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ON UPPER AND LOWER β-CONTINUOUS MULTIFUNCTIONS

Abstract

In this paper the authors define a multifunction $F : X \mapsto Y$ to be upper (respectively, lower) β-continuous if $F^+(V)$ (resp. $F^-(V)$) is β-open in X for every open set V of Y. They obtain some characterizations and several properties concerning upper (resp. lower) β-continuous multifunctions. The relationships between these multifunctions and quasi continuous multifunctions are investigated.

1 Introduction

Abd El-Monsef et al. [1] defined β-continuous functions as a generalization of semi-continuity [15] and precontinuity [17]. Recently, Borsík and Doboš [7] have introduced the notion of almost quasi- continuity which is weaker than that of quasi-continuity [16] and obtained a decomposition theorem of quasi-continuity. The authors [26] of the present paper obtained several characterizations of β-continuity and showed that almost quasi-continuity is equivalent to β-continuity. The equivalence of almost quasi-continuity and β-continuity is also shown by Borsík [6] and Ewert [11]. The purpose of the present paper is to define upper (lower) β-continuous multifunctions and to obtain several characterizations of upper (lower) β-continuous multifunctions and several properties of such multifunctions.

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2 Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be α-open [19] (resp. semi-open [15], preopen [17], β-open [1] or semi-preopen [3]) if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$. The family of all semi-open (resp. β-open) sets of X containing a point $x \in X$ is denoted by $SO(X,x)$ (resp. $\beta(X,x)$). The family of all α-open (resp. semi-open, preopen, semi-preopen) sets in X is denoted by $\alpha(X)$ (resp. $SO(X)$, $PO(X)$, $\beta(X)$). For these four families, it is shown in [20] (see Lemma 3.1) that $SO(X) \cap PO(X) = \alpha(X)$, and it is obvious that $SO(X) \cup PO(X) \subset \beta(X)$. The complement of an α-open (resp. semi-open, preopen, β-open) set is said to be α-closed (resp. semi-closed [9], preclosed [10], β-closed [1]). The intersection of all β-closed sets of X containing A is called the β-closure [2] of A, and it is denoted by ${}_{\beta}\mathrm{Cl}(A)$. Similarly, ${}_{\alpha}\mathrm{Cl}(A)$, $s\mathrm{Cl}(A)$ and $p\mathrm{Cl}(A)$ are defined. The union of all β-open sets of X contained in A is called the β-interior of A and is denoted by ${}_{\beta}\mathrm{Int}(A)$. Abd El-Monsef et al. [1] defined a function to be β-continuous if the inverse image of every open set is β-open.

Throughout this paper, spaces (X,τ) and (Y,σ) (or simply X and Y) always mean topological spaces, and $F : X \mapsto Y$ (resp. $f : X \mapsto Y$) represents a multivalued (resp. single valued) function. For a multifunction $F : X \mapsto Y$, we shall denote the upper and lower inverse of a set G of Y by $F^+(G)$ and $F^-(G)$, respectively, that is $F^+(G) = \{x \in X : F(x) \subset G\}$ and $F^-(G) = \{x \in X : F(x) \cap G \neq \emptyset\}$.

3 Characterizations

Definition 3.1 *A multifunction $F : X \mapsto Y$ is said to be*

(a) upper β-continuous [27] (briefly u.β.c.) at a point $x \in X$ if for each open set V containing $F(x)$, there exists $U \in \beta(X,x)$ such that $F(U) \subset V$;

(b) lower β-continuous [27] (briefly l.β.c.) at a point $x \in X$ if for each open set V such that $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(X,x)$ such that $U \subset F^-(V)$;

(c) upper (lower) β-continuous if F has this property at every point of X.

Remark 3.1 *According to the referee, the definition of the upper and lower β-continuity at a point can be found in [12], where for the definition it is taken the condition (e) from Theorem 3.1 (resp. Theorem 3.2).*

Theorem 3.1 *The following are equivalent for a multifunction $F : X \mapsto Y$:*

(a) F *is u.β.c. at a point* $x \in X$*;*

(b) for each open neighborhood U *of* x *and each open set* V *of* Y *with* $x \in F^+(V)$*,* $F^+(V) \cap U$ *is not nowhere dense;*

(c) for each open neighborhood U *of* x *and each open set* V *of* Y *with* $x \in F^+(V)$*, there exists an open set* G *of* X *such that* $\emptyset \neq G \subset U$ *and* $G \subset Cl(F^+(V))$*;*

(d) for each open set V *of* Y *with* $x \in F^+(V)$*, there exists* $U \in SO(X,x)$ *such that* $U \subset Cl(F^+(V))$*;*

(e) $x \in Cl(Int(Cl(F^+(V))))$ *for every open set* V *of* Y *with* $x \in F^+(V)$*.*

Proof. (a) ⇒ (b) and (b) ⇒ (c): The proofs are obvious and are thus omitted.

(c) ⇒ (d): Let V be an open set of Y containing $F(x)$. By $\mathcal{U}(x)$ we denote the family of all open neighborhoods of x. For each $U \in \mathcal{U}(x)$, there exists an open set G_U of X such that $\emptyset \neq G_U \subset U$ and $G_U \subset \mathrm{Cl}(F^+(V))$. Put $W = \cup\{G_U : U \in \mathcal{U}(x)\}$. Then W is an open set of X, $x \in \mathrm{Cl}(W)$ and $W \subset \mathrm{Cl}(F^+(V))$. Moreover, we put $U_o = W \cup \{x\}$. Then $W \subset U_o \subset \mathrm{Cl}(W)$ and $U_o \in SO(X,x)$ and also $U_o \subset \mathrm{Cl}(F^+(V))$.

