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ON SELECTORS NONMEASURABLE WITH RESPECT TO QUASIINVARIANT MEASURES

Abstract

We discuss a question on the existence of partial μ-nonmeasurable H-selectors, where μ is a given nonzero σ-finite measure defined on some σ-algebra of subsets of a set E and quasiinvariant under an uncountable group G of transformations of E, and H is an arbitrary countable subgroup of G.

Let E be a nonempty set and G be a group of transformations of E. Let S be a σ-algebra of subsets of E and μ be a measure defined on S. We recall that μ is a G-quasiinvariant measure if

a) the σ-algebra S is a G-invariant class of sets;

b) for each $X \in S$ and for each $g \in G$, the equality $\mu(X) = 0$ implies the equality $\mu(g(X)) = 0$.

In particular, every G-invariant measure μ defined on S is simultaneously a G-quasiinvariant measure.

Let H be an arbitrary subgroup of G. Then, obviously, we have a canonical partition of E consisting of all H-orbits.

We say that a subset Y of E is an H-selector if Y is a selector of the above-mentioned partition. We say that a subset Y of E is a partial H-selector if Y is a selector of a subfamily of this partition. Clearly, every partial H-selector can be extended to an H-selector.

A question on measurability of H-selectors, with respect to the given nonzero σ-finite G-quasiinvariant measure μ, arises naturally. We recall that the first result concerning this question was obtained by Vitali [11], who showed that if E is the set of all real numbers, G is the additive group of

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reals and H is the additive group of rationals, then each H-selector is nonmeasurable with respect to the classical Lebesgue measure on E.

This important result of Vitali was generalized in various directions. In particular, some theorems and facts concerning measurability of H-selectors, with respect to μ, were obtained in the papers [1], [5], [8] and [9]. Notice that the case where H is an uncountable subgroup of the given group G was discussed in those papers, too.

In the present paper we shall consider only the case when $\mathrm{card}\,(H) \leq \omega$. First of all let us remark that, even in the classical situation, we cannot assert the nonmeasurability of all H-selectors. Indeed, in [4] a measure ν is constructed such that

1) ν is defined on some σ-algebra of subsets of the real line;

2) ν is a nonzero nonatomic σ-finite measure;

3) ν is invariant under the group of all isometric transformations of the real line;

4) $\mathrm{dom}\,(\nu)$ contains the family of all Lebesgue measurable subsets of the real line;

5) there exists a Vitali set belonging to $\mathrm{dom}\,(\nu)$.

We thus conclude that, for the above-mentioned measure ν and for the countable group H coinciding with the additive group of rationals, there exists a ν-measurable H-selector.

In connection with this result it is reasonable to pose the following question.

Let E be a set and G be an uncountable group of transformations of E. Let μ be a nonzero σ-finite G-quasiinvariant measure defined on some σ-algebra of subsets of E and let H be an arbitrary countable subgroup of the group G. Denote by $\{H(x) : x \in E\}$ the partition of E into H-orbits of points of E. Does there exist a subfamily of $\{H(x) : x \in E\}$ such that all selectors of this subfamily are nonmeasurable with respect to μ?

Our goal is to show that, under some natural assumptions on G and μ, the answer to this question is positive.

We say that the group G acts freely in the space E, with respect to the given measure μ, if for any two distinct transformations g and h from G, we have the equality

$$\mu^*(\{x \in E : g(x) = h(x)\}) = 0,$$

where μ^* denotes, as usual, the outer measure associated with μ.

For example, if E is a finite-dimensional Euclidean space, G is a group of affine transformations of E, and μ is a measure defined on some σ-algebra of subsets of E and vanishing on all affine hyperplanes of E, then G acts freely in E with respect to μ.

Our further consideration needs the following statement which generalizes a result obtained in [2], [3] and [7].

