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ON PRODUCTS OF A.E. CONTINUOUS DERIVATIVES

Abstract

In this paper I examine products of a.e. continuous derivatives. Denote by $\mathcal{N}_u$ the set of all x for which $u(x) \neq 0$. First I prove that if u is a (bounded and/or non-negative) function with $\mathcal{N}_u$ isolated, then u can be written as the product of two (bounded and/or non-negative) a.e. continuous derivatives. Next I show that if u is a.e. continuous and $\mathcal{N}_u$ is a union of an isolated set and one with a null closure, then u can be written as the product of two a.e. continuous derivatives. I construct an example that we cannot require the factors be bounded in case u is bounded. Using this example I construct a bounded non-negative a.e. continuous function, v, such that $\mathcal{N}_v$ is the union of two isolated sets (so v is a Baire one star function and is the product of two bounded non-negative derivatives) and which cannot be written as the product of two a.e. continuous derivatives.

It is well-known that the class of derivatives is not closed with respect to multiplication (cf., e.g., [15] and [5]). So it is a natural problem to characterize the family of all products of derivatives. There are several papers devoted to this problem, e.g., [1], [12], [3], [10] and [8]. (See also survey papers [5] and [4].) However, no final characterization has been found yet. It is also interesting whether each bounded product of derivatives is a product of bounded derivatives [11, p. 57].

In this paper I consider analogous problems concerning products of a.e. continuous derivatives. As in [3], I focus on functions which vanish a.e.

First we need some notation. The real line $(-\infty, +\infty)$ we denote by $\mathbb{R}$, the set of integers by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$. We consider

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only functions from $\mathbb{R}$ into $\mathbb{R}$. The phrase almost everywhere (a.e.) refers to Lebesgue measure on $\mathbb{R}$. For every set $A \subset \mathbb{R}$ let $\operatorname{cl} A$ be its closure, χ_A its characteristic function and $|A|$ its outer Lebesgue measure. Symbols like $\int_a^b f$ or $\int_A f$ will always mean the corresponding Lebesgue integral. If the sets A and B are non-empty, then we define $\varrho(A, B) = \inf\{|x - t| : x \in A, t \in B\}$.

Let f be a function. We say that f is:

- a Baire one function, if for each open set $U \subset \mathbb{R}$, the pre-image $f^{-1}(U)$ is an F_σ set;
- a Baire one star function, if for each open set $U \subset \mathbb{R}$, the pre-image $f^{-1}(U)$ is a G_δ set;
- a derivative, if there is a function F such that $\lim_{t\to x} \dfrac{F(t) - F(x)}{t - x} = f(x)$ for each $x \in \mathbb{R}$;
- quasi-continuous in the sense of S. Kempisty [7], if for each $x \in \mathbb{R}$ and each $\varepsilon > 0$ there is a non-empty open set $U \subset (x - \varepsilon, x + \varepsilon)$ such that $\big|f - f(x)\big| < \varepsilon$ on U.

The symbol $\mathcal{D}_f$ stands for the set of points of discontinuity of f.

S. Marcus proved in 1958 [9] that a.e. continuous derivatives are quasi-continuous. (See also [13].) T. Natkaniec showed in 1990 that a function u can be factored into a (finite) product of quasi-continuous functions if and only if it is pointwise discontinuous and each of the sets $u^{-1}\big((-\infty, 0)\big)$, $u^{-1}(0)$ and $u^{-1}\big((0, \infty)\big)$ is the union of an open set and a nowhere dense set [14]. (It was shown later by J. Borsík [2] that the Natkaniec's triple condition can be simplified to the following one. $u^{-1}(0)$ is the union of an open set and a nowhere dense set.) This condition implies that if a function u is the product of a.e. continuous derivatives and $u = 0$ a.e., then the set $\mathcal{N}_u = \{x \in \mathbb{R} : u(x) \neq 0\}$ is nowhere dense. So the function ψ below is not a product of a.e. continuous derivatives, though it is a bounded Baire one function which vanishes a.e. (So by Corollary 4.3 of [3], it can be written as the product of two bounded non-negative derivatives.) and $\mathcal{D}_\psi \mathcal{N}_\psi$ is countable [6].

