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A NOTE ON THE GRADIENT PROBLEM

1 Introduction

In [Q] C. E. Weil formulated the following problem: "Assume that f is a differentiable real-valued function of N real variables, $N \geq 2$, and let $g = \nabla f$ denote its gradient, which is a function from $\mathbb{R}^N$ to $\mathbb{R}^N$. Let $G \subset \mathbb{R}^N$ be a nonempty open set and let $g^{-1}(G) \neq \emptyset$. Does $g^{-1}(G)$ have positive N-dimensional Lebesgue measure?" For $N = 1$ the answer is yes as was first proved by Denjoy [D] in 1916. Z. Buczolich in [B] gave a partial answer to this problem showing that $g^{-1}(G)$ has positive one-dimensional Hausdorff measure. In other words, he proved that the gradient has the "one-dimensional Denjoy-Clarkson property".

In the present article, we prove the Buczolich result using quite different method. Moreover, our method gives the following improvements and generalizations.

(i) We prove slightly more about $g^{-1}(G)$; namely that any one-dimensional projection of $g^{-1}(G)$ is of positive one-dimensional Hausdorff measure, which clearly implies that $g^{-1}(G)$ has positive one-dimensional Hausdorff measure.

(ii) We prove also that $g^{-1}(G)$ is not a σ-porous set.

(iii) We prove also that $g^{-1}(G)$ is porous at none of its point, which, together with the Buczolich result, gives that the gradient g has a property which can be called the "one-dimensional Zahorski M_3-property".

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(iv) Our method works also in some infinite-dimensional Banach spaces. We prove the results also in the setting of Banach spaces since we feel that they are of some interest also in this generality. However, for the reader interested in the finite-dimensional case only, we give proofs for $\mathbb{R}^N$ (which are slightly easier) separately.

Unfortunately, it seems that our "perturbation" method based on the notion of subdifferentiability cannot give a complete affirmative answer to the gradient problem due to a deep example of D. Preiss [Pr]. (See Remark 1 below.)

2 Notation and Definitions

We will use the following notation and definitions.

Lebesgue measure on $\mathbb{R}$ will be denoted by λ.

If f is a real function on a Banach space X and $x, v \in X$, then we define the (two-sided) directional derivative by

$$\partial_v f(x) = \lim_{t\to 0} \frac{f(x+t\,v) - f(x)}{t}.$$

Recall that f is Fréchet differentiable at a means there is $f'(a) \in X^*$ such that

$$\lim_{h\to 0} \|h\|^{-1} \left(f(a+h) - f(a) - \big(f'(a)\big)(h)\right) = 0.$$

In addition f is Gâteaux differentiable at a means there is $f'_G(a) \in X^*$ such that for each $v \in X$ we have $\partial_v f(a) = \big(f'_G(a)\big)(v)$.

If $X = \mathbb{R}^N$, then differentiability means Fréchet differentiability and $f'(a)$ is identified with the gradient $\nabla f(a) \in \mathbb{R}^N$.

In a metric space (M, ρ), $B(a, r)$ will denote the open ball with center a and radius r. The closed ball $\{x \mid \rho(a, x) \le r\}$ is denoted by $\overline{B}(a, r)$.

From many equivalent definitions of porosity we choose the following one.

Definition 1 *Let M be a metric space, $E \subset M$ and let $a \in M$. Then E is porous at a means there exists a $c(a) > 0$ and a sequence of balls $\{B(x_n, r_n)\}$ such that $B(x_n, r_n) \cap E = \emptyset$, $r_n > c(a)\rho(a, x_n)$ for each n and $\lim_{n\to\infty} x_n = a$. The set E is porous means it is porous at each of its points. It is uniformly porous means it is porous at each point $a \in E$ and the corresponding $c(a)$ $(= c)$ can be chosen independently of $a \in E$.*

A set is σ-porous means it is a countable union of porous sets.

Definition 2 *A real function on a Banach space X is a bump function means it is a nonzero function with a bounded support* supp f.

The main idea of the article is based on the notion of subdifferentiability of functions, cf. [DGZ, p. 339] or [Ph, p. 65].

Definition 3 *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ and $f(a) \in \mathbb{R}$. Then f is Fréchet subdifferentiable at a means there exists $p \in X^*$ such that*

$$\liminf_{h \to 0} \|h\|^{-1} \left(f(a+h) - f(a) - p(h)\right) \geq 0.$$

Also f is Gâteaux subdifferentiable at a means there exists $p \in X^*$ such that, for each $v \in X$,

$$\liminf_{t \to 0} |t|^{-1} \left(f(a+t\,v) - f(a) - p(t\,v)\right) \geq 0.$$

The following facts will be used below. They are easily proved.

