Real Analysis Exchange
Vol. 22(1), 1996-97, pp. 350–361

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A CONCEPT OF GENERALIZED ABSOLUTE CONTINUITY FOR THE $\mathcal{F}$-INTEGRAL

Abstract

We define a concept of generalized absolute continuity for additive functions of figures. This makes it possible to give a descriptive definition of the $\mathcal{F}$-integral introduced in [6]. Finally, we discuss a possible extension to additive functions of sets of bounded variation.

Two definitions of multidimensional generalized Riemann integral were introduced in [6] and [5], mainly in order to integrate the divergence of vector fields with singularities. We will call these integrals the $\mathcal{F}$-integral and BV-integral respectively. Their definitions are closely related. In fact, they differ only by the class of domains allowed: for the $\mathcal{F}$-integral this is the class of all figures in $\mathbb{R}^N$ (i.e. finite unions of nondegenerate intervals), while the BV-integral integrates functions defined on sets of bounded variation in De Giorgi's sense.

In order to provide descriptive definitions for both integrals, a concept of derivability of continuous additive functions of the domains (shortly: charges) was introduced. Together with a notion of "good behavior on sets of measure zero", the derivability almost everywhere ensures that a charge is an indefinite integral (see [6, Theorem 12.3.4] for the $\mathcal{F}$-integral and [3, Theorem 2.6] for the BV-integral).

The question we address in this paper is that of finding a unifying property of "absolute continuity" equivalent to that of being an indefinite integral. We give an answer in the case of $\mathcal{F}$-integration and discuss the technical difficulties arising from the extension to BV-integration (see [3, Question 2.7] and [1, Question 2.6]).

Mathematical Reviews subject classification: Primary: 26A39
Received by the editors August 8, 1995
*Dedicated to my father

1 Preliminaries

An *interval* in $\mathbb{R}$ is a set of the type $[a,b] := \{x \in \mathbb{R} : a \le x \le b\}$ where $a,b \in \mathbb{R}$; it is called *nondegenerate* whenever $a < b$. An *interval* in $\mathbb{R}^N$ is a cartesian product of intervals in $\mathbb{R}$: $\prod_{i=1}^N [a_i,b_i]$; it is called *nondegenerate* whenever each $[a_i,b_i]$ is nondegenerate. A *figure* in $\mathbb{R}^N$ is a finite union of nondegenerate intervals. The family of all figures in $\mathbb{R}^N$ is denoted $\mathcal{F}(\mathbb{R}^N)$, while if $A \in \mathcal{F}(\mathbb{R}^N)$ we set $\mathcal{F}(A) := \{B \in \mathcal{F}(\mathbb{R}^N) : B \subset A\}$. Two figures $A,B \in \mathcal{F}(\mathbb{R}^N)$ are termed *nonoverlapping* whenever $\mathrm{int}\,(A) \cap \mathrm{int}\,(B) = \emptyset$. Given $A,B \in \mathcal{F}(\mathbb{R}^N)$, $A \cup B \in \mathcal{F}(\mathbb{R}^N)$; we also define

$$A \odot B := \mathrm{cl}\,(\mathrm{int}\,(A) \cap \mathrm{int}\,(B)) \quad \text{and} \quad A \ominus B := \mathrm{cl}\,(A \setminus B)\,.$$

A *charge* in $A \in \mathcal{F}(\mathbb{R}^N)$ is a function $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ such that $F(B \cup C) = F(B) + F(C)$ whenever B and C are nonoverlapping subfigures of A.

Given $x = (x_1,\ldots,x_N) \in \mathbb{R}^N$, we put $|x| := \max\{|x_1|,\ldots,|x_N|\}$. When $E \subset \mathbb{R}^N$, we let $d(E) := \sup\{|x-y| : x,y \in E\}$ and we denote $|E|$ the Lebesgue measure of E. We let $\mathcal{H}^{N-1}$ be the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$ and put $\|A\| := \mathcal{H}^{N-1}(\partial A)$ whenever $A \in \mathcal{F}(\mathbb{R}^N)$. A set is *thin* if it is the union of countably many sets with finite $\mathcal{H}^{N-1}$ measure.