(d) ⇒ (e): Let V be an open set of Y containing $F(x)$. There exists $U \in SO(X,x)$ such that $U \subset \mathrm{Cl}(F^+(V))$. Therefore, we have $x \in U \subset \mathrm{Cl}(\mathrm{Int}(U)) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(V))))$.

(e) ⇒ (a): This is shown in [12]. □

Theorem 3.2 *The following are equivalent for a multifunction* $F : X \mapsto Y$*:*

(a) F *is l.β.c. at a point* $x \in X$*;*

(b) for any open neighborhood U *of* x *and any open set* V *of* Y *with* $x \in F^-(V)$*,* $F^-(V) \cap U$ *is not nowhere dense;*

(c) for any open neighborhood U *of* x *and any open set* V *of* Y *with* $x \in F^-(V)$*, there exists an open set* G *of* X *such that* $\emptyset \neq G \subset U$ *and* $G \subset Cl(F^-(V))$*;*

(d) for any open set V *of* Y *with* $x \in F^-(V)$*, there exists* $U \in SO(X,x)$ *such that* $U \subset Cl(F^-(V))$*;*

(e) $x \in Cl(Int(Cl(F^-(V))))$ *for every open set* V *of* Y *with* $x \in F^-(V)$*.*

Proof. The proof is similar to that of Theorem 3.1. □

Theorem 3.3 *The following are equivalent for a multifunction* $F : X \mapsto Y$*:*

(a) F is u.β.c.,;

(b) $F^+(V) \in \beta(X)$ for every open set V of Y;

(c) $F^-(K)$ is β-closed in X for every closed set K of Y;

(d) ${}_\beta Cl(F^-(B)) \subset F^-(Cl(B))$ for every subset B of Y;

(e) $Int(Cl(Int(F^-(B)))) \subset F^-(Cl(B))$ for every subset B of Y.

Proof. (a) $\Rightarrow$ (b): Let V be any open set of Y and $x \in F^+(V)$. There exists $U \in \beta(X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(U))) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(V))))$. Then $F^+(V) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(V))))$ and hence $F^+(V) \in \beta(X)$.

(b) $\Rightarrow$ (c): This follows immediately from the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for every subset B of Y.

(c) $\Rightarrow$ (d): For any subset B of Y, $\mathrm{Cl}(B)$ is closed in Y and $F^-(\mathrm{Cl}(B))$ is β-closed in X. Therefore, we obtain ${}_\beta\mathrm{Cl}(F^-(B)) \subset F^-(B)$.

(d) $\Rightarrow$ (e): Let B be any subset of Y. By [3] (see Theorem 2.15), we obtain $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(B)))) \subset {}_\beta\mathrm{Cl}(F^-(B)) \subset F^-(\mathrm{Cl}(B))$.

(e) $\Rightarrow$ (b): Let V be any open set of Y. Then $Y \setminus V$ is closed in Y and we have $X \setminus F^+(V) = F^-(Y \setminus V) \supset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^-(Y \setminus V)))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(X \setminus F^+(V)))) = X \setminus \mathrm{Cl}(\mathrm{Int}(F^+(V))))$. We obtain $F^+(V) \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(F^+(V))))$ and hence $F^+(V) \in \beta(V)$.

(b) $\Rightarrow$ (a): Let $x \in X$ and V be an open set of Y containing $F(x)$. By (b), we have $x \in F^+(V) \in \beta(X)$. Put $U = F^+(V)$. Then we obtain $U \in \beta(X, x)$ and $F(U) \subset V$. Therefore, F is upper β-continuous. $\square$

Theorem 3.4 *The following are equivalent for a multifunction $F : X \mapsto Y$:*

(a) F is l.β.c.,;

(b) $F^-(V) \in \beta(X)$ for every open set V of Y;

(c) $F^+(K)$ is β-closed in X for every closed set K of Y;

(d) ${}_\beta Cl(F^+(B)) \subset F^+(Cl(B))$ for every subset B of Y;

(e) $Int(Cl(Int(F^+(B)))) \subset F^+(Cl(B))$ for every subset B of Y.

(f) $F(Int(Cl(Int(A)))) \subset Cl(F(A))$ for every subset A of X;

(g) $F({}_\beta Cl(A)) \subset Cl(F(A))$ for every subset A of X.

Proof. It is shown similarly to the proof of Theorem 3.3 that the statements (a), (b), (c), (d) and (e) are equivalent. We shall prove only the following implications.

(e) $\Rightarrow$ (f): Let A be any subset of X. Then we have $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(F^+(F(A))))) \subset F^+(\mathrm{Cl}(F(A)))$. Therefore $F(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) \subset \mathrm{Cl}(F(A))$.

(f) $\Rightarrow$ (g): Let A be any subset of X. By [3] (see Theorem 2.15), we have $F({}_\beta\mathrm{Cl}(A)) = F(A \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) = F(A) \cup F(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))) \subset \mathrm{Cl}(F(A))$.