1. A real function f is Fréchet (Gâteaux) differentiable at a if and only if both f and $-f$ are Fréchet (Gâteaux) subdifferentiable at a.

2. Let f and g be Fréchet (Gâteaux) subdifferentiable at a and $\lambda > 0$. Then $f+g$ and λf are Fréchet (Gâteaux) subdifferentiable at a.

3. If $\limsup_{h \to 0} \|h\|^{-1}(f(a+h) + f(a-h) - 2\,f(a)) > 0$, then $-f$ is not Fréchet subdifferentiable at a.

4. A real function $f : \mathbb{R} \to \mathbb{R}$ is Fréchet subdifferentiable at a (equivalently, f is Gâteaux subdifferentiable at a) if and only if the Dini derivatives satisfy $f_+(a) \geq f^-(a)$.

5. If $L : X \to \mathbb{R}$ has a (two-sided) directional derivative $\partial_v L(a) \neq 0$ for some $a, v \in X$ and $H : \mathbb{R} \to \mathbb{R}$ is not subdifferentiable at $L(a)$, then $H \circ L$ is not Gâteaux subdifferentiable at a.

3 Supporting Lemmas and Remarks

The following lemma is a refinement of Goffman's construction from [G].

Lemma 1 *Let $Z \subset \mathbb{R}$ with $\lambda Z = 0$. Then there exists a Lipschitz function H on $\mathbb{R}$ which is subdifferentiable at no point of Z.*

Proof. By induction we define a sequence $\{G_n\}_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$ in the following fashion.

(i) Choose an open set $G_1 \supset Z$ such that $\lambda(G_1) < 1$.

(ii) If an open set $G_n \supset Z$ is defined, we choose an open set G_{n+1} so that $G_n \supset G_{n+1} \supset Z$,

$$\lambda G_{n+1} < \frac{1}{n+1} \text{ and } \tag{1}$$

$$\lambda((x-h, x+h) \cap G_{n+1}) < \frac{h}{n+1} \text{ for } x \in \mathbb{R} \setminus G_n \text{ and } h > 0. \tag{2}$$

The existence of G_{n+1} follows e.g. from Principal Lemma 6.32 of [LMZ]. Now put $P = \bigcup_{n=1}^{\infty}(G_{2n-1} \setminus G_{2n})$ and $H(x) = \int_0^x \chi_P(t)\,dt$. Clearly H is a 1-Lipschitz function. Let $x \in Z$. We will show that H is not subdifferentiable at x. To this end consider a positive integer k. Let (a_k, b_k) be the component of G_k which contains x. By (1), $b_k - a_k < \frac{1}{k}$ and (2) implies that

$$\lambda(G_{k+1} \cap (x, b_k)) \leq \frac{b_k - x}{k+1},\ \lambda(G_{k+1} \cap (a_k, x)) \leq \frac{x - a_k}{k+1}. \tag{3}$$

If k is odd, then $G_k \setminus G_{k+1} \subset P$ and therefore (3) gives

$$\frac{H(b_k) - H(x)}{b_k - x} = \frac{\lambda(P \cap (x, b_k))}{b_k - x} \geq 1 - \frac{1}{k+1} \text{ and}$$

$$\frac{H(a_k) - H(x)}{a_k - x} = \frac{\lambda(P \cap (a_k, x))}{x - a_k} \geq 1 - \frac{1}{k+1}.$$

If k is even, then $P \cap (a_k, b_k) \subset G_{k+1}$ and therefore by (3)

$$\frac{H(b_k) - H(x)}{b_k - x} \leq \frac{1}{k+1} \text{ and } \frac{H(a_k) - H(x)}{a_k - x} \leq \frac{1}{k+1}.$$

Since H is nondecreasing and 1-Lipschitz, we easily obtain the values of the Dini derivatives of H at x:

$$H^+(x) = H^-(x) = 1 \text{ and } H_+(x) = H_-(x) = 0.$$

Hence H is not subdifferentiable (nor superdifferentiable) at x. □

Remark 1 Unfortunately, Lemma 1 cannot be generalized to $\mathbb{R}^N$. In fact, D. Preiss in [Pr] has proved that in $\mathbb{R}^N, N \geq 2$, there exists a set A of N-dimensional Lebesgue measure zero such that each Lipschitz function on $\mathbb{R}^N$ is (Fréchet) differentiable at some point of A.