The *regularity* of a figure $A \in \mathcal{F}(\mathbb{R}^N)$, $r(A)$, is defined as follows:

$$r(A) := \begin{cases} \frac{|A|}{\|A\| d(A)} & \text{if } \|A\| d(A) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is related to the usual notion of regularity by the formula $(2Nr(A))^N \le \frac{|A|}{d(A)^N}$. Unless specified otherwise, in the rest of this paper A will be a fixed figure in $\mathbb{R}^N$. A *partition* in A is a collection (possibly empty) $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ where the A_i's are nonoverlapping subfigures of A and $x_i \in A_i$ for $i \in \{1,\ldots,p\}$. A *gage* in A is a function $\delta : A \longrightarrow \mathbb{R}_+$ such that its null set $N_\delta := \{x \in \mathbb{R}_+ : \delta(x) = 0\}$ is thin. A *caliber* is a sequence $(\eta_j)_{j\in\mathbb{N}^*} \subset \mathbb{R}_+ \setminus \{0\}$.

Let $\varepsilon > 0$, $\eta \equiv (\eta_j)_{j\in\mathbb{N}^*}$ a caliber, δ a gage in A and $E \subset A$ an arbitrary set. We say that a partition $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ in A is:

(i) *ε-regular* if $r(A_i) \ge \varepsilon$ for all $i \in \{1,\ldots,p\}$;

(ii) *δ-fine* if $d(A_i) \le \delta(x_i)$ for all $i \in \{1,\ldots,p\}$;

(iii) *(ε,η)-approximating* if $A \ominus \cup_{i=1}^p A_i$ is the union of finitely many nonoverlapping figures $B_1,\ldots,B_k$ with $\|B_j\| \le \frac{1}{\varepsilon}$ and $|B_j| \le \eta_j$ for all $j \in \{1,\ldots,k\}$;

(iv) *E-tagged* if $x_i \in E$ for all $i \in \{1,\ldots,p\}$.

Let $f : A \longrightarrow \mathbb{R}$. We say that f is *$\mathcal{F}$-integrable* in A if there is a real number I (called its *$\mathcal{F}$-integral* and also denoted $\int_A f$) with the following property: given $\varepsilon > 0$, one can find a gage δ in A and a caliber η such that

$$\left| \sum_{i=1}^{p} f(x_i)|A_i| - I \right| \leq \varepsilon$$

for every δ-fine, ε-regular, (ε, η)-approximating partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A. If f is $\mathcal{F}$-integrable in A and $B \in \mathcal{F}(A)$, then f is also $\mathcal{F}$-integrable in B; moreover its *indefinite integral* F, defined by $F(B) := \int_B f$ whenever $B \in \mathcal{F}(A)$, is a charge.

Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be an arbitrary charge and $x \in A$. We define the *lower derivative* (resp. *upper derivative*) of F at x, denoted $F_\star(x)$ (resp. $F^\star(x)$) as follows:

$$\begin{aligned} F_\star(x) &:= \inf_{\eta>0} \sup_{\delta>0} \inf \left\{ \frac{F(X)}{|X|} : X \in \mathcal{F}(A),\, x \in X,\, d(X) \leq \delta,\, r(X) \geq \eta \right\}, \\ F^\star(x) &:= \sup_{\eta>0} \inf_{\delta>0} \sup \left\{ \frac{F(X)}{|X|} : X \in \mathcal{F}(A),\, x \in X,\, d(X) \leq \delta,\, r(X) \geq \eta \right\}. \end{aligned}$$

These are extended real numbers. It is easily seen that $F_\star$ and $F^\star$ are measurable functions and that $F_\star(x) \leq F^\star(x)$ (in $[-\infty, +\infty]$) whenever $x \in \operatorname{int}(A)$. We say that F is *derivable* at x if $F_\star(x) = F^\star(x) \in \mathbb{R}$ and in this case, the common value of the derivatives is denoted $F'(x)$.

A charge $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ is called *continuous* in A if for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|F(B)| \leq \varepsilon$ whenever $B \in \mathcal{F}(A)$ with $|B| \leq \eta$ and $\|B\| \leq \frac{1}{\varepsilon}$.

A charge $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ is called $\mathcal{F}AC_*$ in A if it is continuous in A and, given $E \subset A$ with $|E| = 0$ and $\varepsilon > 0$, there is a gage δ in A such that $\sum_{i=1}^p |F(A_i)| \leq \varepsilon$ for each ε-regular, δ-fine, E-tagged partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

The following result can be found in [6, Theorem 12.3.4]:
Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge. F is the indefinite integral of some $\mathcal{F}$-integrable function if and only if it is $\mathcal{F}AC_*$ in A and derivable almost everywhere in A. In this case, $F(B) = \int_B F'$ for each $B \in \mathcal{F}(A)$.

