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UNIVERSALLY BAD DARBOUX FUNCTIONS IN THE CLASS OF ADDITIVE FUNCTIONS

Abstract

The main result: For every family $\mathcal{G}$ of additive functions with $\operatorname{card}\mathcal{G} = 2^\omega$ if the covering of the family of all level sets of functions from $\mathcal{G}$ is equal to 2^ω, then there exists an additive Darboux function f such that $f + g$ is Darboux for no $g \in \mathcal{G}$.

Definitions. Let us establish some terminology to be used. For a subset A of $\mathbb{R} \times \mathbb{R}$ we denote by $\operatorname{dom}(A)$ and $\operatorname{rng}(A)$ the x–projection and y–projection of A. We say that $f : \mathbb{R} \to \mathbb{R}$ is a Darboux function whenever $f(J)$ is connected for every interval $J \subset \mathbb{R}$. The family of all such functions we will denote by $\mathcal{D}$.

We shall consider $\mathbb{R}$ as a linear space over $\mathbb{Q}$, the set of rationals. Every base of this space will be referred to as a *Hamel basis*. It is evident that the cardinality of every Hamel basis is equal to 2^ω.

If $A \subset \mathbb{R}$ is an arbitrary nonempty set, then by $L(A)$ we mean the linear subspace of $\mathbb{R}$ spanned over A, i.e., the set of all finite linear combinations of elements of A (with coefficients from $\mathbb{Q}$). Analogously, for an arbitrary nonempty planar set $A \subset \mathbb{R} \times \mathbb{R}$ we put the set $L_2(A)$. For any $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ we define $x + A = \{x + a : a \in A\}$.

Let L be a linear subspace of $\mathbb{R}$ over $\mathbb{Q}$. A function $f : L \to \mathbb{R}$ is called *additive* iff it satisfies Cauchy's equation $f(x+y) = f(x) + f(y)$ for all $x, y \in L$ [2]. (See also [5, p. 120], for the history of this notion.) Recall that every additive function $f : \mathbb{R} \to \mathbb{R}$ can be obtained as the unique additive extension of a function defined on a Hamel basis. The class of all additive functions from $\mathbb{R}$ to $\mathbb{R}$ will be denoted by $\mathcal{A}dd$.

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We do not distinguish between a function $f\colon L \to \mathbb{R}$ where $L \subset \mathbb{R}$ and its graph (i.e., a subset of $\mathbb{R} \times \mathbb{R}$). By $f+g$ we mean the function $h\colon \operatorname{dom}(f) \cap \operatorname{dom}(g) \to \mathbb{R}$ such that $h(x) = f(x) + g(x)$ for each $x \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$.

The cardinality of a set A is denote by $\operatorname{card}(A)$. Cardinals are identified with initial ordinals. For an arbitrary cover $\mathcal{B}$ of the real line we define the *covering* of $\mathcal{B}$ as the smallest cardinal κ for which there exists a subfamily $\mathcal{B}_0 \subset \mathcal{B}$ with $\operatorname{card}(\mathcal{B}_0) = \kappa$ and $\mathbb{R} = \bigcup \mathcal{B}_0$.

For a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ we denote by $\mathcal{M}_a(\mathcal{F})$ the *maximal additive family for* $\mathcal{F}$, i.e.,

$$\mathcal{M}_a(\mathcal{F}) = \{g \in \mathbb{R}^{\mathbb{R}}\colon\ f+g \in \mathcal{F} \ \text{for each} \ f \in \mathcal{F}\}.$$

Recall that $\mathcal{M}_a(\mathcal{D})$ is equal to the family of all constant functions [6].

For an infinite cardinal κ we say that a cardinal number λ is the *cofinality* of κ (and write $\lambda = \operatorname{cf}(\kappa)$) if λ is the least cardinal number such that there exists a family of sets $(X_i)_{i\in\lambda}$ with the property that $\bigcup_{i\in\lambda} X_i = \kappa$ and $\operatorname{card}(X_i) < \kappa$ for every $i \in \lambda$. For cardinals κ we say that κ is a regular cardinal if $\kappa = \operatorname{cf}(\kappa)$.