Example 1 *Arrange all rationals in a sequence, (q_n). Define $\psi(x) = 0$ whenever x is irrational and $\psi(q_n) = 1/n$ for $n \in \mathbb{N}$.*

Recall that a set A is isolated, if it contains none of its limit points. Clearly isolated sets are countable (whence F_σ), G_δ, nowhere dense sets. So for every function u, if $\mathcal{N}_u$ is a finite union of isolated sets, then u is a Baire one star function.

Theorem 1 *Suppose $\mathcal{N}_u$ is isolated. Then there are derivatives f and g such that $u = fg$, g is bounded and non-negative, and $\mathcal{D}_f \cup \mathcal{D}_g \subset \mathcal{D}_u$. (So in particular, if u is continuous a.e., then f and g are continuous a.e.as well) Moreover, if u is bounded and/or non-negative, then we can require that f be bounded and/or non-negative also.*

PROOF. Arrange all elements of $\mathcal{N}_u$ in a sequence (finite or not), (a_n). Fix an n. There is a $\delta_n > 0$ such that $(a_n-\delta_n, a_n+\delta_n)\cap\mathcal{N}_u = \{a_n\}$. Let f_n, g_n be derivatives such that $\mathcal{D}_{f_n} = \mathcal{D}_{g_n} = \{a_n\}$, $0 \le f_n, g_n < 2$ on $\mathbb{R}$, $f_n g_n = \chi_{\{a_n\}}$ and $f_n(x) = g_n(x) = 0$ whenever $|x - a_n| > 2^{-n}\delta_n \bigm/ \bigl(|u(a_n)| + 1\bigr)$. (See, e.g., Theorem 4.2 of [12].) Let

$$f = \sum_n f_n \operatorname{sgn}\bigl(u(a_n)\bigr) \max\Bigl\{\sqrt{|u(a_n)|}, |u(a_n)|\Bigr\},\ g = \sum_n g_n \min\Bigl\{\sqrt{|u(a_n)|}, 1\Bigr\}.$$

It is clear that $u = fg$, g is bounded and non-negative, and if u is bounded and/or non-negative, then f is bounded and/or non-negative, too. As $\mathcal{N}_u$ is isolated, we have $\mathcal{D}_f \cup \mathcal{D}_g \subset \operatorname{cl}\mathcal{N}_u$. But $\mathcal{N}_u \subset \mathcal{D}_u$, and $x \in \operatorname{cl}\mathcal{N}_u \setminus \mathcal{N}_u$ implies $u(x) = f(x) = g(x) = 0$ and

$$\limsup_{t\to x}\bigl|g(t)\bigr| \le \limsup_{t\to x}\bigl|f(t)\bigr| \le 2\limsup_{t\to x}\max\Bigl\{\sqrt{|u(t)|}, |u(t)|\Bigr\}.$$

So $\mathcal{D}_f \cup \mathcal{D}_g \subset \mathcal{D}_u$.

Now we will show that f and g are derivatives. Fix an $x \in \mathbb{R}$. If $x \notin \operatorname{cl}\mathcal{N}_u$ or $x \in \mathcal{N}_u$, then f coincides with f_n and g coincides with g_n on some neighborhood of x for some n. So assume that $x \in \operatorname{cl}\mathcal{N}_u \setminus \mathcal{N}_u$. Let $k \in \mathbb{N}$. For each $t \in \mathbb{R}$ with $0 < |t - x| < \min\bigl\{|a_n - x| : n \le k\bigr\}\big/2$ we have

$$\begin{aligned}\max\left\{\left|\frac{\int_x^t g}{t-x}\right|,\ \left|\frac{\int_x^t f}{t-x}\right|\right\} &\le \sum_{n>k} \frac{\max\bigl\{\sqrt{|u(a_n)|}, |u(a_n)|\bigr\}\int_{a_n-\delta_n}^{a_n+\delta_n}\max\{f_n, g_n\}}{\delta_n/2}\\ &\le \sum_{n>k} 2^{3-n} = 2^{3-k}.\end{aligned}$$

