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ON c-HOMOTOPIES

Abstract

In this paper we shall introduce a notion analogous to the notion of homotopy and D-homotopy for connected transformations and present some theorems connected with this problem.

Darboux functions mapping the real line into the real line possess a number of properties close to those of continuous functions. (For many years, these notions were even considered to be equivalent.) In this context, questions concerning the possibilities of constructing structures of Darboux transformations, analogous to structures of continuous functions, are interesting. In this paper we shall introduce a notion analogous to the notion of homotopy and present some theorems connected with this problem.

In many papers, the notion of Darboux functions has been generalized to transformations whose domain and range are topological spaces more general than the real line. (See, for example [6], [1], [3], [8].) Various notions of Darboux functions all involve the image of certain "connected"sets being connected. In this article we adopt the terminology and the method of generalizing the notion of a Darboux function from [2].

> We say that $f : X \to Y$, where X and Y are arbitrary topological spaces, is a Darboux transformation if $f(C)$ is a connected set for each connected set $C \subset X$.

In this paper we use the term *connected function* instead *Darboux function* because, in paper [6], there were investigated similar problems for Darboux functions in this sense: *the image of every arc is a connected set.* A simple consequence of those definition were restrictions to the considerations of *D-homotopies* between continuous functions. In this paper we shall investigate some specific notion of the homotopy between connected functions.

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Throughout the paper, we apply the classical symbols and notation. However, in order to avoid any ambiguities, we shall now present those symbols used in the paper whose meanings are not explained in the main text. By the letter $\mathbb{R}$ we denote the set of all real numbers with the natural topology. The letters $\mathbb{Q}$ and I denote the set of all rational numbers and the interval $[0,1] \subset \mathbb{R}$, respectively. The symbols $(a,b),(a,b]$, etc. denote open intervals, those open at the endpoint a, etc. in the space $\mathbb{R}$.

The family of all connected functions mapping X into Y will be denoted by $C^{X,Y}$.

The notation $f \simeq_h g$ (or shortly, $f \simeq g$) is understood as follows: the function f is homotopic to g, with h being the homotopy between f and g. The restriction f to a set A is denoted by $f_{|A}$.

By $\overline{B}$ (B^d) we denote the closure of a set B (derived set of B).

If X is a metric space, then $\operatorname{dia} A$ denotes the diameter of A.

In this paper we generally assume that the domain of all the functions considered is a separable metric space and the range is a Hausdorff topological space (in some cases, we shall give additional assumptions).

Definition 1 *Let $f, h : X \to Y$ be connected transformations. Then the transformations f and h are called c-homotopic if there exists a connected transformation $\xi : X \times I \to Y$ such that $\xi((x,0)) = f(x)$ and $\xi((x,1)) = h(x)$ for $x \in X$. The transformation ξ is called a c-homotopy between f and h.*

The fact that f and h are c-homotopic, and that ξ is a c-homotopy between f and h, is written down as $f \cong_\xi h$ (or shortly, $f \cong h$).

We shall now show that, by means of c-homotopies, one can construct similar structures as in the case of homotopies. Let us first note down the following:

Theorem 1 *In the family $C^{X,Y}$ the relation of c-homotopy is an equivalence relation.*

PROOF. It is evident that this relation is reflexive and symmetric. Consequently, we shall only show that it is transitive. Let $\{X_t\}_{t\in T}$ be the family of all components of X. Assume that $f \cong_\xi g$ and $g \cong_\eta h$. Let $A_t = \xi(X_t \times (0,1))$ and $B_t = \eta(X_t \times (0,1))$ for each $t \in T$. Thus A_t and B_t are connected sets of cardinality not larger than continuum.

Let us define an equivalence relation on the sets $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ as follows: $x * y \iff x - y \in \mathbb{Q}$. Let $\mathcal{A}(\mathcal{B})$ denote the set of all equivalence classes of $(0, \frac{1}{2})$ $((\frac{1}{2}, 1))$. Then, for each $t \in T$, there exists a surjection: $\varphi_t : \mathcal{A} \overset{onto}{\to} A_t$

and $\psi_t : \mathcal{B} \overset{onto}{\to} B_t$. So, let us define a function $\zeta : X \times I \to Y$ by letting:

$$\zeta((x,d)) = \begin{cases} f(x) & \text{for} \quad d = 0, \\ g(x) & \text{for} \quad d = \frac{1}{2}, \\ h(x) & \text{for} \quad d = 1, \\ \varphi([d]) & \text{for} \quad x \in X_t \text{ and } d \in (0, \frac{1}{2}), \\ \psi([d]) & \text{for} \quad x \in X_t \text{ and } d \in (\frac{1}{2}, 1); \end{cases}$$

where $[d]$ stands for the equivalence class of the relation $*$ to which d belongs.