Given a family $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ consider the condition

$c(\mathcal{F})$: there is a $f \in \mathcal{D}$ such that $f + g \notin \mathcal{D}$ for each $g \in \mathcal{F}$.

Such a function f is called *a universally bad Darboux function* for $\mathcal{F}$. Determining for which families $\mathcal{F}$ the condition $c(\mathcal{F})$ is fulfilled is a problem considered by several authors (see, e.g., [6], [8], [1], [3] and [4]). In particular, if the additivity of the ideal of all first category subsets of $\mathbb{R}$ is equal to 2^ω (e.g., if Martin's Axiom or CH hold), then $c(C^*)$ holds for the family C^* of all nowhere constant, continuous functions [3]. On the other hand, there is a model of set theory in which $c(C^*)$ fails to hold. ([7]) In this paper we study analogous problems for the class of additive functions.

Lemma 1 *Let $f \in \mathcal{A}dd$ be such that* $\ker(f) \neq \{0\}$ *and* $\operatorname{rng}(f) = \mathbb{R}$. *Then f has the Darboux property.*

Proof. Observe that $f \in \mathcal{A}dd$, $\ker(f)$ is a linear subspace of $\mathbb{R}$ and, since $\ker(f) \neq \{0\}$, $\ker(f)$ is dense in $\mathbb{R}$. Moreover, it is well-known that each two level sets of an additive function are congruent under translations, so any level set of f is dense in $\mathbb{R}$. Hence $f(I) = \mathbb{R}$ for every interval $I \subset \mathbb{R}$ and $f \in \mathcal{D}$. □

Lemma 2 *Let $\mathcal{B}$ be a cover of $\mathbb{R}$ such that* $\operatorname{card}(\mathcal{B}) = 2^\omega$ *and the covering of $\mathcal{B}$ is equal to 2^ω. There exists a Hamel basis H such that $H \setminus \bigcup \mathcal{B}^* \neq \emptyset$ for every $\mathcal{B}^* \subset \mathcal{B}$ with* $\operatorname{card}(\mathcal{B}^*) < \operatorname{cf}(2^\omega)$.

PROOF. Let $\mathcal{B} = \{B_\alpha : \alpha < 2^\omega\}$ and $h_0 \neq 0$. Fix $\alpha < 2^\omega$ and assume that we have chosen a linearly independent set $\{h_\beta : \beta < \alpha\}$ such that $h_\beta \notin \bigcup_{\gamma<\beta} B_\gamma$ for each $\beta < \alpha$. Let $E_\alpha = L(\{h_\beta : \beta < \alpha\})$. For each $x \in E_\alpha$ choose $C_x \in \mathcal{B}$ with $x \in C_x$. Since $\operatorname{card}(E_\alpha) < 2^\omega$, by our assumption we obtain that the family $\mathcal{B}_\alpha = \{C_x : x \in E_\alpha\} \cup \{B_\beta : \beta < \alpha\}$ does not cover $\mathbb{R}$. Choose $h_\alpha \in \mathbb{R} \setminus \bigcup \mathcal{B}_\alpha$. Then the set $\{h_\alpha : \alpha < 2^\omega\}$ is linearly independent. Let H be a Hamel basis containing $\{h_\alpha : \alpha < 2^\omega\}$. For every $\mathcal{B}^* \subset \mathcal{B}$ with $\operatorname{card}(\mathcal{B}^*) < \operatorname{cf}(2^\omega)$ there is $\alpha < 2^\omega$ such that $\mathcal{B}^* \subset \{B_\beta : \beta < \alpha\}$, so $h_\alpha \in H \setminus \mathcal{B}^*$. □

If we assume that 2^ω is a regular cardinal, then for each cover $\mathcal{B}$ of $\mathbb{R}$ such that the covering of $\mathcal{B}$ equals 2^ω there exists a Hamel basis H such that $H \setminus \bigcup \mathcal{B}^* \neq \emptyset$ for every $\mathcal{B}^* \subset \mathcal{B}$ with $\operatorname{card}(\mathcal{B}^*) < 2^\omega$. We are unable to determine whether this statement can be proved in ZFC.