We point out the fact that this Theorem implies that each $\mathcal{F}$-integrable function is measurable.

2 Generalized Absolute Continuity

We will define a concept of generalized absolute continuity which differs slightly from that of [1] but resembles the one introduced in [2].

Definition 2.1 *Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge and $X \subset A$. We say that F is $\mathcal{F}-AC^{\nabla}(X)$ if it enjoys the following property: given $\varepsilon > 0$, we can find a gage δ in A and a positive real number $\eta > 0$ such that for every δ-fine, ε-regular, X-tagged partitions in A $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$, if*

$$\left|(\cup_{i=1}^{p} A_i) \triangle \left(\cup_{j=1}^{q} B_j\right)\right| \leq \eta$$

then

$$\left|\sum_{i=1}^{p} F(A_i) - \sum_{j=1}^{q} F(B_j)\right| \leq \varepsilon \, .$$

Definition 2.2 *Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge. By saying that F is $\mathcal{F} - ACG^{\nabla}(A)$, we mean that there is a sequence $(X_n)_{n\in\mathbb{N}}$ of measurable subsets of A such that $A = \cup_{n\in\mathbb{N}} X_n$ and F is $\mathcal{F} - AC^{\nabla}(X_n)$ for every $n \in \mathbb{N}$.*

From definition 2.1 we infer that F is $\mathcal{F} - AC^{\nabla}(Y)$ whenever it is $\mathcal{F} - AC^{\nabla}(X)$ and $Y \subset X$. In particular, it is not a restriction to assume that the sets X_n arising from definition 2.2 are pairwise disjoint.

The following lemma is just a reformulation of [2, Theorem 2.9].

Lemma 2.1 *Let $f : A \longrightarrow \mathbb{R}$ be $\mathcal{F}$-integrable in A and F its indefinite integral. Then F is $\mathcal{F} - ACG^{\nabla}(A)$.*

Proof. Since f is measurable, the sets

$$X_n := \{x \in A : |f(x)| \leq n\}$$

are measurable. Fix $n \in \mathbb{N}$: we need to show that F is $\mathcal{F} - AC^{\nabla}(X_n)$. We consider the function $\tilde{f} : A \longrightarrow \mathbb{R}$ defined as follows:

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \in X_n \\ 0 & \text{otherwise.} \end{cases}$$

Being Lebesgue-integrable in A, $\tilde{f}$ is $\mathcal{F}$-integrable in A ([5, Proposition 3.1]). We denote $\tilde{F}$ its indefinite integral. Let $\varepsilon > 0$, using Henstock's lemma ([6, Theorem 12.3.2]) we can find a gage δ in A such that

$$\sum_{i=1}^{p} |f(x_i)|A_i| - F(A_i)| \leq \frac{\varepsilon}{8}$$

and

$$\sum_{i=1}^{p} |\tilde{f}(x_i)|A_i| - \tilde{F}(A_i)| \leq \frac{\varepsilon}{8}$$

for each δ-fine, $\frac{\varepsilon}{8}$-regular partition in A $\{(A_1,x_1),\ldots,(A_p,x_p)\}$. Since $\tilde{F}$ is absolutely continuous with respect to Lebesgue's measure, there is $\eta>0$ such that $|\tilde{F}(E)|\leq\frac{\varepsilon}{2}$ for every measurable set $E\subset A$ with $|E|\leq\eta$. Now assume that $\{(A_1,x_1),\ldots,(A_p,x_p)\}$ and $\{(B_1,y_1),\ldots,(B_q,y_q)\}$ are δ-fine, ε-regular, X_n-tagged partitions in A (being X_n-tagged implies that $f(x_i)=\tilde{f}(x_i)$ and $f(y_j)=\tilde{f}(y_j)$ for all i and j) and that $|(\cup_{i=1}^p A_i)\triangle(\cup_{j=1}^q B_j)|\leq\eta$:

