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ON "LIPSCHITZ" SUBSPACES OF THE SPACE OF CONTINUOUS FUNCTIONS

Abstract

A theorem of Grothendieck states that every closed subspace of the Banach space $L^p(\mu)$, where μ is a finite measure on a locally compact topological space, $p \geq 1$, consisting of essentially bounded functions must have finite dimension. An analog of this result is proved concerning subspaces of the space of continuous functions on a compact metric space consisting of functions satisfying different Lipschitz-type conditions.

1 Introduction

Consider a Banach space X and a closed subspace S. In some cases we can say that every element of S is "better" than an arbitrary element of X. This happens, for example, if elements of X are functions defined on a metric space and elements of S satisfy some additional estimates or smoothness conditions. A general question is how big such "good" subspaces can be.

A. Grothendieck [1] proved a very interesting result showing that it is natural to expect, at least in some cases, that these subspaces are small. Let M be a locally compact topological space equipped with a finite measure μ and let $1 \leq p < \infty$. If S is a vector subspace of $L^\infty(\mu)$, closed in $L^p(\mu)$, then S is finite-dimensional.

2 Preliminaries

Let K be a compact metric space and let $C(K)$ be a Banach space of all continuous complex valued functions on K. For any $f \in C(K)$ the *modulus of continuity* $\omega_f : (0, \infty) \to [0, \infty)$ is defined as

$$\omega_f(\delta) = \sup_{\rho(x,y)<\delta} |f(x) - f(y)|.$$

Clearly ω_f is an increasing function and $\lim_{\delta \to 0} \omega_f(\delta) = 0$.

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Definition 1 *Let $\omega : (0, \infty) \to (0, \infty)$. We will say that ω is modulus-type if*

(i) $\lim\limits_{\delta \to 0} \omega(\delta) = 0$,

(ii) ω *is increasing.*

By S we will denote a closed, linear subspace of $C(K)$; its dimension is denoted by $\dim S$. The following statement is a variation on a theme of Grothendieck.

Proposition 1 *Let ω be a modulus-type function. If for every $f \in S$*

$$\sup_{\delta > 0} \frac{\omega_f(\delta)}{\omega(\delta)} < \infty,$$

then $\dim S < \infty$.

3 First Proof of Proposition 1

A shorter proof of the Proposition using Ascoli's theorem and Riesz lemma will be outlined later. We would like to present a more direct approach. We start with the following simple observation.

Lemma 1 *Suppose* $\dim S = \infty$ *and* $\{x_1, x_2, \ldots, x_n\} \subset K$. *Then there exists a function $f \in S$, such that $f \neq 0$, but $f(x_1) = f(x_2) = \cdots = f(x_n) = 0$.*

PROOF. Suppose not. Consider a linear map $B : S \to \mathbb{R}^n$ given by

$$B(f) = (f(x_1), f(x_2), \ldots, f(x_n)).$$

Clearly, B is linear and $\ker B = \{0\}$. Hence

$$\dim S = \dim(\ker B) + \dim(B(S)) = 0 + n = n.$$

But this contradicts the assumption that $\dim S = \infty$. □

PROOF OF THE PROPOSITION Let ω be a modulus type function. We shall show that there is a constant M such that for all $\delta > 0$

$$\frac{\omega_f(\delta)}{\omega(\delta)} \leq M \parallel f \parallel_{C(K)} \tag{1}$$

Let

$$Y = \{f \in C(K) : \sup_{\delta > 0} \frac{\omega_f(\delta)}{\omega(\delta)} < \infty\}.$$

If

$$\parallel f \parallel_Y = \sup_{x \in K} |f(x)| + \sup_{\delta > 0} \frac{\omega_f(\delta)}{\omega(\delta)},$$

then $(Y, \| \cdot \|_Y)$ is a Banach space.

Consider an identity map $I : S \to Y$ i.e. $If = f$ for $f \in S$. (S is treated as a subspace of a Banach space $C(K)$.) Obviously I is linear. It is easy to deduce from the Closed Graph Theorem that I is also bounded as a linear operator. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in S, such that $\| f_n - f \|_S \to 0$ and $\| If_n - g \|_Y \to 0$. Then

$$\| f_n - g \|_{C(K)} \leq \| f_n - g \|_Y = \| If_n - g \|_Y .$$

Hence $g = \lim_{n\to\infty} f_n$ and trivially $g = f$. Therefore by the Closed Graph Theorem I is a bounded linear operator, i.e. there exists a constant M_1 such that for all $f \in S$ we have $\| f \|_Y \leq M_1 \| f \|_{C(K)}$ Since $\| f \|_Y = \| f \|_{C(K)} + \sup_{\delta>0} \frac{\omega_f(\delta)}{\omega(\delta)}$, for $M = M_1 - 1$ we obtain (1).