The proof is completed by showing that ζ is a connected function. Let $C \subset X \times I$ be an arbitrary connected set. Then there exists $t_0 \in T$ such that $C \subset X_{t_0} \times I$. If $C \subset X_{t_0} \times \{d\}$ for $d \in [0,1]$, then, of course, $\zeta(C)$ is a connected set. Now, without loss of generality we may assume that $C \cap (X_{t_0} \times (0, \frac{1}{2})) \neq \emptyset$ and C is not a subset of $X_{t_0} \times \{d\}$ for any $d \in (0, \frac{1}{2})$. Thus $A_{t_0} = \zeta(C \cap (X_{t_0} \times (0, \frac{1}{2}))) \neq \emptyset$.

Let us consider the following cases:

1^o $C \cap (X_{t_0} \times \{0\}) \neq \emptyset$. Then, by the connectedness of ξ, $\xi((X_{t_0} \times (0,1)) \cup \{(x,0)\})$ is a connected set for each $x \in X_{t_0}$. This means that $A_{t_0} \cup \{\zeta((x,0))\}$ is a connected set, and consequently, $\zeta((x,0)) \in \overline{A_{t_o}}$ for each $x \in X_{t_0}$. The last observation implies that $\zeta(C \cap (X_{t_0} \times \{0\})) \subset \overline{A_{t_0}}$, so $A_{t_0} \cup \zeta(C \cap (X_{t_0} \times \{0\}))$ is a connected set.

2^o $C \cap (X_{t_0} \times \{\frac{1}{2}\}) \neq \emptyset$. Then , as above, it can be shown that $A_{t_0} \cup \zeta(C \cap (X_{t_0} \times \{\frac{1}{2}\}))$ is a connected set.

3^o $C \cap (X_{t_0} \times (\frac{1}{2}, 1)) \neq \emptyset$. Then, of course, $C \cap (X_{t_0} \times \{\frac{1}{2}\}) \neq \emptyset$, $B_{t_0} \subset \zeta(C)$ and $B_{t_0} \cup \zeta(C \cap (X_{t_0} \times \{\frac{1}{2}\})) \cup \zeta(C \cap (X_{t_0} \times \{1\}))$ is a connected set.

Observe that $\zeta(C) = \zeta(C \cap (X_{t_0} \times \{0\})) \cup A_{t_0} \cup \zeta(C \cap (X_{t_0} \times \{\frac{1}{2}\})) \cup B_{t_0} \cup \zeta(C \cap (X_{t_0} \times \{1\}))$ is a connected set, which completes the proof. $\square$

The equivalence class of the relation of c-homotopy will be denoted by $[f]_c$.

It is well known that any two continuous transformations taking their values in some convex subset of Euclidean space are homotopic. A similar fact also takes place in the case of c-homotopy, which is stated by the following

Theorem 2 *Let $f, h : X \to Y$ be connected functions. Then $f \cong h$ if and only if for each component Z of X there exists a connected set $C_Z \subset Y$ such that the cardinality of C_Z is less than or equal to that of the continuum and $f(Z) \cap C_Z \neq \emptyset \neq h(Z) \cap C_Z$.*

PROOF. The proof of necessity is obvious. The proof of sufficiency is similar to that of Theorem 1. $\square$

The theorem below shows that it is possible to use the terms introduced in this paper, to characterize some properties of sets. This characterization is connected with some generalization of the notion of Darboux (in our terminology — connected) retracts ([9], [7]): *A subspace A of a space X is said to be a connected retract (c-retract) of X if there exists a connected function (called a c-retraction) $r : X \to A$ such that $r(x) = x$ for each $x \in A$.*

Theorem 3 *A set C of a metric space X is a component of X if and only if C is a maximal c-retract (with respect to the inclusion) such that each c-retraction of C is c-homotopic to each connected function $f : X \to C$.*

Proof. *Necessity.* Let $p \in C$. Consider the function

$$r(x) = \begin{cases} x & \text{if} \quad x \in C, \\ p & \text{if} \quad x \notin C. \end{cases}$$

Then $r : X \to C$ is a c-retraction. By Theorem 2, r is c-homotopic to each connected function $f : X \to C$.

Now, we shall show that C is the maximal set which possesses this property. Suppose that K is a c-retract such that $K \supset C$ and $K \neq C$. Let $x_0 \in K \setminus C$ and let $g : X \to K$ be defined by the formula $g(x) \equiv x_0$. It is easy to see that the c-retraction $r_K : X \to K$ is not c-homotopic to g.

Sufficiency. First we shall show that C is a connected set. Let t be a fixed point from C and let r_C be a c-retraction of C. Then $r_C \cong_\eta h$ where $h(x) \equiv t \in C$ for $x \in X$. Thus $C = \bigcup_{x \in C} \eta(\{x\} \times I)$ is, in fact, a connected set. Now, we shall prove that C is a component of X. Suppose to the contrary that $C \subset K$ and $C \neq K$ where K is some component of X. According to the necessity of this theorem, K is a c-retract such that each c-retraction of K is c-homotopic to each connected function $h : X \to C$, which contradicts the fact that C is the maximal set (with respect to the inclusion) which possesses this property. $\square$

Homotopy and D-homotopy have been considered between continuous function. In this paper we investigate c-homotopies between connected functions. The following theorem implies, in particular, the equivalence class of D-homotopy can be a proper subset of the equivalence class of c-homotopy.