(g) $\Rightarrow$ (c): Let K be any closed set of Y. Then we have $F({}_\beta\mathrm{Cl}(F^+(K))) \subset \mathrm{Cl}(F(F^+(K))) \subset \mathrm{Cl}(K) = K$. Therefore, we have ${}_\beta\mathrm{Cl}(F^+(K)) \subset F^+(K)$, and hence $F^+(K)$ is β-closed in X. □

For a multifunction $F : X \mapsto Y$, the graph multifunction $G_F : X \mapsto X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$.

Lemma 3.1 (Noiri and Popa, [22]) *For a multifunction $F : X \mapsto Y$, the following hold:*

$$(a)\quad G_F^+(A \times B) = A \cap F^+(B) \quad and \quad (b)\quad G_F^-(A \times B) = A \cap F^-(B)$$

for any subsets $A \subset X$ and $B \subset Y$.

Theorem 3.5 *Let $F : X \mapsto Y$ be a multifunction such that $F(x)$ is compact for each $x \in X$. Then F is u.β.c. if and only if $G_F : X \mapsto X \times Y$ is u.β.c..*

Proof. *Necessity.* Suppose that $F : X \mapsto Y$ is *u.β.c.*. Let $x \in X$ and W be any open set of $X \times Y$ containing $G_F(X)$. For each $y \in F(x)$, there exist open sets $U(y) \subset X$ and $V(y) \subset Y$ such that $(x, y) \in U(y) \times V(y) \subset W$. The family $\{V(y) : y \in F(x)\}$ is an open cover of $F(x)$, and $F(x)$ is compact. Therefore, there exist a finite number of points $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subset \cup\{V(y_i) : 1 \leq i \leq n\}$. Set $U = \cap\{U(y_i) : 1 \leq i \leq n\}$ and $V = \cup\{V(y_i) : 1 \leq i \leq n\}$. Then U and V are open in X and Y, respectively, and $\{x\} \times F(x) \subset U \times V \subset W$. Since F is *u.β.c.*, there exists $U_o \in \beta(X, x)$ such that $F(U_o) \subset U$. By Lemma 3.1, we have $U \cap U_o \subset U \cap F^+(V) = G_F^+(U \times V) \subset G_F^+(W)$. Therefore, we obtain $U \cap U_o = \beta(X, x)$ and $G_F(U \cap U_o) \subset W$. This shows that G_F is *u.β.c.*.

Strong sufficiency. Suppose that $G_F : X \mapsto X \times Y$ is *u.β.c.*. Let $x \in X$ and V be any open set of Y containing $F(x)$. Since $X \times V$ is open in $X \times Y$ and $G_F(x) \subset X \times V$, there exists $U \in \beta(X, x)$ such that $G_F(U) \subset X \times V$. By Lemma 3.1, we have $U \subset G_F^+(X \times V) = F^+(V)$ and $F(U) \subset V$. This shows that F is *u.β.c.*. □

Theorem 3.6 *A multifunction $F : X \mapsto Y$ is l.β.c. if and only if $G_F : X \mapsto X \times Y$ is l.β.c..*

Proof. *Necessity.* Suppose that F is $l.\beta.c.$. Let $x \in X$ and W be any open set of $X \times Y$ such that $x \in G_F^-(W)$. Since $W \cap (\{x\} \times F(x)) \neq \emptyset$, there exists $y \in F(x)$ such that $(x, y) \in W$, and hence $(x, y) \in U \times V \subset W$ for some open sets $U \subset X$ and $V \subset Y$. Since $F(x) \cap V \neq \emptyset$, there exists $G \in \beta(X, x)$ such that $G \subset F^-(V)$. By Lemma 3.1, we have $U \cap G \subset U \cap F^-(V) = G_F^-(U \times V) \subset G_F^-(W)$. Moreover, $x \in U \cap G \subset \beta(X)$ and hence G_F is $l.\beta.c.$.

Sufficiency. Suppose that G_F is $l.\beta.c.$. Let $x \in X$ and V be an open set of Y such that $x \in F^-(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is $l.\beta.c.$, there exists $U \in \beta(X, x)$ such that $U \subset G_F^-(X \times V)$. By Lemma 3.1, we obtain $U \subset F^-(V)$. This shows that F is $l.\beta.c.$. □

Definition 3.2 *A subset A of a topological space X is said to be*

(a) α-paracompact [28] if every cover of A by open sets of X is refined by a cover of A which consists of open sets of X and is locally finite in X;

(b) α-regular [14] if for each $a \in A$ and each open set U of X containing a there exists an open set G of x such that $a \in G \subset Cl(G) \subset U$.

Lemma 3.2 (Kovacević [14]) *If A is an α-regular α-paracompact set of a topological space X and U is an open neighborhood of A, then there exists an open set G of X such that $A \subset G \subset Cl(G) \subset U$.*

For a multifunction $F : X \mapsto Y$, by ClF : $X \mapsto Y$ (see [4]) we denote a multifunction defined as follows: $(\mathrm{ClF})(x) = \mathrm{Cl}(F(x))$ for each $x \in X$. Similarly, we can define ${}_\beta$ClF : $X \mapsto Y$, sClF : $X \mapsto Y$, pClF : $X \mapsto Y$ and ${}_\alpha$ClF : $X \mapsto Y$.