This completes the proof. □

Theorem 2 *Let A be isolated, u be a function, $B = \mathcal{N}_u \setminus A$ be non-empty and $A \cap \operatorname{cl} B = \emptyset$. Suppose moreover that there is a differentiable function, Φ, such that $\Phi' = u$ on $\operatorname{cl} B$. Then there are derivatives f and g such that $u = fg$, g is non-negative, and $\mathcal{D}_f \cup \mathcal{D}_g \subset \mathcal{D}_u \cup \operatorname{cl} B$.*

Proof. Write $U = \mathbb{R} \setminus \operatorname{cl} B$ as the union $\bigcup_{k\in\mathbb{N}} I_k$ of non-overlapping compact intervals such that each $x \in U$ belongs to the interior of $I_k \cup I_l$ for some $k, l \in \mathbb{N}$ and

$$\max\bigl\{|I_k|, \sup\{|\Phi(y) - \Phi(z)| : y, z \in I_k\}\bigr\} \le \bigl[\varrho(I_k, B)\bigr]^2 \tag{$*$}$$

for each $k \in \mathbb{N}$. Proceeding as in the proof of Theorem 1 construct derivatives f_0 and g_0 such that $u\chi_U = f_0 g_0$, $\mathcal{D}_{f_0} \cup \mathcal{D}_{g_0} \subset \mathcal{D}_u$, $f_0 = g_0 = 0$ on $\operatorname{cl} B$ and for each $k \in \mathbb{N}$ the set $\bigl\{x \in I_k : f_0(x) = g_0(x) = 0\bigr\}$ contains an interval. For each $k \in \mathbb{N}$ construct continuous functions f_k and g_k such that f_k does not change its sign, g_k is non-negative, $f_k = g_k = 0$ outside of I_k, $f_k g_0 = f_k g_k = f_0 g_k = 0$ on I_k, $\int_{I_k} f_k = \Phi(b_k) - \Phi(a_k)$ and $\int_{I_k} g_k = |I_k|$, where $[a_k, b_k] = I_k$. Define

$$f = f_0 + \sum_{k\in\mathbb{N}} f_k + \Phi' \chi_{\operatorname{cl} B} \quad \text{and} \qquad g = g_0 + \sum_{j\in\mathbb{N}} g_j + \chi_{\operatorname{cl} B}.$$

Clearly g is non-negative and $\mathcal{D}_f \cup \mathcal{D}_g \subset \mathcal{D}_u \cup \operatorname{cl} B$. Moreover

$$\begin{aligned} fg = f_0 g_0 + \sum_{k\in\mathbb{N}} f_k g_0 + \Phi' g_0 \chi_{\operatorname{cl} B} + \sum_{j\in\mathbb{N}} f_0 g_j + \sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} f_k g_j + \sum_{j\in\mathbb{N}} \Phi' g_j \chi_{\operatorname{cl} B} \\ + f_0 \chi_{\operatorname{cl} B} + \sum_{k\in\mathbb{N}} f_k \chi_{\operatorname{cl} B} + \Phi' \chi_{\operatorname{cl} B} = u\chi_U + u\chi_{\operatorname{cl} B} = u. \end{aligned}$$

Now we will show that f and g are derivatives. Let

$$\begin{aligned} &F(x) = \Phi(a_k) + \int_{a_k}^{x} f_k, \quad G(x) = a_k + \int_{a_k}^{x} g_k, \text{ if } x \in I_k,\ k \in \mathbb{N}, \\ &F(x) = \Phi(x), \qquad\qquad\quad G(x) = x, \qquad\qquad\quad \text{if } x \in \operatorname{cl} B. \end{aligned}$$

We will show that $F' = f - f_0$ and $G' = g - g_0$ on $\mathbb{R}$.