Theorem 1 *Let E be a set, G be a group of transformations of E and let μ be a nonzero σ-finite G-quasiinvariant measure defined on a σ-algebra of subsets of E. Suppose also that G contains an uncountable subgroup Γ acting freely in E with respect to μ. Let H be an arbitrary countable subgroup of Γ and let $\{H(x) : x \in E\}$ be a partition of E into H-orbits. Then there exists a subfamily of $\{H(x) : x \in E\}$ such that its union is a μ-nonmeasurable set in E.*

Proof. We may assume, without loss of generality, that

a) μ is a probability measure,

b) the group Γ coincides with the original group G,

c) $\mathrm{card}\,(\Gamma) = \mathrm{card}\,(G) = \omega_1$.

Let us denote by $\{g_\xi : \xi < \omega_1\}$ a family of elements of G such that

$$g_\xi H \neq g_\zeta H \qquad (\xi < \omega_1,\ \zeta < \omega_1,\ \xi \neq \zeta).$$

The existence of such a family is obvious since $\mathrm{card}\,(G) = \omega_1$ and $\mathrm{card}\,(H) \leq \omega$. Next, since H is a countable group, we can write $H = \{h_n : n \in \omega\}$. Let F be a subset of E for which the family $\{G(y) : y \in F\}$ is injective and consists of all G-orbits in E (in other words, F is a G-selector). Consider a family $\{H(y) : y \in F\}$ and put $Y = \cup\{H(y) : y \in F\}$. If the set Y is nonmeasurable with respect to μ, then there is nothing to prove. Suppose now that $Y \in \mathrm{dom}\,(\mu)$. Then it is not hard to check that, for any two distinct ordinals $\xi < \omega_1$ and $\zeta < \omega_1$, we have the inclusion

$$g_\xi(Y) \cap g_\zeta(Y) \subseteq g_\xi(\cup\{h_n(X_{nm}) : n \in \omega,\ m \in \omega\}),$$

where a set X_{nm} is defined by the formula

$$X_{nm} = \{x \in E : g_\xi h_n(x) = g_\zeta h_m(x)\}.$$

But $\mu^*(X_{nm}) = 0$, since G acts freely in E with respect to μ. Taking into account the fact that μ is a G-quasiinvariant measure, we obtain

$$\mu(g_\xi(Y) \cap g_\zeta(Y)) = 0,$$

for all $\xi < \omega_1$, $\zeta < \omega_1$, $\xi \neq \zeta$. The latter relation implies the equality $\mu(Y) = 0$, since μ (being a σ-finite measure) satisfies the countable chain condition.

Now, it is easy to see that we can represent the set E in the form

$$E = \cup\{Y_\alpha : \alpha < \omega_1\},$$

where

1) the sets Y_α ($\alpha < \omega_1$) are pairwise disjoint,

2) for each $\alpha < \omega_1$, the set Y_α is the union of a family of H-orbits in E,

3) for each $\alpha < \omega_1$, we have $\mu(Y_\alpha) = 0$.

According to the classical theorem of Ulam [10], there exists a subset A of ω_1 such that the set $\cup\{Y_\alpha : \alpha \in A\}$ does not belong to $\mathrm{dom}\,(\mu)$. But it is clear that $\cup\{Y_\alpha : \alpha \in A\}$ can be represented as the union of a family of H-orbits in E. Thus, the theorem is proved. □

Now, we can easily deduce from Theorem 1 the following statement.

Theorem 2 *Let E be a set and G be an uncountable group of transformations of E. Let μ be a nonzero σ-finite G-quasiinvariant measure defined on some σ-algebra of subsets of E. Suppose that G acts freely in E with respect to μ. Fix a countable subgroup H of G and denote by $\{H(x) : x \in E\}$ the partition of E consisting of all H-orbits. Then there exists a subfamily of $\{H(x) : x \in E\}$ such that all its selectors are nonmeasurable with respect to μ.*