We prove that the following modification of [PT, Proposition 1] holds.

Lemma 2 *Let X be a Banach space which admits a bump function b which is Lipschitz and Fréchet differentiable at every element of X. Let E be a σ-porous subset of X. Then there is a Lipschitz function $h : X \to \mathbb{R}$ which is Fréchet subdifferentiable at no point of E.*

Remark 2 We follow the part of proof of [PT, Proposition 1] concerning the case of separable X with separable dual. The difference which requires a few slight changes in that proof is that we do not suppose that E is contained in a countable union of *closed* porous sets. Thereby we partially answer the question posed in [PT] before Proposition 1.

We note first that the following stronger formulation of [PT, Lemma 1] was in fact proved there.

Lemma 3 *Let M be a metric space and let E be a closed uniformly porous subset of M. Then there are a $C > 1$ and a set $S \subset M \times (0, 1)$ such that the family $\mathcal{B} = \{\overline{B}(x, r) \mid (x, r) \in S\}$ is disjoint, $\bigcup \mathcal{B} \cap E = \emptyset$, and, for each $\delta > 0$,*

$$\mathcal{B} \cup \bigcup \{\overline{B}(x, Cr) \mid (x, r) \in S, r < \delta\} = M. \tag{4}$$

PROOF OF LEMMA 2. Assume, as we may, that $b(0) = \beta > 0$, $\operatorname{supp} b \subset B(0, 1)$, b is K-Lipschitz for some $K > 0$ and $b \geq 0$. (By taking $\tilde{b}(x) = b^2(tx - a)$ for suitable $t \in \mathbb{R}$ and $a \in X$ we get obviously a Lipschitz Fréchet differentiable function.) It is easy to see that E can be written in the form $E = \bigcup_{i=1}^{\infty} E_i$ where each E_i is a uniformly porous set (cf. [Z, Lemma 3.5]). (For a stronger result which we will not need here see [Z, Theorem 4.5].) We apply Lemma 3 to each E_i as a subset of the metric space $M_i = X \setminus (\overline{E_i} \setminus E_i)$. Obviously, E_i is uniformly porous and closed in M_i. We find S_i and $C_i > 1$ according to Lemma 3.

Note that the family $\mathcal{B}_i^* = \{B(x, r) \mid (x, r) \in S_i\}$ is disjoint where as expected $B(x, r) = \{y \in X \mid \rho(y, x) < r\}$. In fact, if an intersection I of two members of $\mathcal{B}_i^*$ contains a point, it must belong to $\overline{E}_i$; consequently I contains a point from $E_i \subset M_i$, which is impossible. From (4) we get that, for each $\delta > 0$,

$$\bigcup \{\overline{B}(x, C_i r) \mid (x, r) \in S_i, r < \delta\} \supset E_i. \tag{5}$$

For each i we define f_i by

$$f_i(x) = \begin{cases} 0 & \text{if } x \notin \bigcup \mathcal{B}_i^* \\ r\, b(\frac{x-y}{r}) & \text{if } x \in B(y, r),\, (y, r) \in S_i. \end{cases}$$

Then f_i, being the supremum of K-Lipschitz functions, is K-Lipschitz, is Fréchet differentiable on $\bigcup \mathcal{B}_i^*$, $0 \le f_i \le 2K$ (Here we use $S_i \subset X \times (0,1)$.) and $f_i(x) = r\beta$ for $(x,r) \in S_i$. Therefore, for $x \in E_i$, we have

$$\limsup_{h\to 0} \frac{f_i(x+h)+f_i(x-h)-2f_i(x)}{\|h\|} \ge \frac{\beta}{C_i} > 0 \tag{6}$$

due to (5) and to the fact that $E_i \cap \bigcup \mathcal{B}_i^* = \emptyset$. For $x \in \bigcup \mathcal{B}_i^*$ since f_i is Fréchet differentiable at x,

$$\lim_{h\to 0} \frac{f_i(x+h)+f_i(x-h)-2f_i(x)}{\|h\|} = 0 \tag{7}$$

and for $x \notin \bigcup \mathcal{B}_i^*$ because $f_i(x) = \min f_i = 0$ in such a case,

$$\liminf_{h\to 0} \frac{f_i(x+h)+f_i(x-h)-2f_i(x)}{\|h\|} \ge 0. \tag{8}$$

For $x \in X$ let $f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x)$. For $x \in E_j$ we get

