RESEARCH

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ON CHARACTERIZING EXTENDABLE CONNECTIVITY FUNCTIONS BY ASSOCIATED SETS

Abstract

We answer two questions in [11]. We show that the class of extendable connectivity functions from I into I ($I = [0,1]$) cannot be characterized in terms of associated sets, and we show that one of Jones' functions obeying $f(x+y) = f(x) + f(y)$ is an example of an almost continuous function from $\mathbb{R}$ into $\mathbb{R}$ which is not the uniform limit of any sequence of extendable connectivity functions.

A class K of real-valued functions defined on an interval is *characterized by associated sets* if there exists a family P of subsets of $\mathbb{R}$ such that $f \in K$ if and only if for every $\alpha \in \mathbb{R}$, the "associated" sets $E^{\alpha}(f) = \{x : f(x) < \alpha\}$ and $E_{\alpha}(f) = \{x : f(x) > \alpha\}$ belong to P. For example, the family P of open sets characterizes the class K of continuous functions.

A *Darboux* function $f : I \to I$ has the intermediate value property. A function $f : I \to I$ has the *Weak Cantor Intermediate Value Property* ($WCIVP$) if for each subinterval (x, y) of I with $f(x) \neq f(y)$, there exists a Cantor set C in (x, y) such that $f(C)$ lies between $f(x)$ and $f(y)$. A *connectivity* function $f : I \to I$ or $F : I^2 \to I$ obeys the property that the graph of its restriction to each connected subset of its domain is a connected set. A connectivity function $f : I \to I$ is *extendable* if there exists a connectivity function $F : I^2 \to I$ such that $F(x, 0) = f(x)$ for all $x \in I$. Each neighborhood of the graph of an *almost continuous* function $f : I \to I$ in $I \times I$ contains the graph of a continuous function from I into I. Similar definitions hold for $\mathbb{R}$ replacing I.

Of the above classes of functions, these have been shown by the following to be not characterizable by associated sets:

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Darboux functions	Bruckner [1]
connectivity functions	Cristian and Tevy [3]
almost continuous functions	Kellum [9]

We continue this pattern for extendable connectivity functions.

Theorem 1 *The class K of extendable connectivity functions from I into I cannot be characterized by associated sets.*

PROOF. Assume K is characterized by a family P of associated sets. It follows from [12] or [2] that there exists an extendable connectivity function $f : I \to I$ whose graph is dense in I^2 and $f(I) = (0,1)$. By [11], there exists a dense G_δ-subset A of $(0,1)$ that is f-negligible. This means that every function from I into I obtained by arbitrarily redefining f on A is still an extendable connectivity function. Therefore the function $g : I \to I$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in I \setminus A \\ 0 & \text{if } x \in A \end{cases}$$

belongs to K, and so $E_0(g) = \{x \in I : g(x) > 0\} = I \setminus A \in P$. As in [12], we show that $I \setminus A$ is negligible for some extendable connectivity function. Since $I \setminus A$ is of the first category, it follows from Lemma 3 in [10] that there exists a homeomorphism $h : I \to I$ such that $(I \setminus A) \cap h(I \setminus A) = \emptyset$. $I \setminus A \subset h^{-1}(A)$ and $f \circ h : I \to (0,1)$. According to Corollary 1 and Lemma 2 in [10], $f \circ h$ is an extendable connectivity function, and $h^{-1}(A)$ is $(f \circ h)$-negligible. Therefore $I \setminus A$ is $(f \circ h)$-negligible. Then the function $\phi : I \to I$ defined by

$$\phi(x) = \begin{cases} (f \circ h)(x) & \text{if } x \in A \\ 0 & \text{if } x \in I \setminus A \end{cases}$$

belongs to K, and so $E_0(\phi) = \{x \in I : \phi(x) > 0\} = A \in P$. Define the function $\psi : I \to I$ by

$$\psi(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in I \setminus A. \end{cases}$$

Then for every $\alpha \in \mathbb{R}$, $E^\alpha(\psi)$, $E_\alpha(\psi) \in P$, yet $\psi \notin K$, a contradiction. □

Theorem 2 *The class K of uniform limits of sequences of extendable connectivity functions $f_n : \mathbb{R} \to \mathbb{R}$ is not characterized by associated sets.*