PROOF. According to Theorem 1, there exists a subset D of E such that the family $\{H(x) : x \in D\}$ is injective and the set $\cup\{H(x) : x \in D\}$ is nonmeasurable with respect to μ. Let us show that all selectors of $\{H(x) : x \in D\}$ are μ-nonmeasurable, too. Denote by Z an arbitrary selector of $\{H(x) : x \in D\}$. Obviously, we have the equality

$$\cup\{H(x) : x \in D\} = \cup\{h(Z) : h \in H\}.$$

Suppose that $Z \in \mathrm{dom}\,(\mu)$. Then, taking into account the fact that H is a countable group and μ is a G-quasiinvariant measure, we obtain

$$\cup\{h(Z) : h \in H\} \in \mathrm{dom}\,(\mu)$$

and, consequently,

$$\cup\{H(x) : x \in D\} \in \mathrm{dom}\,(\mu),$$

which contradicts the definition of the family $\{H(x) : x \in D\}$. This contradiction finishes the proof. □

Remark 1 *It is essential for validity of Theorem 2 that the partition $\{H(x) : x \in E\}$ of the set E consists of all H-orbits, where H is a countable subgroup of the original transformation group G. In order to show this, let us take an arbitrary group G with card$(G) = \omega_1$ and let us put $E = G$. Clearly, we can* *identify* G *with the group of all left translations of* E*, which acts freely in* E*. Further, we can represent* G *in the form*

$$G = \cup\{G_\xi : \xi < \omega_1\},$$

where a family $\{G_\xi : \xi < \omega_1\}$ *satisfies the following conditions:*

1. *for each* $\xi < \omega_1$*, we have card*$(G_\xi) = \omega$*,*
2. *for each* $\xi < \omega_1$*, the set* G_ξ *is a subgroup of the group* G*,*
3. *for each* $\xi < \omega_1$*, the set* $\cup\{G_\zeta : \zeta < \xi\}$ *is a proper subset of* G_ξ*.*

Let us fix a point $e \in E$ and let us put

$$E_\xi = G_\xi(e) \setminus \cup\{G_\zeta(e) : \zeta < \xi\},$$

for all ordinals $\xi < \omega_1$. Then we obtain a partition $\{E_\xi : \xi < \omega_1\}$ of the set E such that $\mathrm{card}\,(E_\xi) = \omega$, for any $\xi < \omega_1$. We assert now that an analogue of Theorem 2 is not true for the above-mentioned partition. Indeed, let λ be a probability diffused measure defined on the σ-algebra of subsets of the set E, generated by the family of all countable subsets of E, and let Z be a fixed selector of $\{E_\xi : \xi < \omega_1\}$. Denote by J the G-invariant σ-ideal of subsets of E, generated by the one-element family $\{Z\}$. It is easy to check that, for any set $X \in J$, we have the equality $\lambda_*(X) = 0$, where λ_* denotes, as usual, the inner measure associated with λ. Starting with this property of J we can easily extend the measure λ to a measure μ such that

a) $\mathrm{dom}\,(\mu)$ coincides with the σ-algebra of subsets of E, generated by $\mathrm{dom}\,(\lambda) \cup J$,

b) $\mu(X) = 0$ for all sets $X \in J$,

c) μ is a G-invariant measure.

Now, it is clear that, for any subset Ξ of ω_1, there exists a μ-measurable selector of the family $\{E_\xi : \xi \in \Xi\}$.

A similar argument shows us that if H is an arbitrary uncountable subgroup of our group G and $\{H(x) : x \in E\}$ is a partition of E into H-orbits, then for every selector Z of $\{H(x) : x \in E\}$ there exists a measure ν satisfying the following conditions:

(1) ν is a complete probability diffused G-invariant measure defined on some σ-algebra of subsets of E,

(2) Z belongs to $\mathrm{dom}\,(\nu)$ and $\nu(Z) = 0$.

In particular, we obtain immediately from (2) that, for each subset F of E, there is a selector of $\{H(x) : x \in F\}$ belonging to $\mathrm{dom}\,(\nu)$.

The next result is an easy consequence of Theorem 2.