Lemma 3.3 *If $F : X \mapsto Y$ is a multifunction such that $F(x)$ is α-paracompact α-regular for each $x \in X$, then for each open set V of Y, $G^+(V) = F^+(V)$, where G denotes ${}_\beta$ClF, sClF, pClF, ${}_\alpha$ClF or ClF.*

Proof. Let V be any open set of Y. Let $x \in G^+(V)$. Then $G(x) \subset V$ and $F(x) \subset G(x) \subset V$. We have $x \in F^+(V)$, and hence $G^+(V) \subset F^-(V)$. Conversely, let $x \in F^+(V)$, then $F(x) \subset V$. By Lemma 3.2, there exists an open set H of Y such that $F(x) \subset H \subset \mathrm{Cl}(H) \subset V$; hence $G(x) \subset \mathrm{Cl}(H) \subset V$. Therefore, we have $x \in G^+(V)$ and $F^+(V) \subset G^+(V)$. □

Theorem 3.7 *Let $F : X \mapsto Y$ be a multifunction such that $F(x)$ is α-paracompact and α-regular for each $x \in X$. Then the following are equivalent: a) F is $u.\beta.c.$; b) ${}_\beta$ClF is $u.\beta.c.$; c) sClF is $u.\beta.c.$; d) pClF is $u.\beta.c.$; e) ${}_\alpha$ClF is $u.\beta.c.$; f) ClF is $u.\beta.c.$.*

Proof. Similarly to Lemma 3.3, we put $G = {}_{\beta}\mathrm{ClF}$, $s\mathrm{ClF}$, $p\mathrm{ClF}$, ${}_{\alpha}\mathrm{ClF}$ or ClF. Suppose that F is $u.\beta.c.$. Let $x \in X$ and V be any open set of Y containing $G(x)$. By Lemma 3.3, $x \in G^+(V) = F^+(V)$ and there exists $U \in \beta(X, x)$ such that $F(U) \subset V$. Since $F(u)$ is α-paracompact and α-regular for each $u \in U$, by Lemma 3.2 there exists an open set H such that $F(u) \subset H \subset \mathrm{Cl}(H) \subset V$; hence $G(u) \subset \mathrm{Cl}(H) \subset V$ for each $u \in U$. Therefore, we obtain $G(U) \subset V$. This shows that G is $u.\beta.c.$.

Conversely, suppose that G is $u.\beta.c.$. Let $x \in X$ and V be any open set of Y containing $F(x)$. By Lemma 3.3, $x \in F^+(V) = G^+(V)$ and hence $G(x) \subset V$. There exists $U \in \beta(X, x)$ such that $G(U) \subset V$. Thus $U \subset G^+(V) = F^+(V)$, and hence $F(U) \subset V$. This shows that F is $u.\beta.c.$. □

Lemma 3.4 *If $F : X \mapsto Y$ is a multifunction, then for each open set V of Y, $G^-(V) = F^-(V)$, where G denotes ${}_{\beta}\mathrm{ClF}$, $s\mathrm{ClF}$, $p\mathrm{ClF}$, ${}_{\alpha}\mathrm{ClF}$ or ClF.*

Proof. Let V be any open set of Y and $x \in G^-(V)$. Then $G(x) \cap V \neq \emptyset$, and hence $F(x) \cap V \neq \emptyset$ since V is open. Thus, we obtain $x \in F^-(V)$ and hence $G^-(V) \subset F^-(V)$. Conversely, let $x \in F^-(V)$. Then we have $\emptyset \neq F(x) \cap V \subset G(x) \cap V$ and hence $x \in G^-(V)$. Thus, we have $F^-(V) \subset G^-(V)$. Consequently, we obtain $G^-(V) = F^-(V)$. □

Theorem 3.8 *For a multifunction $F : X \mapsto Y$, the following are equivalent: a) F is $l.\beta.c.$; b) ${}_{\beta}\mathrm{ClF}$ is $l.\beta.c.$; c) $s\mathrm{ClF}$ is $l.\beta.c.$; d) $p\mathrm{ClF}$ is $l.\beta.c.$; e) ${}_{\alpha}\mathrm{ClF}$ is $l.\beta.c.$; f) ClF is $l.\beta.c.$.*

Proof. By using Lemma 3.4, this is shown similarly as in Theorem 3.7. □

Theorem 3.9 *Let $\{U_\alpha : \alpha \in \bigtriangledown\}$ be an α-open cover of a topological space X. A multifunction $F : X \mapsto Y$ is u.β.c. if and only if the restriction $F/U_\alpha : U_\alpha \mapsto Y$ is u.β.c. for each $\alpha \in \bigtriangledown$.*

Proof. *Necessity.* Let $\alpha \in \bigtriangledown$ and $x \in U_\alpha$. Let V be an open set of Y such that $(F/U_\alpha)(x) \subset V$. Since F is $u.\beta.c.$ and $F(x) = (F/U_\alpha)(x) \subset V$, there exists $G \in \beta(X, x)$ such that $F(G) \subset V$. Set $U = G \cap U_\alpha$, then we have $U \in \beta(U_\alpha, x)$ (see [1], Lemma 2.5) and $(F/U_\alpha)(U) = F(U) \subset V$. Therefore, F/U_α is $u.\beta.c.$.