Fix an $x \in \mathbb{R}$. If $x \in I_k$ for some $k \in \mathbb{N}$. Then clearly $F'(x) = f_k(x)$ and $G'(x) = g_k(x)$. So assume that $x \in \operatorname{cl} B$. For each $t \in \mathbb{R}$, if $t \in \operatorname{cl} B$, then $F(t) - F(x) = \Phi(t) - \Phi(x)$ and $G(t) - G(x) = t - x$, and if $t \in I_k$ for some $k \in \mathbb{N}$, then by $(*)$,

$$\begin{aligned} \left| \frac{\bigl(F(t) - F(x)\bigr) - \bigl(\Phi(t) - \Phi(x)\bigr)}{t - x} \right| &\le \frac{\int_{I_k} |f_k| + \bigl|\Phi(t) - \Phi(a_k)\bigr|}{|t - x|} \\ &\le \frac{2\sup\bigl\{|\Phi(y) - \Phi(z)| : y, z \in I_k\bigr\}}{\varrho(I_k, \operatorname{cl} B)} \le \varrho(I_k, \operatorname{cl} B) \le |t - x| \end{aligned}$$

and

$$\left| \frac{G(t) - G(x)}{t - x} - 1 \right| \le \frac{\int_{I_k} |g_k - 1|}{|t - x|} \le \frac{|I_k|}{\varrho(I_k, \operatorname{cl} B)} \le \varrho(I_k, \operatorname{cl} B) \le |t - x|.$$

This completes the proof. $\square$

Remark 1 *In the above theorem, if the function u is a.e. continuous and the closure of the set B has Lebesgue measure zero (e.g., if B is finite), then the derivatives f and g are a.e. continuous. The example below shows that we cannot require that they are bounded if u is bounded, even in case B is a singleton.*

Example 2 *There is a bounded non-negative function u and an isolated set A such that $\mathcal{D}_u = \mathcal{N}_u = A \cup \{0\}$, and f is unbounded whenever f and g are derivatives, f is a.e. continuous and $u = fg$.*

CONSTRUCTION. Set $I_n = \big[1/(n+1), 1/n\big]$ for $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. Let $F_n \subset I_n$ be a nowhere dense closed set of measure $(1 - 1/n)|I_n|$ such that both end points of I_n belong to F_n. Let $A_n = \{x_{n,k} : k \in \mathbb{N}\} \subset I_n \setminus F_n$ be an isolated set with $F_n \subset \operatorname{cl} A_n$. Let $A = \bigcup_{n\in\mathbb{N}} A_n$ and define

$$u(x) = \begin{cases} 1/(n+k) & \text{if } x = x_{n,k},\ n,k \in \mathbb{N}, \\ 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Evidently the set A is isolated and $\mathcal{D}_u = \mathcal{N}_u = A \cup \{0\}$. Moreover u is bounded and non-negative. Suppose that there exist derivatives f and g such that f is bounded and a.e. continuous, and $u = fg$. Observe first that f is equal to 0 a.e. on $\operatorname{cl} A$.

Indeed, otherwise there would be a point $x \in \operatorname{cl} A$ at which f is non-zero and continuous. So f is non-zero in some neighborhood U of x, and since u is equal to 0 a.e., g is equal to 0 a.e. in U. As g is a derivative, it must be equal to 0 everywhere in U. So u also equals 0 everywhere in U. But $x \in \operatorname{cl} A$ and $u(t) \neq 0$ for $t \in A$, a contradiction.

Since $f(0) \neq 0$ (because $u(0) = f(0)g(0) = 1$), we may assume that $f(0) = 1$. Find a $\delta > 0$ such that $\big|h^{-1}\int_0^h f - 1\big| < 1/2$ for $h \in (0,\delta)$. Set $M = \sup\big\{|f(x)| : x \in \mathbb{R}\big\}$ and take an $m > \max\{2M, 1/\delta\}$. Then $1/m \in (0,\delta)$ and

$$\begin{aligned} \left| m\int_0^{1/m} f - 1 \right| &\geq 1 - m\int_0^{1/m} |f| \geq 1 - mM \cdot \big|(0,1/m) \setminus \operatorname{cl} A\big| \\ &= 1 - mM \sum_{n\geq m} |I_n \setminus F_n| = 1 - mM \sum_{n\geq m} |I_n|/n \\ &> 1 - M \sum_{n\geq m} |I_n| = 1 - M \cdot \big|(0,1/m)\big| = 1 - M/m > 1/2. \end{aligned}$$

We obtained a contradiction with the previous inequality. □

Example 3 *There is a bounded non-negative function* v *such that* $\mathcal{D}_v = \mathcal{N}_v$ *is the union of two isolated sets, and* f *is not a.e. continuous whenever* f *and* g *are derivatives and* $v = fg$.