$$\left|\sum_{i=1}^p F(A_i)-\sum_{j=1}^q F(B_j)\right|\leq\sum_{i=1}^p|F(A_i)-f(x_i)|A_i||+\sum_{i=1}^p|\tilde{f}(x_i)|A_i|-\tilde{F}(A_i)|$$
$$+\left|\tilde{F}\left((\cup_{i=1}^p A_i)\triangle\left(\cup_{j=1}^q B_j\right)\right)\right|\sum_{j=1}^q|\tilde{F}(B_j)-\tilde{f}(y_j)|B_j||$$
$$+\sum_{j=1}^q|f(y_j)|B_j|-F(B_j)|\leq\frac{\varepsilon}{8}+\frac{\varepsilon}{8}+\frac{\varepsilon}{2}+\frac{\varepsilon}{8}+\frac{\varepsilon}{8}\leq\varepsilon\,. \qquad \square$$

The aim of the three following results is to prove that a charge F is derivable almost everywhere in X provided it is $\mathcal{F}-AC^{\nabla}(X)$.

Lemma 2.2 *Let $C\subset int(A)$ be compact, $0<\varepsilon\leq\frac{1}{2N}$ and $\gamma:C\longrightarrow\mathbb{R}_+\setminus\{0\}$. There is an ε-regular, γ-fine, C-tagged partition in A $\{(C_1,x_1),\ldots,(C_m,x_m)\}$ such that $C\subset\cup_{i=1}^m C_i$.*

Proof. Let P be any dyadic cube containing C. We define a positive function $\delta:P\longrightarrow\mathbb{R}_+\setminus\{0\}$ as follows:

$$\delta(x):=\begin{cases}\min\left\{\gamma(x),\frac{1}{2}\mathrm{dist}\,(x,\partial A)\right\} & \text{if } x\in C\\ \frac{1}{2}\mathrm{dist}\,(x,C) & \text{if } x\notin C.\end{cases}$$

Let $\{(C_1,x_1),\ldots,(C_l,x_l)\}$ be any δ-fine partition of A such that the C_i's are dyadic cubes. It is easily seen that the family $\{(C_i,x_i):x_i\in C\}$ satisfies the required conditions. $\square$

Proposition 2.1 *Let $F:\mathcal{F}(A)\longrightarrow\mathbb{R}$ be a charge, $E\subset int(A)$ a measurable set and assume that F is $\mathcal{F}-AC^{\nabla}(E)$. Given $0<\varepsilon\leq\frac{1}{2N}$, $\theta>0$, $\alpha\in\mathbb{R}$, $\beta\in\mathbb{R}$ and a gage δ in A, one can find an ε-regular, δ-fine, E-tagged partition in A $\{(C_1,x_1),\ldots,(C_m,x_m)\}$ such that $|\cup_{i=1}^m C_i|\leq|E|+\theta$ and satisfying the two following conditions:*

(1) *if $E\subset\{x\in A:F^{\star}(x)\geq\beta\}$ then $F(\cup_{i=1}^m C_i)+\varepsilon\geq\beta|E|$;*

(2) *if* $E \subset \{x \in A : F_\star(x) \leq \alpha\}$ *then* $F(\cup_{i=1}^m C_i) - \varepsilon \leq \alpha|E|$.

PROOF. First, we use the continuity of the map

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} : (x, y) \longmapsto x \cdot y$$

at the points $(\alpha, |E|)$ and $(\beta, |E|)$ to find some $\eta_1 > 0$ sufficiently small for the following to be true: if $\max\{|x - \alpha|, |y - |E||\} \leq \eta_1$ then

$$|xy - \alpha|E|| \leq \frac{\varepsilon}{2},$$

and if $\max\{|x - \beta|, |y - |E||\} \leq \eta_1$ then

$$|xy - \beta|E|| \leq \frac{\varepsilon}{2}.$$

To simplify the notations, put $A_\beta^\star := \{x \in A : F^\star(x) \geq \beta\}$ and $A_\star^\alpha := \{x \in A : F_\star(x) \leq \alpha\}$.