Sufficiency. Let $x \in X$ and V be any open set of Y such that $F(x) \subset V$. There exists $\alpha \in \bigtriangledown$ such that $x \in U_\alpha$. Since F/U_α is $u.\beta.c.$ and $(F/U_\alpha)(x) = F(x) \subset V$, there exists $U \in \beta(U_\alpha, x)$ such that $(F/U_\alpha)(U) \subset V$. Thus, we have $U \in \beta(X, x)$ (see [1], Lemma 2.7) and $F(U) = (F/U_a)(U) \subset V$. This shows that F is $u.\beta.c.$. □

Theorem 3.10 *Let $\{U_\alpha : \alpha \in \bigtriangledown\}$ be an α-open cover of a topological space X. A multifunction $F : X \mapsto Y$ is l.β.c. if and only if the restriction $F/U_\alpha : U_\alpha \mapsto Y$ is l.β.c. for each $\alpha \in \bigtriangledown$.*

Proof. The proof is similar to that of Theorem 3.9. □

4 Some properties

Definition 4.1 *A multifunction $F : X \mapsto Y$ is said to be upper rarely continuous [25] at a point x of X if for each open set G of Y containing $F(x)$, there exists a rare set R_G with $Cl(R_G) \cap G = \emptyset$ and an open set U containing x such that $F(U) \subset G \cup R_G$. A multifunction is said to be upper rarely continuous if it has the property at each point of X.*

Theorem 4.1 *If a multifunction $F : X \mapsto Y$ is upper rarely continuous at each point $x \in X$ and for each open set G containing $F(x)$, $F^-(Cl(R_G))$ is a β-closed set of X, where R_G is the rare set of Definition 4.1, then F is u.β.c..*

Proof. Let $x \in X$ and G be an open set such that $F(x) \subset G$. Since F is upper rarely continuous, there exist an open set V of X containing x and a rare set R_G with $Cl(R_G) \cap G = \emptyset$ such that $F(V) \subset G \cup R_G$. Let $U = V \cap (X \setminus F^-(\mathrm{Cl}(R_G)))$. Then we have $U \in \beta(X)$ (see [3], Theorem 2.7) and $x \in U$, since $x \in V$ and $x \in X \setminus F^-(\mathrm{Cl}(R_G))$. If we suppose that $x \in F^-(\mathrm{Cl}(R_G))$ then $F(x) \cap \mathrm{Cl}(R_G) \neq \emptyset$, but $F(x) \subset G$ and $G \cap \mathrm{Cl}(R_G) = \emptyset$. Let $s \in U$. Then $F(s) \subset G \cup R_G$ and $F(s) \cap \mathrm{Cl}(R_G) = \emptyset$. Therefore, we have $F(s) \cap R_G = \emptyset$, and hence $F(s) \subset G$. Since U is a β-open set containing x, it follows that F is *u.β.c.*. □

Definition 4.2 *A multifunction $F : X \mapsto Y$ is said to be upper α- continuous [18] if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists an α-open set U containing x such that $F(U) \subset V$.*

Theorem 4.2 *If $F, G : X \mapsto Y$ are multifunctions and Y is a normal space such that*

a) F and G are punctually closed;

b) F is u.β.c.;

c) G is upper α-continuous,

then the set $\{x \in X : F(x) \cap G(x) \neq \emptyset\}$ is β-closed in X.

Proof. Put $A = \{x \in X : F(x) \cap G(x) \neq \emptyset\}$ and let $x \in X \setminus A$. Then $F(x) \cap G(x) = \emptyset$. Since Y is normal, there exist disjoint open sets V and W such that $F(x) \subset V$ and $G(x) \subset W$. Since F is *u.β.c.*, there exists $U_1 \in \beta(X, x)$ such that $F(U_1) \subset V$. Since G is upper α-continuous, there exists an α-open set U_2 containing x such that $G(U_2) \subset W$. Put $U = U_1 \cap U_2$. Then $U \in \beta(X, x)$ (see [3], Corollary 2.14) and $F(U) \cap G(U) = \emptyset$. Therefore, we have $U \cap A = \emptyset$ and hence A is β-closed in X. □

Definition 4.3 *The β-frontier of a subset A of X, denoted by ${}_\beta Fr(A)$, is defined by ${}_\beta Fr(A) = {}_\beta Cl(A) \cap {}_\beta Cl(X \setminus A) = {}_\beta Cl(A) - {}_\beta Int(A)$.*

Theorem 4.3 *The set of all points x of X at which a multifunction $F : X \mapsto Y$ is not u.β.c. (l.β.c.) is identical with the union of the β-frontier of the upper (lower) inverse images of open sets containing (meeting) $F(x)$.*

Proof. Let x be a point of X at which F is not *u.β.c.*. Then there exists an open set V of Y containing $F(x)$ such that $U \cap (X \setminus F^+(V)) \neq \emptyset$ for every $U \in \beta(X, x)$. Therefore, we have $x \in {}_\beta\mathrm{Cl}(X \setminus F^+(V)) = X \setminus {}_\beta\mathrm{Int}(F^+(V))$ and $x \in F^+(V)$. Thus we obtain $x \in {}_\beta\mathrm{Fr}(F^+(V))$. Conversely, suppose that V is an open set containing $F(x)$ and that $x \in {}_\beta\mathrm{Fr}(F^+(V))$. If F is *u.β.c*, at x, then there exists $U \in \beta(X, x)$ such that $U \subset F^+(V)$; hence $x \in {}_\beta\mathrm{Int}(F^+(V))$. This is a contradiction, hence F is not *u.β.c.* at x. The case for *l.β.c.* is similarly shown. □

5 β-continuity and quasi-continuity

Definition 5.1 *A multifunction $F : X \mapsto Y$ is said to be*

(a) upper quasi continuous [23] if for each $x \in X$, each open set U containing x and each open set V containing $F(x)$, there exists a nonempty open set G of X such that $G \subset U$ and $F(G) \subset V$;

(b) lower quasi continuous [23] if for each $x \in X$, each open set U containing x and each open set V such that $F(x) \cap V \neq \emptyset$, there exists a nonempty open set G of X such that $G \subset U$ and $F(g) \cap V \neq \emptyset$ for every $g \in G$.