Theorem 1 *Assume that 2^ω is a regular cardinal and $\mathcal{G} = \{g_\alpha : \alpha < 2^\omega\} \subset \mathcal{A}dd$ satisfies the condition*

($\star$) *the covering of the family $\mathcal{B} = \{g^{-1}(y) : g \in \mathcal{G}\ \&\ y \in \mathbb{R}\}$ is equal to 2^ω.*

Then there is $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $f + g_\alpha \notin \mathcal{D}$ for each $\alpha < 2^\omega$.

PROOF. Let $H = \{h_\alpha : \alpha < 2^\omega\}$ be the Hamel basis constructed in Lemma 2 for the cover $\mathcal{B}$. For every $\alpha < 2^\omega$ we will construct an additive function f_α and choose $u_\alpha \in H$ such that

(i) $f_\beta \subset f_\alpha$ for $\beta < \alpha$,

(ii) $h_\alpha \in \operatorname{dom}(f_\alpha) \cap \operatorname{rng}(f_\alpha)$ and $\ker(f_\alpha) \neq \{0\}$,

(iii) $\operatorname{rng}(f_\alpha + g_\alpha) \neq \{0\}$,

(iv) $u_\beta \notin \operatorname{rng}(f_\alpha + g_\beta)$ for each $\beta \leq \alpha$,

(v) $\operatorname{card}(\operatorname{dom}(f_\alpha)) = \max(\omega, \operatorname{card}(\alpha))$.

First assume that $\alpha = 0$. Choose $h \in H$ such that $h \neq -g_0(h_2)$ and set $f_0 = L_2(\{(h_0, 0), (h_1, h_0), (h_2, h)\})$. Since $\operatorname{card}(\operatorname{rng}(f_0 + g_0)) = \omega$, we can choose $u_0 \in H \setminus \operatorname{rng}(f_0 + g_0)$. It is easy to verify that f_0 and u_0 fulfill conditions (i)–(v).

Now fix $\alpha < 2^\omega$ and assume that we have chosen for $\beta < \alpha$ functions f_β and points u_β which satisfy conditions (i)–(v). Set $f_\alpha^{(0)} = \bigcup_{\beta<\alpha} f_\beta$. We consider two cases. If $h_\alpha \in \operatorname{rng}(f_\alpha^{(0)})$, then define $f_\alpha^{(1)} = f_\alpha^{(0)}$. Otherwise we will choose $y_\alpha \in H$ such that $y_\alpha \notin \operatorname{dom}(f_\alpha^{(0)})$ and

$$u_\beta \notin L\left(\operatorname{rng}\left(f_\alpha^{(0)} + g_\beta\right) \cup \{g_\beta(y_\alpha) + h_\alpha\}\right) \text{ for } \beta < \alpha. \tag{1}$$

To choose y_α observe that the family

$$\mathcal{B}_\alpha = \bigcup_{\beta<\alpha} \left\{ g_\beta^{-1}(y)\colon\ y \in L\left(\operatorname{rng}\left(f_\alpha^{(0)} + g_\beta\right) \cup \{u_\beta\}\right) - h_\alpha \right\} \cup \left\{ g_\alpha^{-1}(h_\alpha) \right\}$$

has cardinality less than 2^ω. For each $x \in \operatorname{dom}(f_\alpha^{(0)})$ choose $C_x \in \mathcal{B}$ with $x \in C_x$. Since $\operatorname{card}(\operatorname{dom}(f_\alpha^{(0)})) < 2^\omega$, $H \setminus \left(\bigcup \mathcal{B}_\alpha \cup \bigcup_{x\in\operatorname{dom}(f_\alpha^{(0)})} C_x\right) \neq \emptyset$. Take an arbitrary

$$y_\alpha \in H \setminus \left(\bigcup \mathcal{B}_\alpha \cup \bigcup_{x\in\operatorname{dom}(f_\alpha^{(0)})} C_x\right) \tag{2}$$

To prove (1) fix $\beta < \alpha$ and suppose that $u_\beta = q_1 y + q_2(g_\beta(y_\alpha) + h_\alpha)$, where $q_1, q_2 \in \mathbb{Q}$ and $y \in \operatorname{rng}(f_\alpha^{(0)} + g_\beta)$. By (2) we obtain $q_2 = 0$. So $u_\beta \in \operatorname{rng}(f_\tau + g_\beta)$, where $\tau = \min\{\gamma\colon\ u_\beta \in \operatorname{rng}(f_\gamma + g_\beta)\}$. By (iv), $\tau < \beta < \alpha$. But then $u_\beta \in \operatorname{rng}(f_\tau + g_\beta) \subset \operatorname{rng}(f_\beta + g_\beta)$, contrary to (iv).