If $E \subset A_\beta^\star$ then to each $j \in \mathbb{N}^*$ we associate a set

$$E_j^\star := \Big\{x \in E \ : (\forall\delta > 0)(\exists X \in \mathcal{F}(A)) \text{ such that } x \in X,\, d(X) \leq \delta, \\ r(X) \geq \frac{1}{j} \text{ and } F(X) \geq (\beta - \eta_1)|X|\Big\}.$$

It is easily observed that

$$E_j^\star = E \cap \left(\cap_{k\in\mathbb{N}^*} E_{j,k}^\star\right)$$

where we have put

$$E_{j,k}^\star := \bigcup \left\{X \in \mathcal{F}(A) : d(X) \leq \frac{1}{k},\, r(X) \geq \frac{1}{j} \text{ and } F(X) \geq (\beta - \eta_1)|X|\right\}$$

for $k \in \mathbb{N}^*$. Since the $E_{j,k}^\star$'s are measurable (see [7, Ch. IV, Lemma 4.1]), so are also the $E_j^\star$'s. Moreover, the relation $E \subset A_\beta^\star$ implies that $\cup_{j\in\mathbb{N}^*} E_j^\star = E$.

If $E \not\subset A_\beta^\star$, we let $E_j^\star := E$ for all $j \in \mathbb{N}^*$.

If $E \subset A_\star^\alpha$ then to each $j \in \mathbb{N}^*$ we associate a set

$$E_\star^j := \Big\{x \in E : (\forall\delta > 0)(\exists X \in \mathcal{F}(A)) \text{ such that } x \in X,\, d(X) \leq \delta, \\ r(X) \geq \frac{1}{j} \text{ and } F(X) \leq (\alpha + \eta_1)|X|\Big\}.$$

We also infer that the $E_\star^j$'s are measurable and that $\cup_{j\in\mathbb{N}^*} E_\star^j = E$.

If $E \not\subset A_\star^\alpha$, we let $E_\star^j := E$ for all $j \in \mathbb{N}^*$.

Now, put $E_j := E_j^\star \cap E_\star^j$. This defines an increasing sequence of measurable sets such that $\cup_{j\in\mathbb{N}^*} E_j = E$. Hence there is a $j_0 \in \mathbb{N}^*$ with the property

$$|E \setminus E_{j_0}| \leq \frac{\eta_1}{3}\,. \tag{1}$$

Let $\eta_2 := \min\{\frac{\varepsilon}{2}, \frac{1}{j_0}\}$. Since F is $\mathcal{F} - AC^\nabla(E)$, there is a gage δ_0 in A and a $\eta_3 > 0$ such that for every δ_0-fine, η_2-regular, E-tagged partitions in A $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$, the relation

$$\left|\left(\cup_{i=1}^p A_i\right) \triangle \left(\cup_{j=1}^q B_j\right)\right| \leq \eta_3$$

implies

$$\left|\sum_{i=1}^p F(A_i) - \sum_{j=1}^q F(B_j)\right| \leq \eta_2.$$

Since E_{j_0} is measurable, there is an open set V and a closed set C with the following properties:

$$C \subset E_{j_0} \setminus (N_{\delta_0} \cup N_\delta) \text{ and } |E_{j_0} \setminus C| \leq \frac{\eta_3}{4}\,, \tag{2}$$

$$E_{j_0} \subset V \text{ and } |V \setminus E_{j_0}| \leq \min\left\{\theta, \frac{\eta_1}{3}, \frac{\eta_3}{4}\right\}\,, \tag{3}$$

(recall that $|N_{\delta_0} \cup N_\delta| = 0$).

We can now define a function $\gamma : C \longrightarrow \mathbb{R}_+ \setminus \{0\}$ by means of the following formula:

$$\gamma(x) := \min\left\{\delta(x), \delta_0(x), \frac{1}{2}\operatorname{dist}(x, \partial V)\right\}.$$

Apply Lemma 2.2 to C, γ and ε to find a partition in A $\{(C_1, x_1), \ldots, (C_m, x_m)\}$ ε-regular (also η_2-regular since $\eta_2 \leq \frac{\varepsilon}{2}$), γ-fine (also δ- and δ_0-fine), C-tagged (also E-tagged); moreover,

$$C \subset \cup_{i=1}^m C_i \subset V\,. \tag{4}$$

We claim that this partition enjoys all the required properties. First, from (4) and (3) we deduce that

$$|\cup_{i=1}^m C_i| \leq |V| = |E_{j_0}| + |V \setminus E_{j_0}| \leq |E| + \theta\,.$$