Lemma 5.1 (Noiri and Popa [21]) *If A is an α-regular set of a topological space X, then for every open set U which intersects A there exists an open set U_A such that $A \cap U_A \neq \emptyset$ and $Cl(U_A) \subset U$.*

The set of all points at which a multifunction $F : X \mapsto Y$ is *u.β.c.* and *l.β.c.* (resp. upper quasi continuous and lower quasi continuous) will be denoted by B_F (resp. Q_F).

Theorem 5.1 *If a multifunction $F : X \mapsto Y$ is punctually α-regular and α-paracompact, then $B_F \cap Int(Cl(Q_F)) \subset Q_F$.*

Proof. Let $x \in B_F \cap \mathrm{Int}(\mathrm{Cl}(Q_F))$. First, we show that F is upper quasi continuous. Let U and V be open sets such that $x \in U$ and $F(x) \subset V$. Since $F(x)$ is α-regular and α-paracompact, by Lemma 3.2 there exists an open set W such that $F(x) \subset W \subset \mathrm{Cl}(W) \subset V$. The upper β-continuity of F at x implies that there exists a nonempty open set $G \subset U \cap \mathrm{Int}(\mathrm{Cl}(Q_F))$ such that $G \subset \mathrm{Cl}(F^+(W))$. It follows from $G \subset \mathrm{Int}(\mathrm{Cl}(Q_F))$ that $G \cap Q_F \neq \emptyset$.

$$\text{If } \ s \in G \cap Q_F\,, \ \text{ then } \ s \in F^+(\mathrm{Cl}(W))\,. \tag{1}$$

Suppose that (1) does not hold. Then there exists $s \in G \cap Q_F$ such that $s \in F^-(Y \setminus \mathrm{Cl}(W))$. The lower quasi continuity of F implies that there exists a nonempty open set $G_1 \subset G$ such that $G_1 \subset F^-(Y \setminus \mathrm{Cl}(W)) \subset F^-(Y \setminus W)$ (see [24], Theorem 2.2). This contradicts that $G \subset \mathrm{Cl}(F^+(W))$. It follows from (1) that if $s \in G \cap Q_F$ then $F(s) \subset \mathrm{Cl}(W) \subset V$. The upper quasi continuity of F at s implies that there exists a nonempty open set $H \subset U$ such that $F(H) \subset V$. Thus F is upper quasi continuous at x. Next, we show that F is lower quasi continuous. Let U and V be open sets such that $x \in U$ and $F(x) \cap V \neq \emptyset$. Since $F(x)$ is α-regular, by Lemma 5.1 there exists an open set W such that $F(x) \cap W \neq \emptyset$ and $\mathrm{Cl}(W) \subset V$. The lower β-continuity of F at x implies that there exists a nonempty open set $G \subset U \cap \mathrm{Int}(\mathrm{Cl}(Q_F))$ such that $G \subset \mathrm{Cl}(F^-(W))$. It follows from $G \subset \mathrm{Int}(\mathrm{Cl}(Q_F))$ that $G \cap Q_F \neq \emptyset$.

$$\text{If } \ s \in G \cap Q_F \ \text{ then } \ s \in F^-(\mathrm{Cl}(W))\,. \tag{2}$$

Suppose that (2) does not hold. Then, there exists $s \in G \cap Q_F$ such that $s \in F^+(Y \setminus \mathrm{Cl}(W))$. The upper quasi continuity of F at s implies that there exists a nonempty open set $G_2 \subset G$ such that $G_2 \subset F^+(Y \setminus \mathrm{Cl}(W)) \subset F^+(Y \setminus W)$ (see [24], Theorem 2.1). This is in contradiction with $G \subset \mathrm{Cl}(F^-(W))$. It follows from (2) that if $s \in G \cap Q_F$, then $F(s) \cap \mathrm{Cl}(W) \neq \emptyset$ and hence $F(s) \cap V \neq \emptyset$. The lower quasi continuity of F are s implies that there exists a nonempty open set $H \subset U$ such that $F(H) \cap V \neq \emptyset$ for each $h \in H$. Thus F is lower quasi continuous at x. Consequently we obtain $x \in Q_F$. □

Corollary 5.1 (Borsík and Doboš [7]) *Let Y be a regular space and $f : X \mapsto Y$ be a function. Then $B_f \cap Int(Cl(Q_f) \subset Q_f$.*

Corollary 5.2 *If a multifunction $F : X \mapsto Y$ is u.β.c. and l.β.c., and $F(x)$ is α-regular α-paracompact for each $x \in X$, then Q_F is semi-closed in X.*

Corollary 5.3 *Let Y be a regular space and $F : X \mapsto Y$ an u.β.c. and l.β.c. multifunction. If $F(x)$ is compact for each $x \in X$, then Q_F is semi-closed in X.*