Now set

$$f_\alpha^{(1)} = L_2(f_\alpha^{(0)} \cup \{(y_\alpha, h_\alpha)\}).$$

If $h_\alpha \in \operatorname{dom}(f_\alpha^{(1)})$, then put $f_\alpha = f_\alpha^{(1)}$. Otherwise choose

$$v_\alpha \in H \setminus \bigcup_{\beta<\alpha} \left(L\left(\operatorname{rng}\left(f_\alpha^{(1)} + g_\beta\right) \cup \{u_\beta\}\right) - g_\beta(h_\alpha)\right)$$

and put

$$f_\alpha = L_2(f_\alpha^{(1)} \cup \{(h_\alpha, v_\alpha)\}).$$

Finally, since $\operatorname{card}(\operatorname{dom}(f_\alpha)) < 2^\omega$, we can choose $u_\alpha \in H \setminus \operatorname{rng}(f_\alpha + g_\alpha)$. It can be easily seen that f_α fulfills conditions (i)–(v).

Define f by

$$f = \bigcup_{\alpha<2^\omega} f_\alpha.$$

Because $H \subset \operatorname{dom}(f)$, $f \in \mathcal{A}dd$. Since $\ker(f) \neq \{0\}$ and $H \subset \operatorname{rng}(f)$, by Lemma 1, $f \in \mathcal{D}$. Notice that $f + g_\beta \notin \mathcal{D}$ for each $\beta < 2^\omega$. Indeed, fix an arbitrary $\beta < 2^\omega$. Then by (iv) $u_\beta \notin \operatorname{rng}(f + g_\beta)$. But, by (iii), $\operatorname{rng}(f + g_\beta)$ is dense in $\mathbb{R}$, which shows that $f + g_\beta \notin \mathcal{D}$. □

Remark. Since all level sets of an additive function are congruent under translations, the condition ($\star$) is equivalent to the following:

($\star\star$) the covering of the family $\mathcal{B} = \{\ker(g) + y\colon g \in \mathcal{G}\ \&\ y \in \mathbb{R}\}$ is equal to 2^ω.

Corollary 1 *Assume that 2^ω is a regular cardinal, $\mathcal{G} \subset \mathcal{A}dd$,* $\operatorname{card}(\mathcal{G}) = 2^\omega$ *and there exists an ideal $\mathcal{J} \supset \{\ker(g)\colon g \in \mathcal{G}\}$ satisfies the following conditions:*

(i) $\mathcal{J}$ is invariant under translations, i.e., $A + x \in \mathcal{J}$ for all $A \in \mathcal{J}$ and $x \in \mathbb{R}$;

(ii) the covering of $\mathcal{J}$ is equal to 2^ω.

Then there is $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $f + g \notin \mathcal{D}$ for each $g \in \mathcal{G}$.

If Martin's Axiom (MA) or the Continuum Hypothesis (CH) hold, then the ideals $\mathcal{K}$ of all meager sets and $\mathcal{N}$ of all null sets fulfill the statements (i) and (ii). Therefore we have the following corollary.

Corollary 2 (MA) *Assume that $\mathcal{G}$ is a family of additive functions such that* $\operatorname{card}(\mathcal{G}) = 2^\omega$ *and either* $\ker(g) \in \mathcal{N}$ *for each $g \in \mathcal{G}$ or* $\ker(g) \in \mathcal{K}$ *for each $g \in \mathcal{G}$. Then there is $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $f + g \notin \mathcal{D}$ for each $g \in \mathcal{G}$.*

Proposition 1 *The covering of the family $\mathcal{S}(\mathbb{R})$ of all proper linear subspaces of $\mathbb{R}$ over $\mathbb{Q}$ is equal to ω.*