Proof. Assume K is characterizable in terms of a family P of associated sets. By [12] or [2], there exists an extendable connectivity function $g : \mathbb{R} \to \mathbb{R}$ whose graph is dense in $\mathbb{R}^2$. According to Theorem 1 in [11] which still holds there for $\mathbb{R}$ replacing I, there exists a dense G_δ-subset A of $\mathbb{R}$ that is g-negligible. We may suppose $A \cap g^{-1}(1/2) = \emptyset$. Therefore $g^{-1}(1/2) = \cup_{i=1}^{\infty} C_i$, where each C_i is nowhere dense in $\mathbb{R}$ and if $i \neq j$, then $C_i \cap C_j = \emptyset$. Like Example 1 in [11], for each positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} g(x) & \text{if } x \in \mathbb{R} \setminus \cup_{i=1}^{n} C_i \\ \frac{i+2}{2i} & \text{if } x \in C_i \text{ and } i = 1, 2, \ldots, n \end{cases}$$

Then the sequence of extendable connectivity functions f_n converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$ with range $\mathbb{R} \setminus \{1/2\}$. Then $f \in K$ and so $E^{1/2}(f)$, $E_{1/2}(f) \in P$. Define a function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 1 & \text{if } x \in E^{1/2}(f) \\ 0 & \text{if } x \in E_{1/2}(f). \end{cases}$$

Clearly, h cannot be a uniform limit of a sequence of Darboux functions, and so $h \notin K$. Yet for every $\alpha \in \mathbb{R}$, $E^{\alpha}(h)$, $E_{\alpha}(h) \in P$, and so $h \in K$, a contradiction. □

In [5], Gibson and Roush gave an example of a Darboux function $f : I \to \mathbb{R}$ which is not the uniform limit of a sequence of connectivity functions, and in [6], Jastrzebski gave an example of a connectivity function $f : I \to \mathbb{R}$ which is not the uniform limit of a sequence of almost continuous functions. We continue this trend with an example of an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not the uniform limit of any sequence of extendable connectivity functions. We prove two preliminary results.

Theorem 3 *There exists a function $f : \mathbb{R} \to \mathbb{R}$ obeying $f(x+y) = f(x)+f(y)$ that is almost continuous but does not have the $WCIVP$.*

Proof. Jones constructed a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ and such that its graph intersects every perfect subset P of $\mathbb{R}^2$ with x-projection $\pi_1(P)$ having c-many points [7]. Kellum showed it was almost continuous [8].

Suppose $x < y$, $f(x) \neq f(y)$, and C is an arbitrary Cantor set between x and y. By construction, f meets the perfect subset $C \times \{f(x)\}$ of $\mathbb{R}^2$, and so $f(C)$ does not lie between $f(x)$ and $f(y)$. Therefore f does not have the $WCIVP$. □

According to [4], an extendable connectivity function has the $WCIVP$. Therefore the example given by Theorem 3 is not an extendable connectivity function.

Theorem 4 *The uniform limit f of a sequence of extendable connectivity functions $f_n : \mathbb{R} \to \mathbb{R}$ has the $WCIVP$.*

PROOF. Suppose $x < y$ and $f(x) \neq f(y)$. Choose x_1 and y_1 such that $x < x_1 < y_1 < y$, $f(x_1) \neq f(y_1)$, and $f(x_1)$ and $f(y_1)$ lie between $f(x)$ and $f(y)$. We may suppose $f(x) < f(x_1) < f(y_1) < f(y)$. Let

$$\epsilon = (1/2)\min\{f(x_1) - f(x),\ f(y) - f(y_1),\ f(y_1) - f(x_1)\}\,.$$

There is an integer N such that for every $n \geq N$ and for every $z \in \mathbb{R}$, $|f(z) - f_n(z)| < \epsilon$. Since $f_N(x_1) \neq f_N(y_1)$, there exists a Cantor set $C \subset (x_1, y_1)$ such that $f_N(C) \subset (f_N(x_1), f_N(y_1))$. Then $C \subset (x, y)$ and $f(C) \subset (f(x), f(y))$. This shows f has the $WCIVP$. □

Since Jones' function f in Theorem 3 does not have the $WCIVP$, then according to Theorem 4, f cannot be the uniform limit of a sequence of extendable connectivity functions. This proves the following result.

Theorem 5 *There exists an almost continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not the uniform limit of a sequence of extendable connectivity functions from $\mathbb{R}$ into $\mathbb{R}$.*

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