Corollary 5.4 (Borsík and Doboš [7]) *If Y is a regular space and $f : X \mapsto Y$ a β- continuous function, then Q_f is semi-closed in X.*

Theorem 5.2 *Let $F : X \mapsto Y$ be a multifunction such that $F(x)$ is α-regular α-paracompact for each $x \in X$. Then F is upper and lower quasi continuous if and only if F is u.β.c. and l.β.c. and Q_F is a dense set in X.*

Corollary 5.5 (Borsík and Doboš [7]) *Let Y be a regular space. Then $f : X \mapsto Y$ is quasi continuous if and only if it is β-continuous and Q_f is dense in X.*

Definition 5.2 *A multifunction $F : X \mapsto Y$ is said to be upper (resp. lower) s-quasi continuous [13] at a point $x \in X$ if for each open set V of Y containing (resp. meeting) $F(x)$ and having the connected complement, there exists $U \in SO(X, x)$ such that $F(U) \subset V$ (resp. $U \subset F^-(V)$).*

By $Q_s(F)$ we shall denote the set of all points of X at which a multifunction $F : X \mapsto Y$ is upper and lower s-quasi continuous.

Theorem 5.3 *Let Y be a locally connected regular space. If $F : X \mapsto Y$ is a multifunction such that $F(x)$ is connected and compact for each $x \in X$, then $B_F \cap Int(Cl(Q_s(F))) \subset Q_s(F)$.*

Proof. The proof is similar to that of Theorem 5.1 and is thus omitted. □

Corollary 5.6 *Let Y be a locally connected regular space. If a multifunction $F : X \mapsto Y$ is u.β.c. and l.β.c. and $F(x)$ is connected compact for each $x \in X$, then $Q_s(F)$ is semi-closed in X.*

6 Nets for multifunctions

In what follows $(D, >)$ is a directed set, (F_α) is a net of multifunctions $F_\alpha : X \mapsto Y$, $\alpha \in D$ and F is a multifunction on X into Y.

Definition 6.1 *(1) (F_α) converges upper pointwise to F on X [8] if for each $x \in X$ and each open set $G \subset Y$ containing $F(x)$, there exists $\beta(x, G) \in D$ such that $F_\alpha(x) \subset G$ for all $\alpha > \beta(x, G)$;*

(2) (F_α) converges lower pointwise to F on X [8] if for each $x \in X$ and each open set $G \subset Y$ which intersects $F(x)$, there exists $\beta(x, G) \in D$ such that $F_\alpha(x) \cap G \neq \emptyset$ for all $\alpha > \beta(x, G)$;

(3) (F_α) converges pointwise to F on X [8] if it converges upper pointwise and lower pointwise to F.

Definition 6.2 *(1) (F_α) converges quasi upper r-uniformly (q.u.r.u.) to F on X [8] if*

(i) (F_α) converges pointwise to F on X,

(ii) for each open set G of Y with $F^+(G) \neq \emptyset$ and each $\beta \in D$, there exists $\alpha > \beta$ such that $F_\alpha(x) \subset G$ for all $x \in F^+(G)$;

(2) (F_α) converges quasi lower r-uniformly (q.l.r.u.) to F on X [8] if

(i) (F_α) converges pointwise to F on X,

(ii) for each open set G of Y with $F^-(G) \neq \emptyset$ and each $\beta \in D$, there exists $\alpha > \beta$ such that $F_\alpha(x) \cap G \neq \emptyset$ for all $x \in F^-(G)$;

(3) (F_α) converges quasi r-uniformly (q.r.u.) to F on X if it converges q.u.r.u. and q.l.r.u..

Theorem 6.1 *Let (F_α) be a net which converges q.l.r.u. to $F : X \mapsto Y$ and $F(x)$ be compact for each $x \in X$. If Y is regular and F_α is u.β.c. for each $\alpha \in D$, then F is u.β.c..*

Proof. We suppose that F is not $u.\beta.c.$ at $x_o \in X$ but all F_α are $u.\beta.c.$ at x_o. Then there exists an open set G of Y containing $F(x_o)$ such that for every β-open set V of X containing x_o, there exists $x_V \in V$ such that $F(x_V)$ is not contained in G. But $F(x_o) \cap (Y \setminus G) = \emptyset$, $F(x_o)$ is compact, $Y \setminus G$ is closed and Y is regular. Therefore, it follows that there exist two disjoint open sets G_1 and G_2 such that $F(x_o) \subset G_1$, $Y \setminus G \subset G_2$. From the pointwise convergence of (F_α) to F it follows that there exists $\alpha_o \in D$ such that $F_\alpha(x_o) \subset G_1$ for all $\alpha > \alpha_o$. But $F^-(G_2) \neq \emptyset$ since $x_V \in F^-(G_2)$ and (F_α) converges $q.l.r.u.$ to F. Therefore, it follows that there exists $\gamma > \alpha_o$ such that $F_\gamma(x) \cap G_2 \neq \emptyset$ for each $x \in F^-(G_2)$; hence $F_\gamma(x_V) \cap G_2 \neq \emptyset$. This implies that $F_\gamma(x_V)$ is not contained in G_1. Therefore, F_γ is not $u.\beta.c.$ in x_o. This contradicts the hypothesis. □