Proof. Let $\mathcal{B}_0 \subset \mathcal{S}(\mathbb{R})$ be such that $\bigcup \mathcal{B}_0 = \mathbb{R}$. We will show that $\operatorname{card}(\mathcal{B}_0) \geq \omega$. By way of contradiction suppose that $\mathcal{B}_0 = \{V_1, \dots, V_n\}$ for some $n \in \mathbb{N}$. We may assume that $V_i \setminus \bigcup_{k \neq i} V_k \neq \emptyset$ for every $i \leq n$. Note that $n \geq 2$, because all V_i are proper. For $i = 1, 2$ choose

$$v_i \in V_i \setminus \bigcup_{k \neq i} V_k \tag{3}$$

and set $v_k = (k-2)v_2 + v_1$ for $k > 2$ Then there exists $i \leq n$ for which the set $N_i = \{k\colon\ v_k \in V_i\}$ is infinite. Fix $j, k \in N_i$ with $2 < j < k$. Then we have

$$v_k - v_j = (k-j)v_2 \in V_2 \cap V_i.$$

Therefore $v_2 \in V_i$ and, by (3), $i = 2$. But then $v_1 = v_k - (k-2)v_2 \in V_2$, contrary to the choice of v_1.

Now we will construct a family $\mathcal{B}_0 \subset \mathcal{S}(\mathbb{R})$ such that $\operatorname{card}(\mathcal{B}_0) = \omega$ and $\bigcup \mathcal{B}_0 = \mathbb{R}$. Let $H \subset \mathbb{R}$ be an arbitrary Hamel basis and let $\{H_n\colon n \in \mathbb{N}\}$ be a partition of H into proper subsets. Put

$$\mathcal{B}_0 = \left\{ L\left(\bigcup_{n \in A} H_n \right)\colon\ A \subset \mathbb{N},\ \operatorname{card}(A) < \omega \right\}$$

It is obvious that $\operatorname{card}(\mathcal{B}_0) = \omega$ and $\mathbb{R} \notin \mathcal{B}_0$. Fix an arbitrary $x \in \mathbb{R}$. Then $x = \sum_{n=1}^{k} q_n h_n$ for some $k \in \mathbb{N}$, $q_n \in \mathbb{Q}$, and $h_n \in H$ $n = 1, \dots k$. Since $A_x = \{j\colon \exists_{n \leq k}\, h_n \in H_j\}$ is finite, $x \in L(\bigcup_{n \in A_x} H_n) \in \mathcal{B}_0$. Consequently, $\bigcup \mathcal{B}_0 = \mathbb{R}$. □

Proposition 2 *If a family $\mathcal{G} \subset \mathcal{A}dd$ satisfies the condition $(\star)$, then* $\operatorname{card}(\operatorname{rng}(g)) = 2^\omega$ *for each $g \in \mathcal{G}$.*

Proof. If $\operatorname{card}(\operatorname{rng}(g)) < 2^\omega$ for some $g \in \mathcal{G}$, then $\bigcup_{y \in \operatorname{rng}(g)} g^{-1}(y) = \mathbb{R}$. So the covering of $\mathcal{G}$ is less than 2^ω. □

Lemma 3 *Assume that $n \in \mathbb{N}$ and $\mathcal{G} = \{g_i : i \leq n\}$ is a family of additive functions such that* $\operatorname{card}(\operatorname{rng}(g_i)) = 2^\omega$ *for $i \leq n$. Then $\mathcal{G}$ satisfies condition $(\star)$.*

Proof. Let $\mathcal{B}_0$ be a subfamily of $\{g^{-1}(y) : g \in \mathcal{G}$ and $y \in \mathbb{R}\}$ such that $\operatorname{card}(\mathcal{B}_0) < 2^\omega$. Define $Y_i = L(\{y \in \mathbb{R} : g_i^{-1}(y) \in \mathcal{B}_0\})$ and $V_i = g_i^{-1}(Y_i)$. Note that $\operatorname{card}(Y_i) < 2^\omega$ for $i = 1, \ldots, n$. Because $\operatorname{card}(\operatorname{rng}(g_i)) = 2^\omega$, $V_i \neq \mathbb{R}$ for every $i = 1, \ldots, n$. So by Proposition 1, $V = \bigcup_{i=1}^n V_i \neq \mathbb{R}$. But $\bigcup \mathcal{B}_0 \subset V$. Thus $\bigcup \mathcal{B}_0 \neq \mathbb{R}$, which completes the proof. □