Theorem 4 *Let X contain a point x_0 for which there exists an open local base $B(x_0) = \{V_n\}_{n=1}^{\infty}$ such that each V_n is a nonsingleton connected set and let Y be a metric space such that $\operatorname{dia} Y < \infty$ and Y has a nonconstant path through every point. Then, for each connected function $f : X \to Y$ in the space $[f]_c$ with the metric of uniform convergence, the set of all continuous functions is nowhere dense.*

Proof. Let $C^{[f]}$ denote the set of all continuous functions belonging to $[f]_c$. Since continuous functions are closed under uniform limits, it is sufficient to show that every continuous function $g \in [f]_c$ has a "nearby" function $h \in [f]_c \setminus C^{[f]}$. Let $g \in [f]_c$ and $\varepsilon > 0$. Let $\alpha_0 = g(x_0)$. Thus there exists a positive integer n_0 such that $g(\overline{V_{n_0}}) \subset K(\alpha_0, \frac{\varepsilon}{3})$. Let $\delta \in (0, \frac{1}{2})$ be a number such that $\overline{K(x_0,\delta)} \subset V_{n_0}$ and $V_{n_0} \setminus \overline{K(x_0,\delta)} \neq \emptyset$. Then $\overline{K(x_0,\xi)} \setminus K(x_0,\xi) \neq \emptyset$ for $\xi < \delta$. Let Z_{α_0} be a nonconstant path through α_0 such that $Z_{\alpha_0} \subset K(\alpha_0, \frac{\varepsilon}{3})$.

On the set $[0,\delta)$ we define the equivalence relation $*$ as follows.

$$x * y \text{ if and only if } x - y \in \mathbb{Q}.$$

Let $\mathcal{A}$ be the set of all equivalence classes of $*$. Then there exists a surjection $\varphi : \mathcal{A} \to g(\overline{V_{n_0}}) \cup Z_{\alpha_0}$ such that $\varphi([0]) = \alpha_0$. Let us now define a function $h : X \to Y$ by letting

$$h(x) = \begin{cases} g(x) & \text{if} \quad x \in X \setminus K(x_0,\delta) \\ \varphi([\delta_x]) & \text{if} \quad x \in K(x_0,\delta) \end{cases}$$

where $\delta_x = \varrho_X(x_0, x)$ (ϱ_X denotes the metric in the space X.) and $[\delta_x]$ is the equivalence class of $*$ containing the element δ_x.

Observe first that h is a connected function. Indeed, let C be an arbitrary connected set in X. If $C \cap K(x_0,\delta) = \emptyset$ then $h(C) = g(C)$ is a connected set. If $C \subset \{x : \varrho_X(x_0,x) = \beta < \delta\}$, then $h(C)$ is a singleton. If $C \subset K(x_0,\delta)$, and C is not a subset of $\{x : \varrho_X(x_0,x) = \beta\}$ for any $\beta < \delta$, then $h(C) = g(\overline{V_{n_0}}) \cup Z_{\alpha_0}$, and thus, it is connected. If $C \cap K(x_0,\delta) \neq \emptyset \neq C \setminus K(x_0,\delta)$, then $h(C) = g(C) \cup g(\overline{V_{n_0}}) \cup Z_{\alpha_0}$ is a connected set, too.

Now, we shall show that h and g are c-homotopic.

Let us consider $K(x_0,\delta) \times I$. For each $\xi \in (0,\delta)$, let $K_\xi = (\overline{K(x_0,\xi)} \setminus K(x_0,\xi)) \times [\delta - \xi, 1 - \delta + \xi]$ and let $K_0 = \{x_0\} \times [\delta, 1-\delta]$. On the set $\{K_\xi : \xi \in (0,\delta)\}$ we define the equivalence relation $**$ as follows:

$$K_{\xi_1} ** K_{\xi_2} \text{ if and only if } \xi_1 - \xi_2 \in \mathbb{Q}.$$

Let $\mathcal{B}$ be the set of all equivalence classes of $**$. Then there exists a surjection $\psi : \mathcal{B} \to g(\overline{V_{n_0}}) \cup Z_{\alpha_0}$. Let us define a function η by letting

$$\eta((x,d)) = \begin{cases} g(x) & \text{if} \quad d = 0 \text{ or } x \notin K(x_0,\delta), \\ h(x) & \text{if} \quad d = 1, \\ \alpha_0 & \text{if} \quad (x,d) \in K_0, \\ \psi([K_{\xi_x}]) & \text{if} \quad (x,d) \in K_{\xi_x} \text{ and } x \neq x_0; \end{cases}$$

where $[K_{\xi_x}]$ denotes the equivalence class of $**$ containing the element K_{ξ_x}.

We can show that η is a connected function, and so, g and h are c-homotopic. Moreover, it is easy to see that $h \in K(g,\varepsilon)$ and $h \notin C^{[f]}$. $\square$

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