Theorem 6.2 *Let (F_α) be a net which converges q.u.r.u. to $F : X \mapsto Y$. If Y is regular and F_α is l.β.c. for each $\alpha \in D$, then F is l.β.c..*

Proof. We suppose that F is not $l.\beta.c.$ at $x_o \in X$, but all F_α are $l.\beta.c.$ at x_o. Then there exists an open set G of Y intersecting $F(x_o)$ such that for every β-open set V of X containing x_o, there exists $x_V \in V$ such that $F(x_V) \cap G = \emptyset$. Let y_o be an arbitrary point of $F(x_o) \cap G$. Then $y_o \in Y \setminus (Y \setminus G)$ and Y is regular. Therefore, it follows that there exist two disjoint open sets G_1 and G_2 such that $y_o \in G_1$, $Y \setminus G \subset G_2$. Hence $F(x_o) \cap G_1 \neq \emptyset$. ¿From the pointwise convergence of (F_α) to F it follows that there exists $\alpha_o \in D$ such that $F_\alpha(x_o)\cap G_1 \neq \emptyset$ for all $\alpha > \alpha_o$. But $F^+(G_2) \neq \emptyset$ since $x_V \in F^+(G_2)$ and (F_α) converges $q.u.r.u.$ to F. Therefore, it follows that there exists $\gamma > \alpha_o$ such that $F_\gamma(x) \subset G_2$ for each $x \in F^+(G_2)$; hence $F_\gamma(x_V) \subset G_2$. This implies that $F_\gamma(x_v) \cap G_1 = \emptyset$. Therefore, F_γ is not $l.\beta.c.$ in x_o. This contradicts the hypothesis. □

Definition 6.3 *Let $(F_\alpha)_{\alpha\in D}$ be a net of multifunctions on X into Y. A multifunction $F^* : X \mapsto Y$ defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y$: for each open neighborhood of y and each $\beta \in D$, there exists $\alpha \in D$ such that $\alpha > \beta$ and $V \cap F_\alpha(x) \neq \emptyset\}$ is called the upper topological limit [5] of the net (F_α).*

Definition 6.4 *A net $(F_\alpha)_{\alpha\in D}$ is said to be equally $u.\beta.c.$ at $x_o \in X$ if for every open set V_α containing $F_\alpha(x_o)$ there exists a β-open set U containing x_o such that $F_\alpha(U) \subset V_\alpha$ for all $\alpha \in D$.*

Theorem 6.3 *Let $(F_\alpha)_{\alpha\in D}$ be a net of multifunctions from a topological space (X,τ) into a compact topological space (Y,σ). If the following are satisfied:*

(1) $\cap\{(Y \setminus F_\beta(x)) : \beta > \alpha\} \in \sigma$ for each $\alpha \in D$ and each $x \in X$,

(2) (F_α) is equally $u.\beta.c.$ on X ,

then F^* is $u.\beta.c.$ on X.

Proof. It is known that $F^*(x) = \cap\{\mathrm{Cl}(\cup\{F_\beta(x) : \beta > \alpha\}) : \alpha \in D\}$. From (1) we have $F^\star(x) = \cap\{[\cup\{F_\beta(x) : \beta > \alpha\}] : \alpha \in D\}$. Since the net $(\cup\{F_\beta(x) : \beta > \alpha\})_{\alpha\in D}$ is a family of closed sets having the finite intersection property and Y is compact, it follows that $F^*(x) \neq \emptyset$ for each $x \in X$. Now, let $x_o \in X$ and let $V \in \sigma$ such that $V \neq Y$ and $F^*(x_o) \subset V$. Then $F^*(x_o) \cap (Y \setminus V) = \emptyset$, $F^*(x_o) \neq \emptyset$ and $Y \setminus V \neq \emptyset$. It results that $\cap\{[\cup\{F_\beta(x_o) : \beta > \alpha\}] : \alpha \in D\} \cap (Y \setminus V) = \emptyset$ and hence $\cap\{[\cup\{F_\beta(x_o) \cap (Y \setminus V) : \beta > \alpha\}] : \alpha \in D\} = \emptyset$. Since Y is compact and the family $\{[\cup\{F_\beta(x_o) \cap (Y \setminus V) : \beta > \alpha\}] : \alpha \in D\}$ is a family of closed sets with the empty intersection, there exists $\alpha \in D$ such that for each $\beta \in D$ with $\beta > \alpha$ we have $F_\beta(x_o) \cap (Y \setminus V) = \emptyset$; hence $F_\beta(x_o) \subset V$. Since the net $(F_\alpha)_{\alpha\in D}$ is equally $u.\beta.c.$ on X, it results that there exists a β-open set U containing x_o such that $F_\beta(U) \subset V$ for each $\beta > \alpha$; hence $F_\beta(x) \cap (Y \setminus V) = \emptyset$ for each $x \in U$. Then we have $\cup\{F_\beta(x) \cap (Y \setminus V) : \beta > \alpha\} = \emptyset$; hence $\cap\{[\cup\{F_\beta(x) : \beta > \alpha\}] : \alpha \in D\} \cap (Y \setminus V) = \emptyset$. This implies that $F^*(U) \subset V$. If $V = Y$ then it is clear that for each β- open set U containing x_o we have $F^*(U) \subset V$. Hence F^* is $u.\beta.c.$ at x_o. Since x_o is arbitrary, the proof is complete. □

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