Lemma 4 *Assume that $n \in \mathbb{N}$, $\mathcal{G} = \{g_i : i \leq n\} \subset \mathcal{A}dd$ and* $\operatorname{card}(\operatorname{rng}(g)) = 2^\omega$ *for each $g \in \mathcal{G}$. Then there exists a linearly independent set $H_1 \subset \mathbb{R}$ such that* $\operatorname{card}(H_1) = 2^\omega$ *and $g_i | L(H_1)$ is an injection for each $i \leq n$.*

Proof. Choose an arbitrary $h_0 \neq 0$. Fix $\alpha < 2^\omega$ and assume that we have chosen a linearly independent set $\{h_\beta : \beta < \alpha\}$ such that $g_i | L(\{h_\beta : \beta < \alpha\})$ is an injection for each $i \leq n$. For each i we have $\operatorname{card}(L(\{g_i(h_\beta) : \beta < \alpha\})) < 2^\omega$ and $\operatorname{card}(\operatorname{rng}(g_i)) = 2^\omega$. So $g_i^{-1}(L(\{g_i(h_\beta) : \beta < \alpha\}))$ is a proper linear subspace of $\mathbb{R}$. By Theorem 1, we obtain

$$\mathbb{R} \setminus \bigcup_{i=1}^n g_i^{-1}(L(\{g_i(h_\beta) : \beta < \alpha\})) \neq \emptyset.$$

Choose $h_\alpha \in \mathbb{R} \setminus \bigcup_{i=1}^n g_i^{-1}(L(\{g_i(h_\beta) : \beta < \alpha\}))$. Then the set $H_1 = \{h_\alpha : \alpha < 2^\omega\}$ is linearly independent and $g_i | L(H_1)$ is an injection for $i = 1, \ldots, n$. □

Assuming $\operatorname{cf}(2^\omega) = 2^\omega$, the next theorem is a consequence of Theorem 1 and Lemma 3. We shall prove it in ZFC, without additional set-theoretical assumptions.

Theorem 2 *Assume that $\mathcal{G} = \{g_i : i = 1, \ldots, n\} \subset \mathcal{A}dd$ and* $\operatorname{card}(\operatorname{rng}(g)) = 2^\omega$ *for each $g \in \mathcal{G}$. Then there is $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $f + g \notin \mathcal{D}$ for $g \in \mathcal{G}$.*

Proof. Let $H_1 \subset \mathbb{R}$ be the set constructed in Lemma 4 for the family $\mathcal{G}$ and let $H = \{h_\alpha : \alpha < 2^\omega\} \supset H_1$ be a Hamel basis. Choose

$$h \in H \setminus \{g_i(h_2) : i = 1, \leq n\} \tag{4}$$

and set $f_0 = L_2(\{(h_0, 0), (h_1, h_0), (h_2, h)\})$. Clearly $\operatorname{card}(\operatorname{rng}(f_0 + g_i)) = \omega$ for $i = 1, \ldots, n$.

For $i = 1, \ldots, n$ choose $u_i \in H \setminus \operatorname{rng}(f_0 + g_i)$. For every $0 < \alpha < 2^\omega$ we will construct an additive function f_α such that

(i) $f_0 \subset f_\beta \subset f_\alpha$ for $\beta < \alpha$,

(ii) $h_\alpha \in \operatorname{dom}(f_\alpha) \cap \operatorname{rng}(f_\alpha)$,

(iii) $u_i \notin \operatorname{rng}(f_\alpha + g_i)$ for each $i = 1, 2, \ldots, n$,

(iv) $\operatorname{card}(\operatorname{dom}(f_\alpha)) = \max(\omega, \alpha)$.