Since $\lim_{\delta\to 0} \omega(\delta) = 0$, let us fix $\delta_0 > 0$ such that $\omega(\delta_0) < \frac{1}{M}$ K is a compact metric space, so there exists a δ_0-net, i.e. a set
$\{x_1, x_2, \ldots, x_n\} \subset K$ such that for every $x \in K$ there is an $i \in \{1, 2, \ldots, n\}$ such that $\rho(x, x_i) < \delta_0$.

Suppose that $\dim S = \infty$. By Lemma 1, there is a function $f \in S$, $f \neq 0$, such that $f(x_1) = f(x_2) = \cdots = f(x_n) = 0$. Since K is compact, $\| f \|_{C(K)} = |f(x_0)|$ for some $x_0 \in K$. Therefore, by (3) we obtain for some $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned} \omega_f(\delta_0) &= \sup_{\rho(x,y)<\delta_0} |f(x) - f(y)| \geq |f(x_0) - f(x_i)| \\ &= |f(x_0) - 0| = \| f \|_{C(K)} = M \| f \|_{C(K)} \frac{1}{M} \\ &> M \| f \|_{C(K)} \omega(\delta_0) \end{aligned}$$

or $\frac{\omega_f(\delta_0)}{\omega(\delta_0)} > M \| f \|_{C(K)}$ contrary to (1). □

4 Second Proof of Proposition 1

We used the Closed Graph Theorem to show that norms $\| \cdot \|_{C(K)}$ and $\| \cdot \|_Y$ are equivalent on S. The same fact can be established if we apply Banach's theorem on isomorphism. Let us consider $\Phi = S \cap \{f \in C(K) : \| f \|_{C(K)} \leq 1\}$. From the equivalence of norms $\| \cdot \|_{C(K)}$ and $\| \cdot \|_Y$ we obtain that Φ is an equicontinuous family of functions in $C(K)$, bounded by 1. Since Φ is closed, Ascoli's theorem gives compactness of Φ. But the unit ball in a Banach space is compact if and only if its dimension is finite (the Riesz lemma). Hence $\dim S < \infty$.

5 Consequences

Let us notice some straightforward corollaries from the Proposition:

Corollary 1 *If* $S \subset Lip_\alpha(K)$*, then* $\dim S < \infty$.

Proof. Take $\omega(\delta) = \delta^\alpha$ and apply Proposition. □

Corollary 2 *If* S *is a closed linear subspace of* $C[0,1]$ *and every* $f \in S$ *is continuously differentiable* ($S \subset C^1[0,1]$)*, then* $\dim S < \infty$.

Proof. Since $C^1[0,1] \subset Lip[0,1]$, this follows from Corollary 1. □

The above corollary is one more statement illustrating the well known fact that a “typical” continuous function is not differentiable ([2]).

6 Example

It is required in Corollary 2 that all functions in S have continuous derivatives on $[0,1]$. This example shows that it is not enough to assume only the existence of such derivatives on a subset of $[0,1]$.

It is well known that given an interval (a,b), there exists a function $f_{a,b} \in C^\infty(\mathbb{R})$ such that

(i) $f_{a,b}(x) = 0$ for all $x \in \mathbb{R}\backslash(a,b)$,

(ii) $\sup_{x\in\mathbb{R}} |f_{a,b}(x)| = 1$.

We can take for instance

$$f_{a,b}(x) = \begin{cases} c_{a,b}e^{-\frac{1}{(x-a)^2(x-b)^2}}, & x \in (a,b) \\ 0, & x \in \mathbb{R}\backslash(a,b) \end{cases}$$

where $c_{a,b} = e^{\frac{16}{(a-b)^4}}$. Let $g_n = f_{2^{-n},2^{-n+1}}$ and let S be the closed linear manifold generated by $\{g_n\}_{n=1}^\infty$. Then $\dim S = \infty$ and every function in S has continuous derivatives in $(0,1]$. Also $g = \sum_{n=1}^\infty \sqrt{n} g_n \in S$ and g has no derivative at 0.

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