We now want to show that $\{(C_1, x_1), \ldots, (C_m, x_m)\}$ satisfies condition (1) of the thesis. We consider the family

$$\mathcal{C} := \Big\{(X, x) : X \in \mathcal{F}(A),\, x \in E^{\star}_{j_0},\, x \in X,\, r(X) \geq \frac{1}{j_0}, \\ d(X) \leq \min\Big\{\delta_0(x),\, \frac{1}{2}\operatorname{dist}(x, \partial V)\Big\},\, F(X) \geq (\beta - \eta_1)|X|\Big\}.$$

Since we assume $E \subset A^{\star}_{\beta}$, it is clear that $\{X : (X, x) \in \mathcal{C}$ for some $x\}$ is a Vitali cover of E_{j_0} in the sense of [7, Ch. IV '3] and consequently there is a countable subfamily $\{(X_i, x_i) : i \in \mathbb{N}^*\} \subset \mathcal{C}$ such that the X_i's are pairwise disjoint and

$$|E_{j_0} \setminus \cup_{i \in \mathbb{N}^*} X_i| = 0. \tag{5}$$

We choose an index $k \in \mathbb{N}^*$ sufficiently large for

$$|(\cup_{i \in \mathbb{N}^*} X_i) \setminus (\cup_{i=1}^{k} X_i)| \leq \min\Big\{\frac{\eta_1}{3}, \frac{\eta_3}{4}\Big\}. \tag{6}$$

We observe that $\{(X_1, x_1), \ldots, (X_k, x_k)\}$ is a δ_0-fine, η_4-regular (because $\eta_4 \leq \frac{1}{j_0}$), E-tagged partition in A and that

$$\cup_{i=1}^{k} X_i \subset \cup_{i \in \mathbb{N}^*} X_i \subset V\,. \tag{7}$$

On the other hand, relations (4), (5), (6) and (3) imply that

$$\begin{aligned} |(\cup_{i=1}^{m} C_i) \setminus (\cup_{i=1}^{k} X_i)| &\leq |V \setminus (\cup_{i=1}^{k} X_i)| \\ &\leq |E_{j_0} \setminus (\cup_{i=1}^{k} X_i)| + |V \setminus E_{j_0}| \\ &\leq |E_{j_0} \setminus (\cup_{i \in \mathbb{N}^*} X_i)| + |(\cup_{i \in \mathbb{N}^*} X_i) \setminus (\cup_{i=1}^{k} X_i)| + |V \setminus E_{j_0}| \leq \frac{\eta_3}{2}, \end{aligned}$$

and relations (7), (4), (2) and (3) imply that

$$|(\cup_{i=1}^{k} X_i) \setminus (\cup_{i=1}^{m} C_i)| \leq |V \setminus C| = |V \setminus E_{j_0}| + |E_{j_0} \setminus C| \leq \frac{\eta_3}{2}.$$

Put these estimations together to obtain

$$|(\cup_{i=1}^{m} C_i) \triangle (\cup_{i=1}^{k} X_i)| \leq \eta_3\,.$$

Hence, from the choice of η_3 and δ_0 we infer that

$$\left|\sum_{i=1}^{k} F(X_i) - \sum_{i=1}^{m} F(C_i)\right| \leq \frac{\varepsilon}{2}\,.$$

Furthermore, from the properties of the X_i's we deduce that

$$F(\cup_{i=1}^m C_i) + \frac{\varepsilon}{2} \geq \sum_{i=1}^k F(X_i) \geq (\beta - \eta_1)| \cup_{i=1}^k X_i| \tag{8}$$

Next, we want to show that

$$||E| - | \cup_{i=1}^k X_i|| \leq \eta_1 \tag{9}$$

First check that

$$|E_{j_0}| - | \cup_{i\in\mathbb{N}^*} X_i| \leq |E_{j_0} \setminus (\cup_{i\in\mathbb{N}^*} X_i)| = 0\,,$$

next, using (7) and (3),

$$| \cup_{i\in\mathbb{N}^*} X_i| - |E_{j_0}| \leq |V| - |E_{j_0}| \leq \frac{\eta_1}{3}$$

so that

$$||E_{j_0}| - | \cup_{i\in\mathbb{N}^*} X_i| \leq \frac{\eta_1}{3}\,. \tag{10}$$