Proposition 1 *Let the assumptions of Theorem* 2 *be satisfied and let, in addition, a countable subgroup H of G be such that card$(H(x)) \geq 2$, for all points $x \in E$. Then there exists an H-selector nonmeasurable with respect to μ.*

Proof. Indeed, it immediately follows from Theorem 2 that there exists a partial H-selector Z nonmeasurable with respect to μ. Evidently, we can find two H-selectors Z_1 and Z_2 which extend Z and satisfy the equality $Z = Z_1 \cap Z_2$. Now, since the set Z is μ-nonmeasurable, at least one of the sets Z_1 and Z_2 is μ-nonmeasurable. Thus, we see that there exists a μ-nonmeasurable H-selector. □

In fact, the preceding argument shows that, for a measure μ defined on some σ-algebra of subsets of a set E, the following two assertions are equivalent:

a) there exists a subset of E nonmeasurable with respect to μ,

b) if $\{E_i : i \in I\}$ is a partition of E such that $2 \leq \text{card}\,(E_i) \leq \omega$, for all $i \in I$, then there exists a selector of $\{E_i : i \in I\}$ nonmeasurable with respect to μ.

Remark 2 *Let E be a set and let G be an uncountable group of transformations of E, acting freely in E with respect to a nonzero σ-finite G-invariant measure μ defined on some σ-algebra of subsets of E. It was proved in* [9] *that there always exists a countable subgroup H of G such that all H-selectors are nonmeasurable with respect to every G-invariant measure extending μ. In other words, the group H plays a role similar to the role played by the additive group of rationals in the classical Vitali construction* [11]*. In connection with this result, we wish to notice that the method of* [9] *is essentially based on the assumption of G-invariance of the measure μ and, therefore, it does not work for nonzero σ-finite G-quasiinvariant measures.*

Remark 3 *Let $(G, +)$ be an uncountable commutative group equipped with a nonzero σ-finite G-quasiinvariant measure μ. It was shown in* [6] *that there always exists a μ-nonmeasurable subgroup of G. Starting with this result it is not difficult to prove that, if H is an arbitrary countable subgroup of G, then there exists a family of H-orbits whose union is a μ-nonmeasurable subgroup of G. Notice also that an analogous assertion is not true, in general, for uncountable noncommutative groups (see* [6]*).*

Remark 4 *It is easy to see that the results presented above can be formulated and proved in a more general form, namely, in terms of the pair (S, J), where*

1) *S is a G-invariant σ-algebra of subsets of E,*

2) *J is a G-invariant σ-ideal of subsets of E,*

3) *J is contained in S,*

4) *(S, J) satisfies the countable chain condition.*

In particular, we have the respective analogues of Theorems 1*,* 2 *and Proposition for the Baire property.*

References

[1] J. Cichoń, A. B. Kharazishvili and B. Weglorz, *On sets of Vitali's type*, Proc. Amer. Math. Soc., **118** (1993), 1221–1228.

[2] P. Erdös and R. D. Mauldin, *The nonexistence of certain invariant measures*, Proc. Amer. Math. Soc., **59** (1976), 321–322.

[3] A. B. Kharazishvili, *Certain types of invariant measures*, Dokl. Akad. Nauk SSSR, **222**(3) (1975), 538–540, (in Russian).

[4] A. B. Kharazishvili, *Some applications of Hamel bases*, Bull. Acad. Sci. Georgian SSR, **85**(1) (1977), 17–20 (in Russian).

[5] A. B. Kharazishvili, *Martin's axiom and* Γ*-selectors*, Bull. Acad. Sci. Georgian SSR, **137**(2) (1990),(in Russian).

[6] A. B. Kharazishvili, *Selected Topics of Point Set Theory*, Łódź University Press, Łódź, 1996.

[7] C. Ryll-Nardzewski and R. Telgarsky, *The nonexistence of universal invariant measures*, Proc. Amer. Math. Soc., **69** (1978), 240–242.

[8] S. Solecki, *On sets nonmeasurable with respect to invariant measures*, Proc. Amer. Math. Soc., **119**(1) (1993), 115–124.

[9] S. Solecki, *Measurability properties of sets of Vitali's type*, Proc. Amer. Math. Soc., **119**(3) (1993), 897–902.

[10] S. Ulam, *Zur Masstheorie in der allgemeinen Mengenlehre*, Fund. Math., **16** (1930), 140–150.

[11] G. Vitali, *Sul Problema della Misura dei Gruppi di Punti di una Retta*, Bologna, Italy, 1905.