$$\begin{aligned}
&\limsup_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{\|h\|} \\
&\ge \liminf_{h\to 0} \sum_{i=1}^{j-1} \frac{1}{2^i} \frac{f_i(x+h)+f_i(x-h)-2f_i(x)}{\|h\|} \\
&\quad + \limsup_{h\to 0} \frac{1}{2^j} \frac{f_j(x+h)+f_j(x-h)-2f_j(x)}{\|h\|} \\
&\quad + \liminf_{h\to 0} \sum_{i=j+1}^{J} \frac{1}{2^i} \frac{f_i(x+h)+f_i(x-h)-2f_i(x)}{\|h\|} \\
&\quad - \sum_{i=J+1}^{\infty} (2K)\frac{1}{2^i} \ge \frac{\beta}{C_j} - 2K\frac{1}{2^J} > 0
\end{aligned}$$

for J sufficiently large. Here we have used (6), (7) and (8) for $i \le J$, and the fact that each f_i is K-Lipschitz (for $i > J$). Hence $-f$ is not Fréchet subdifferentiable at x and is K-Lipschitz. □

Roughly speaking, the following lemma says that under some assumptions on the space a function can't be "very small" on a ball while its derivative is "big" on it except for a "small" set.

For the reader interested just in the case $X = \mathbb{R}^N$, only (α) is relevant. The items (β) and (γ) concern some "bad" Banach spaces. More precisely, (β)

is concerned with spaces which admit Gâteaux differentiable Lipschitz bump functions but not Fréchet differentiable ones, and (γ) pertains to spaces X which don't admit any Gâteaux differentiable Lipschitz bump functions.

Lemma 4 *Let X be a Banach space, $B(z,r) \subset X$ an open ball and $E \subset B(z,r)$. Let v, d, r, m be positive real numbers such that $8\,m < r\,d\,v$. Suppose that $b : X \to \mathbb{R}$ is 1-Lipschitz, $b(0) = v$ and $\operatorname{supp} b \subset B(0,1)$. Let F be a continuous function on the closed ball $\overline{B}(z,r)$ such that*

$$|F(x)| \le m \text{ for each } x \in B(z,r), \tag{9}$$
$$F'(x) \text{ exists at each } x \in B(z,r), \|F'(x)\| > d \text{ for each } x \in B(z,r) \setminus E.$$

Then none of the following three assertions holds.

(α) *b is Fréchet differentiable and there exists a Lipschitz function h on X which is Fréchet subdifferentiable at no point of E.*

(β) *b is Gâteaux differentiable and there exists a Lipschitz function h on X which is Gâteaux subdifferentiable at no point of E.*

(γ) *$E = \emptyset$.*

PROOF. The proof in general Banach spaces is based on variational principles. For readers who are interested only in the case $X = \mathbb{R}^N$ (where no variational principle is necessary and only (α) is interesting) we give the proof separately.

I. The case $X = \mathbb{R}^N$

Suppose that (α) holds. Put $b^*(x) = b(\frac{x-z}{r})$. It is easy to see that

$$b^*(z) = v, \operatorname{supp} b^* \subset B(z,r) \text{ and } b^* \text{ is } \frac{1}{r}\text{-Lipschitz.} \tag{10}$$

Since $8\,m < r\,d\,v$, there is a number ω such that

$$\frac{4\,m}{v} < \omega < \frac{d\,r}{2} \tag{11}$$

and put $G = F - \omega\, b^*$. Applying (9), (10) and (11) we obtain that

$$|G(x)| = |F(x)| \le m \text{ for each } x \in \partial B(z,r) \text{ and } G(z) \le m - \omega\, v < -3m. \tag{12}$$

We can clearly find $c > 0$ so small that for the function $h^* = ch$ we have

$$|h^*(x)| < \frac{m}{2} \text{ for } x \in \overline{B}(z,r) \text{ and } h^* \text{ is } \frac{d}{4}\text{-Lipschitz on } \overline{B}(z,r). \tag{13}$$

Now put $G^*(x) = G(x) + h^*(x)$ for $x \in \overline{B}(z,r)$. By (12) and (13) we easily get

$$G^*(x) \geq -\frac{3}{2}m \text{ for } x \in \partial B(z,r) \text{ and } G^*(z) < -\frac{5}{2}m. \tag{14}$$

