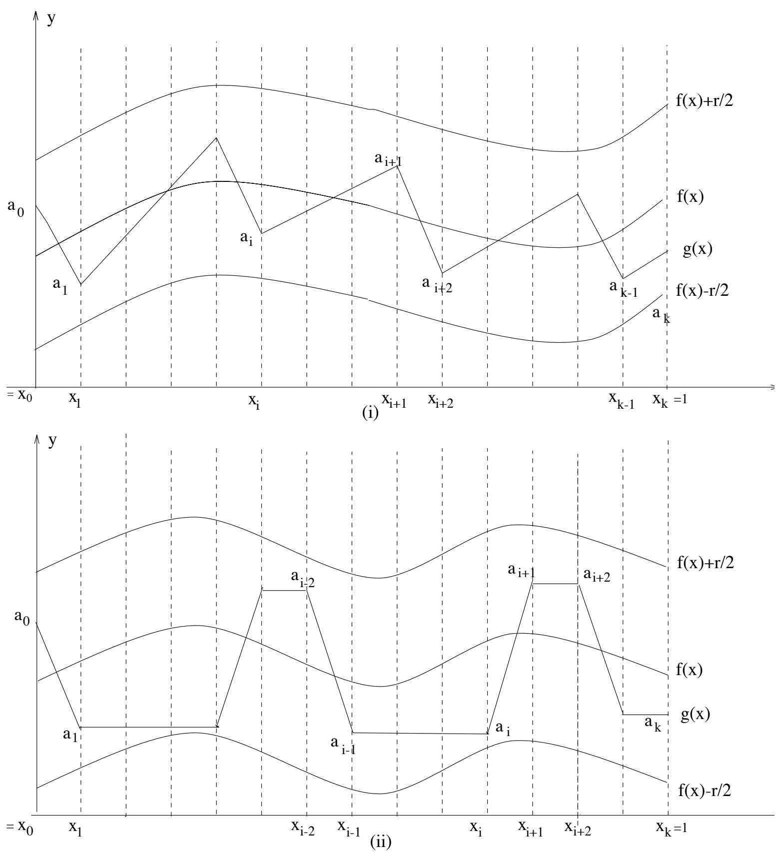
For $x \in [x^2, x_{i+1}]$, choose $h = x - x_i$. Then

$$|D^2 g(x,h)| = \left| | \frac{g(x-h)-g(x)}{h} \right| - \left| | \frac{g(x+h)-g(x)}{h} \right|$$
$$\geq (1 - 1/3) \frac{(3/4)\epsilon - (1/16)\epsilon}{\delta} > m.$$

For $x \in [x_{i+1}, x_{i+2}]$, choose $h = \min\{x - x_i, x_{i+3} - x\}$. Then

$$|D^2 g(x,h)| = \left| | \frac{g(x+h)-g(x)}{h} \right| + \left| | \frac{g(x-h)-g(x)}{h} \right| \geq \frac{(3/4)\epsilon - (1/16)\epsilon}{2\delta} = \frac{11\epsilon}{32\delta} > m.$$

For $x \in [x_{i-2}, x_{i-1}]$ using the same method for $x \in [x_i, x_{i+1}]$ we can show that the function g satisfies our requirements. Hence the lemma follows. □



Theorem 6 *The typical function $f \in C[0,1]$ satisfies (2) for all $x \in (0,1)$.*

PROOF. Let

$$A = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exist some point } x \in (0,1) \text{ and constant } C \\ \text{such that } \limsup_{h\to 0} |D^2 f(x,h)| \le C \end{array} \right\},$$

$$A_{nm} = \left\{ f \in C[0,1] : \begin{array}{l} \text{there exists some } x \in [1/n, 1-1/n] \text{ such that} \\ |D^2 f(x,h)| \le m \ \text{ whenever } \ 0 < |h| < 1/n, \end{array} \right\}$$

Then $A = \bigcup_{n,m=1}^{\infty} A_{nm}$. Using the same standard arguments as in Theorem 2 and Lemma 5 we can show that each A_{nm} is an open dense set in $C[0,1]$ and therefore the theorem follows. □

Note that the analogous statement to Lemma 5 but using D^1 in place of D^2 is easier to prove and can be obtained by choosing a saw-tooth function with suitable slopes as in figure (i). Similar methods can be used to prove the following theorem.

Theorem 7 *The typical function $f \in C[0,1]$ satisfies (1) for all $x \in (0,1)$.*

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