Then we obtain (9) in the following way, using (1),(10) and (6):

$$\begin{aligned} ||E| - | \cup_{i=1}^k X_i|| &\leq ||E| - |E_{j_0}|| + ||E_{j_0}| - | \cup_{i\in\mathbb{N}^*} X_i|| \\ &\quad + || \cup_{i\in\mathbb{N}^*} X_i| - | \cup_{i=1}^k X_i|| \leq \eta_1\,. \end{aligned}$$

From (9) and the choice of η_1 we get

$$(\beta - \eta_1)| \cup_{i=1}^k X_i| \geq \beta|E| - \frac{\varepsilon}{2}$$

which, together with (8), implies that

$$F(\cup_{i=1}^m C_i) + \varepsilon \geq \beta|E|.$$

This completes the proof of the first part of Proposition 2.1.

In order to prove that $\{(C_1, x_1), \ldots, (C_m, x_m)\}$ satisfies condition (2) of the thesis, we should construct another partition $\{(Y_1, y_1), \ldots, (Y_l, y_l)\}$ with, among many other ones, the property that $F(Y_j) \leq (\alpha + \eta_1)|Y_j|$. Since the construction is analogous to the first one, we omit the details. □

Lemma 2.3 *Let $E \subset \mathbb{R}^N$ be a measurable set and $\alpha > 0$. If $|E| > 0$, then there exists a measurable set $\tilde{E} \subset E$ such that $0 < |\tilde{E}| \leq \alpha$.*

Proof. We may assume E is bounded. Given $t \in \mathbb{R}$, define $T_t := \{x \in \mathbb{R}^N : x_1 < t\}$. Observe that the map $\varphi(t) := |E \cap T_t|$ is lipschitz-continuous and $\lim_{t\to\infty} \varphi(t) = |E| > 0$, $\lim_{t\to-\infty} \varphi(t) = 0$. Hence, there is some $t_0 \in \mathbb{R}$ such that $\varphi(t_0) = \min\{\alpha, |E|\}$. The set $\tilde{E} := E \cap T_{t_0}$ satisfies the thesis. $\square$

Proposition 2.2 *Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge, $X \subset A$ a measurable set and assume that F is $\mathcal{F} - AC^{\nabla}(X)$. Then F is derivable at almost every point of X.*

Proof. Let E be the set of points in $X \cap \operatorname{int}(A)$ where F is not derivable. We define

$$\begin{aligned} E_- &:= \{x \in X \cap \operatorname{int}(A) : F_\star(x) = -\infty\} \\ E^+ &:= \{x \in X \cap \operatorname{int}(A) : F^\star(x) = +\infty\} \\ E_{\alpha,\beta} &:= \{x \in X \cap \operatorname{int}(A) : F_\star(x) \le \alpha < \beta \le F^\star(x)\} \end{aligned}$$

so that

$$E = E_- \cup E^+ \cup (\cup_{\alpha,\beta\in\mathbb{Q}} E_{\alpha,\beta}) .$$

First, we want to prove that $|E_-| = 0$. In order to get a contradiction, suppose $|E_-| > 0$. Using the fact that F is $\mathcal{F} - AC^{\nabla}(X)$, choose any gage δ in A and $\eta > 0$ in order that for each $\frac{1}{4N}$-regular, δ-fine partitions in A $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ and $\{(B_1, y_1), \ldots, (B_q, y_q)\}$, if $|(\cup_{i=1}^p A_i) \triangle (\cup_{j=1}^q B_j)| \le \eta$ then $|\sum_{i=1}^p F(A_i) - \sum_{j=1}^q F(B_j)| \le \frac{1}{4N}$. Apply Lemma 2.3 to E_- and $\alpha = \frac{\eta}{2}$ to find a measurable set $\tilde{E}_- \subset E_-$ such that $0 < |\tilde{E}_-| \le \frac{\eta}{2}$. Next apply Proposition 2.1 to F, $\varepsilon = \frac{1}{4N}$, $\theta = |\tilde{E}_-|$, $\alpha = -\frac{1}{N|\tilde{E}|}$ and the gage δ we kept from the fact that F is $\mathcal{F} - AC^{\nabla}(X)$. We get a $\frac{1}{4N}$-regular, δ-fine, X-tagged partition in A $\{(C_1, x_1), \ldots, (C_m, x_m)\}$ such that $|\cup_{i=1}^m C_i| \le 2|\tilde{E}_-| \le \eta$ and hence, from the choice of δ and η,

$$|F(\cup_{i=1}^m C_i)| \le \frac{1}{4N} . \tag{11}$$

On the other hand, from Proposition 2.1 (2), this partition satisfies also

$$F(\cup_{i=1}^m C_i) \le \alpha|\tilde{E}_-| + \varepsilon = -\frac{3}{4N} .$$

This is in contradiction with (11). The proof that $|E^+| = 0$ is very similar and we omit the details.