The continuity of G^* and (14) give that G^* attains its minimum at point $x_0 \in B(z,r)$. Thus G^* is Fréchet subdifferentiable at a point x_0. Since G is Fréchet differentiable at x_0, we obtain that h^* is Fréchet subdifferentiable at x_0 and therefore $x_0 \notin E$ by (α). Consequently, (9) implies that we can choose a vector $y \in X$ with $\|y\| = 1$, such that $\partial_y F(x_0) < -d$. Formulas (10) and (11) imply that $\omega\, b^*$ is $\frac{d}{2}$-Lipschitz, we obtain that $\partial_y G(x_0) < -\frac{d}{2}$. This inequality and (13) imply that G^* does not attain its minimum at x_0, which is a contradiction.

II. The case of an arbitrary Banach space X

Suppose that at least one of the statements (α), (β), (γ) holds. We define G^* on $\overline{B}(z,r)$ in the same manner as in case **I**. (We put $h^* = h = 0$ in case (γ) holds.) Further we put $G^*(x) = \infty$ for $x \notin \overline{B}(z,r)$. Now, choose $\lambda > 0$ such that

$$0 < \lambda < \min(\frac{d}{8}, \frac{m}{3}, \frac{m}{3r}). \tag{15}$$

We add, to the proper lower semicontinuous function G^* a third "perturbation function" φ such that $G^* + \varphi$ attains its minimum at a point $x_0 \in X$, where

> if (α) holds, then φ is a Fréchet differentiable function on X such that $|\varphi(x)| \leq \lambda$ and $\|\varphi'(x)\| \leq \lambda$ for each $x \in X$, (The existence of such φ and x_0 follows from the smooth variational principle. See [Ph, Theorem 4.10].),
>
> if (β) holds and (α) does not hold, then φ is a Gâteaux differentiable function on X such that $|\varphi(x)| \leq \lambda$ and $\|\varphi'_G(x)\| \leq \lambda$ for each $x \in X$ ([Ph, Theorem 4.10]) and
>
> if (γ) holds and (β) does not hold, then $\varphi(x) = \lambda\|x - x_0\|$. (The existence of such a φ and x_0 follows from Ekeland's variational principle. See [Ph, Lemma 3.13].)

In all three cases (15) implies that $|\varphi(x)| \leq \frac{m}{3}$ on $\overline{B}(z,r)$ and therefore (14) gives $x_0 \in B(z,r)$. We know that $G^* + \varphi$ is Fréchet subdifferentiable at x_0. If (α) (resp. (β)) holds, we obtain that h^* is Fréchet (resp. Gâteaux) subdifferentiable at x_0 and thus (α) (resp. (β)) implies $x_0 \notin E$. Thus in any case (9) implies that we can choose a vector $y \in X$ with $\|y\| = 1$ such that $\partial_y F(x_0) < -d$. Since (10) and (11) imply that ωb^* is $\frac{d}{2}$-Lipschitz, we obtain that $\partial_y G(x_0) < -\frac{d}{2}$. By (13), h^* is $\frac{d}{4}$-Lipschitz on $B(z,r)$ and (15) implies that φ is $\frac{d}{8}$-Lipschitz on $B(z,r)$. Thus $G^* + \varphi = G + h^* + \varphi$ does not attain its minimum at x_0, which is a contradiction. □

4 Main Results

Theorem 5 *Let $D \subset \mathbb{R}^N$ be an open set and let f be a (Fréchet) differentiable function on D. Denote $g(x) = \nabla f(x)$ for $x \in D$ and suppose that $G \subset \mathbb{R}^N$ is an open set such that $g^{-1}(G) \neq \emptyset$. Then the following assertions hold.*

(i) *$g^{-1}(G)$ is porous at none of its point.*

(ii) *If $T \subset \mathbb{R}^N$ is open and $T \cap g^{-1}(G) \neq \emptyset$, then $L(T \cap g^{-1}(G))$ is of one-dimensional Lebesgue measure zero for no nonzero linear function $L : \mathbb{R}^N \to \mathbb{R}$. In particular, the one-dimensional Hausdorff measure of $T \cap g^{-1}(G)$ is positive.*

(iii) *If $T \subset \mathbb{R}^N$ is open and $T \cap g^{-1}(G) \neq \emptyset$, then $T \cap g^{-1}(G)$ is not σ-porous.*

Remark 3 It is easy to see that (i), (ii) and (iii) hold iff the following statement is true.