Fix $\alpha < 2^\omega$ and assume that for $\beta < \alpha$ we have chosen the function f_β which satisfies conditions (i)–(iv). Set

$$f_\alpha^{(0)} = \bigcup_{\beta<\alpha} f_\beta. \tag{5}$$

We consider two cases. If $h_\alpha \in \operatorname{rng}(f_\alpha^{(0)})$, then define $f_\alpha^{(1)} = f_\alpha^{(0)}$. Otherwise we will choose $y_\alpha \in H$ such that $y_\alpha \notin \operatorname{dom}(f_\alpha^{(0)})$ and

$$u_i \notin L\left(\operatorname{rng}\left(f_\alpha^{(0)} + g_i\right) \cup \{g_i(y_\alpha) + h_\alpha\}\right) \text{ for } i \leq n. \tag{6}$$

To choose y_α, observe that $g_i|H_1$ is an injection, and by (iv),

$$\operatorname{card}\left(L\Big(\operatorname{rng}(f_\alpha^{(0)} + g_i)\{u_i\}\Big) - h_\alpha\right) < 2^\omega.$$

So the cardinality of the set

$$A_{\alpha,i} = (g_i|H_1)^{-1}\left(L\Big(\operatorname{rng}(f_\alpha^{(0)} + g_i)\{u_i\}\Big) - h_\alpha\right)$$

is less than 2^ω. Therefore, the cardinality of $A_\alpha = H_1 \setminus \bigcup_{i=1}^n A_{\alpha,i}$ is equal to 2^ω. Take an arbitrary $y_\alpha \in A_\alpha$. Now the proof of (6) is analogous to the proof of condition (1) in Theorem 1.

Next let $f_\alpha^{(1)} = L_2(f_\alpha^{(0)} \cup \{(y_\alpha, h_\alpha)\})$. If $h_\alpha \in \operatorname{dom}(f_\alpha^{(1)})$, then put $f_\alpha = f_\alpha^{(1)}$. Otherwise choose

$$v_\alpha \in H \setminus \bigcup_{i=1}^{n} \left[L\left(\operatorname{rng}\left(f_\alpha^{(1)} + g_i\right) \cup \{u_i\}\right) - g_i(h_\alpha)\right]$$

and let $f_\alpha = L_2(f_\alpha^{(1)} \cup \{(h_\alpha, v_\alpha)\})$. It can be seen that f_α fulfills conditions (i)–(iv).

Finally, let $f = \bigcup_{\alpha<2^\omega} f_\alpha$. Because $H \subset \mathrm{dom}\,(f)$, $f \in \mathcal{A}dd$. Since $\ker(f) \neq \{0\}$ and $H \subset \mathrm{rng}\,(f)$, by Lemma 1, $f \in \mathcal{D}$.

Notice that $f + g_i \notin \mathcal{D}$ for each $i = 1, \ldots, n$. Indeed, fix arbitrary $i \leq n$. Then by (iii) $u_i \notin \mathrm{rng}\,(f+g_i)$. But conditions (4) and (i) imply that $\mathrm{rng}\,(f+g_i)$ is dense in $\mathbb{R}$. Thus $f + g_i \notin \mathcal{D}$. □

Corollary 3 $\mathcal{M}_a(\mathcal{A}dd \cap \mathcal{D}) = \{0\}$.

Proof. The inclusion "$\supset$" is obvious. To prove the inclusion "$\subset$" assume that $f \in \mathcal{M}_a(\mathcal{A}dd \cap \mathcal{D}) \setminus \{0\}$. Note that $f \in \mathcal{A}dd \cap \mathcal{D}$, because the constant function $g \equiv 0$ belongs to the class $\mathcal{A}dd \cap \mathcal{D}$. So, $\mathrm{rng}\,(f) = \mathbb{R}$. By Theorem 2, $f + h \notin \mathcal{A}dd \cap \mathcal{D}$ for some $h \in \mathcal{A}dd \cap \mathcal{D}$. Hence $f \notin \mathcal{M}_a(\mathcal{A}dd \cap \mathcal{D})$, an impossibility. □

The importance of the assumptions in Theorem 1 is not clear. In particular, the following problem is open.

Problem 1 *Assume CH and $\mathcal{G} = \{g_\alpha \colon \alpha < 2^\omega\}$ is a family of additive functions. Does there exists $f \in \mathcal{A}dd \cap \mathcal{D}$ such that $f + g_\alpha \notin \mathcal{D}$ for each α?*

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