Finally, fix α, $\beta \in \mathbb{Q}$ such that $\alpha < \beta$. We need to prove that $|E_{\alpha,\beta}| = 0$. Given any $\frac{1}{4N} \ge \varepsilon > 0$, we find through Proposition 2.1 a figure $\cup_{i=1}^m C_i$ in A

such that

$$F(\cup_{i=1}^{m} C_i) - \varepsilon \leq \alpha |E_{\alpha,\beta}| \leq \beta |E_{\alpha,\beta}| \leq F(\cup_{i=1}^{m} C_i) + \varepsilon.$$

Hence,

$$|\alpha - \beta|.|E_{\alpha,\beta}| \leq 2\varepsilon$$

since $\varepsilon > 0$ is arbitrary small and $\alpha \neq \beta$, we get $|E_{\alpha,\beta}| = 0$. □

The following is just a reformulation of [1, Lemma 2.2]. We omit the proof.

Lemma 2.4 *Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a continuous charge. If F is $\mathcal{F} - ACG^{\nabla}(A)$, then F is $\mathcal{F}AC^{\star}$ in A.*

Theorem 2.1 *Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge. The following conditions are equivalent:*

(1) *F is continuous and $\mathcal{F} - ACG^{\nabla}(A)$;*

(2) *F is $\mathcal{F}AC^{\star}$ in A and almost everywhere derivable in A;*

(3) *F is the indefinite integral of some $\mathcal{F}$-integrable function in A.*

PROOF. (1) implies (2) in view of Proposition 2.2 and Lemma 2.4. (2) was already known to be equivalent to (3). Finally, (3) implies (1) from Lemma 2.1. □

Question 2.1 *Here we define generalized absolute continuity as in [1]. Let $F : \mathcal{F}(A) \longrightarrow \mathbb{R}$ be a charge and $X \subset A$. We say that F is $\mathcal{F} - AC^{\star}(X)$ if given $\varepsilon > 0$, one can find a gage δ in A and $\eta > 0$ such that for each ε-regular, δ-fine, X-tagged partition in A $\{(A_1, x_1), \ldots, (A_p, x_P)\}$ satisfying $|\cup_{i=1}^{p} A_i| \leq \eta$ one has*

$$\sum_{i=1}^{p} |F(A_i)| \leq \varepsilon.$$

Moreover, we say that F is $\mathcal{F} - ACG^{\star}(A)$ if there is a sequence $(X_n)_{n\in\mathbb{N}}$ of measurable subsets of A such that $A = \cup_{n\in\mathbb{N}} X_n$ and A is $\mathcal{F} - AC^{\star}(X_n)$.

Obviously, if F is $\mathcal{F} - ACG^{\nabla}(A)$, then it is also $\mathcal{F} - ACG^{\star}(A)$. From [1], it turns out that these two concepts are equivalent in case $N = 1$. What about the case $N \geq 2$?

Remark 2.1 *Extension to BV-integration. In the rest of this paper, the word "integrable" should be understood in the sense defined in* [5]*,* $A \subset \mathbb{R}^N$ *will be a fixed BV-set and a "charge" will mean an additive function* $F : BV_A \longrightarrow \mathbb{R}$ *as defined in* [5]*.*

We may define the derivatives of a charge as in [3]*. For instance,*

$$F_\star(x) := \inf_{\eta>0} \sup_{\delta>0} \inf \left\{ \frac{F(X)}{|X|} : X \in BV_A, x \in cl_e X, d(X) \leq \delta, r(X) \geq \eta \right\} .$$

This gives rise to a good concept of derivability for extending the Theorem 2.1 *(see* [3, Theorem 2.6]*). However, it is not suitable for proving an analogous version of Proposition* 2.1*. Indeed, in the proof of this proposition, we used Vitali's covering theorem, assuming that the sets "X", taken into account in the definition of* $F_\star$ *and* $F^\star$*, are closed.*

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