(iv) Suppose that $a \in g^{-1}(G)$ and $B(z_n, r_n)$, $n = 1, 2, \ldots$, is a sequence of open balls such that $z_n \to a$ and $r_n > c\|z_n - a\|$ for some $c > 0$ and all n. Then there exists an n_0 such that, for all $n \geq n_0$,

(a) $L(g^{-1}(G) \cap B(z_n, r_n))$ is of one-dimensional Lebesgue measure zero for no nonzero linear function L on $\mathbb{R}^N$. In particular the one-dimensional Hausdorff measure of $g^{-1}(G) \cap B(z_n, r_n)$ is positive.

(b) $g^{-1}(G) \cap B(z_n, r_n)$ is not σ-porous.

Remark 4 It can be seen from the proof that (ii) holds also if L is a Lipschitz function on $\mathbb{R}^N$ which has a (two-sided) directional derivative $\partial_{v_x} L(x) \neq 0$ at each point $x \in T \cap g^{-1}(G)$ for some $v_x \in \mathbb{R}^N$.

Proof of Theorem 5. We shall prove statement (iv) of Remark 3. To this end suppose that a, z_n, r_n, c are as in (iv). We may suppose that $r_n \to 0$. It is easy to see that there exist $v > 0$ and a Fréchet differentiable function b on $\mathbb{R}^N$ such that b is 1-Lipschitz, $b(0) = v$ and $\operatorname{supp} b \subset B(0, 1)$. Since G is open, there exists $d > 0$ such that $\|g(x) - g(a)\| > d$ for each n and $x \in B(z_n, r_n) \setminus g^{-1}(G)$. Now we can find $\delta > 0$ such that

$$|f(x) - f(a) - g(a)(x - a)| \leq \frac{d\, v\, \|x - a\|}{16\,(1 + 1/c)} \text{ whenever } x \in B(a, \delta).$$

Set $F(x) := f(x) - f(a) - g(a)(x-a)$ and let n_0 be such that $B(z_n, r_n) \subset B(a, \delta)$ for each $n > n_0$. Then, for $n > n_0$ and $x \in B(z_n, r_n)$, we have

$$|F(x)| \leq \frac{d\,v\,\|x-a\|}{16\,(1+1/c)} \leq \frac{d\,v\,(r_n + \|z_n - a\|)}{16\,(1+1/c)} <$$

$$< \frac{d\,v(r_n + r_n/c)}{16(1+1/c)} = \frac{d\,v\,r_n}{16} := m.$$

Further, for $x \in B(z_n, r_n) \setminus g^{-1}(G)$ we have $\|\nabla F(x)\| = \|g(x) - g(a)\| > d$. Thus the assumptions of Lemma 4 are satisfied for $z = z_n$, $r = r_n$ and $E = B(z_n, r_n) \cap g^{-1}(G)$.

To prove (a), suppose to the contrary that $Z := L(E)$ is of measure zero for a nonzero linear function $L : \mathbb{R}^N \to \mathbb{R}$. Thus we can apply Lemma 1 and obtain a Lipschitz function H on $\mathbb{R}$ which is subdifferentiable at no point of Z. Therefore the function $h(x) := H(L(x))$ is not Fréchet subdifferentiable (even Gâteaux subdifferentiable) at any point of E. Thus (α) of Lemma 4 holds, which is a contradiction.

To prove (b), suppose that E is σ-porous. Then Lemma 2 implies that there exists a Lipschitz function h which is Fréchet subdifferentiable at no point of E. Thus (α) of Lemma 4 holds as well, which is a contradiction. $\square$

The corresponding generalization to general Banach spaces is the following.

Theorem 6 *Let X be a Banach space, let $D \subset X$ be an open set and let f be a Fréchet differentiable function on D. Put $g(x) = f'(x)$ for $x \in D$ and suppose that $G \subset X^*$ is an open set such that $g^{-1}(G) \neq \emptyset$. Let (i), (ii), (iii) be the statements from Theorem 5 in which X is substituted for $\mathbb{R}^N$ and L is a continuous linear function (or, more generally, L is as in Remark 4). Then*

(a) *(i) holds,*

(b) *(ii) holds if X admits a Lipschitz Gâteaux differentiable bump function,*

(c) *(iii) holds if X admits a Lipschitz Fréchet differentiable bump function.*

Sketch of the proof. It is easy to see that we can proceed quite analogously as in the proof of Theorem 5. The only difference is that we use not only (α) of Lemma 4, but also (β) and (γ) to prove (b) and (a), respectively.

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