CONSTRUCTION. Let K be a nowhere dense perfect set of positive measure and let $\{(x_n - \delta_n, x_n + \delta_n) : n \in \mathbb{N}\}$ be the family of all bounded components of $\mathbb{R} \setminus K$. Let u be the function defined in Example 2. Define the function v by

$$v(x) = \begin{cases} u\big((x - x_n)/\delta_n\big)/n & \text{if } |x - x_n| < \delta_n,\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{D}_v = \mathcal{N}_v$ is the union of two isolated sets: $B = \{x_n : n \in \mathbb{N}\}$ and $\mathcal{N}_v \setminus B$. Suppose that there are derivatives f and g such that f is a.e. continuous and $v = fg$. By the properties of the function u, f is unbounded on every interval $(x_n - \delta_n, x_n + \delta_n)$ $(n \in \mathbb{N})$. But this implies that $\mathcal{D}_f \supset K$ and $|\mathcal{D}_f| \geq |K| > 0$, a contradiction. □

Remark 2 *Similarly to the proof of Theorem* 2 *it can be proved that both the function* u *of Example* 2 *and the function* v *of Example* 3 *can be written as the product of two bounded non-negative a.e. continuous Darboux quasi-continuous functions.*

References

[1] S. J. Agronsky, R. Biskner, A. M. Bruckner and J. Mařík, *Representations of functions by derivatives*, Trans. Amer. Math. Soc., **263**, no. 2 (1981), 493–500.

[2] J. Borsík, *Products of simply continuous and quasicontinuous functions*, Math. Slovaca, **45** (1995), 445–452.

[3] A. M. Bruckner, J. Mařík and C. E. Weil, *Baire one, null functions*, Contemporary Math., **42** (1985), 29–41.

[4] A. M. Bruckner, J. Mařík and C. E. Weil, *Some aspects of products of derivatives*, Amer. Math. Monthly, **99**, no. 2 (1992), 134–145.

[5] R. J. Fleissner, *Multiplication and the fundamental theorem of calculus — a survey*, Real Anal. Exchange, **2**, no. 1 (1976), 7–34.

[6] Z. Grande, *Sur le fonctions cliquish*, Časopis Pěst. Mat., **110** (1985), 225–236.

[7] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19** (1932), 184–197.

[8] A. Maliszewski, *Characteristic functions and products of bounded derivatives*, Proc. Amer. Math. Soc., **123 No. 7** (1995), 2203–2211.

[9] S. Marcus, *Sur les fonctions dérivées, intégrables au sens de Riemann et sur les dérivées partielles mixtes*, Proc. Amer. Math. Soc., **9** (1958), 973–978.

[10] J. Mařík, *Characteristic functions and products of derivatives*, Real Anal. Exchange, **16**, no. 1 (1990-91), 245–254.

[11] J. Mařík and C. E. Weil, *Products of derivatives and approximate continuity*, Real Anal. Exchange, **7**, no. 1 (1981-82), 54–57.

[12] J. Mařík and C. E. Weil, *Products of powers of nonnegative derivatives*, Trans. Amer. Math. Soc., **276**, no. 1 (1983), 361–373.

[13] A. V. Martin, *A note on derivatives and neighborly functions*, Proc. Amer. Math. Soc., **8** (1957), 465–467.

[14] T. Natkaniec, *Products of quasi-continuous functions*, Math. Slovaca, **40**, no. 4 (1990), 401–405.

[15] W. H. Young, *A note on the property of being a differential coefficient*, Proc. London Math. Soc., **9** (1911), 360–368.