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THE DARBOUX PROPERTY FOR GRADIENTS

Abstract

It is well known that the derivative of a function of one variable has the Darboux property. In this paper it is shown that the gradient of a differentiable function of several variables maps certain closed convex sets to connected sets.

It is well known that any differentiable real function f on an interval $I \subset \mathbb{R}$ has the Darboux property. This means that if $a < b$ are points in I and ξ is a value between $f'(a)$ and $f'(b)$, then there is $x \in [a, b]$ such that $f'(x) = \xi$. It is equivalent to say that for any closed convex subset K of I the image $f'(K)$ is connected. In this note we are going to show that an analogous property holds for the derivative (gradient) of a differentiable function of several variables. Even more generally, we work in infinite-dimensional Banach spaces.

As a special case of the result we obtain the Darboux property of partial derivatives of differentiable functions, which is due to Neugebauer [N] and Weil [W]. Let us mention that if we modify the definition of Darboux property of partial derivatives as in [N], the assumption of differentiability may be weakened.

The one-dimensional Darboux property has been generalized in a variety of other directions as well. There are several papers which are devoted to the Darboux property of derivatives of interval functions of several variables. A general result was proved by Mišik [M], for further development see [N], [B].

We suppose that X is a Banach space. In particular we may consider $X = \mathbb{R}^n$. The symbol $U(x, r)$ is used for the open ball with center at x and

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radius r. We denote the dual space of X by X^*. Both norms in X and X^* are denoted by $\|\dots\|$. We write $x^* \cdot x$ for the duality pairing between $x^* \in X^*$ and $x \in X$. The topological notions in X and in $X*$ are with ¿ respect to the corresponding norm topology. Differentiability is interpreted as Fréchet differentiability. This means that a function f is said to be differentiable at $x \in X$ with respect to $D \subset X$ if there is a unique $x^* \in X^*$ (called the derivative of f at x with respect to D) such that

$$\lim_{y \to x, y \in D} \frac{f(y) - f(x) - x^* \cdot (y - x)}{\|y - x\|} = 0.$$

If f is differentiable with respect to its domain, we say simply that f is differentiable and denote the derivative by f'. The main goal of this note is the following theorem.

Theorem 1 *Let f be a differentiable function on $D \subset X$. Then for any closed convex set $K \subset D$ with nonempty interior, $f'(K)$ is a connected subspace of X^*.*

For the proof of Theorem 1, we may assume that $D = K$. We fix a closed convex set $K \subset X$ with nonempty interior and start with a series of auxiliary results.

Lemma 2 (Ekeland's variational principle) *Let g be a continuous function on $\overline{U}(x, r) \subset X$ and $\varepsilon > 0$. Suppose that*

$$g(y) \le g(x) + \varepsilon r$$

for each $y \in \overline{U}(x, r)$. Then there is $u \in \overline{U}(x, r/2)$ such that

$$g(y) \le g(u) + 2\varepsilon |y - u| \tag{1}$$

for each $y \in \overline{U}(x, r)$.

PROOF. We refer e.g. to [Ph], Lemma 3.13, but for reader's convenience we notice that the finite-dimensional case is easy. Indeed, we find $u \in \overline{U}(x, r)$ such that $y \mapsto g(y) - 2\varepsilon|y - x|$ attains a maximum at u relative to $\overline{U}(x, r)$. Then an exercise in handling the triangle inequality shows that in fact $u \in \overline{U}(x, r/2)$ and (1) holds. □

Since K has a nonempty interior, we may fix a ball $\overline{U}(x_0, r_0)$ inside K.

Lemma 3 *Let $x \in K$ and $r > 0$. Then there are $x_1 \in K$ and $r_1 > 0$ such that*

$$r_1 \geq \min\Big\{\frac{rr_0}{2\|x-x_0\|},\ \frac{r}{2},\ r_0\Big\} \text{ and } \overline{U}(x_1,r_1) \subset K \cap \overline{U}(x,r).$$

Proof. If $\|x-x_0\| \leq r/2$, then it is enough to set

$$x_1 = x_0, \qquad r_1 = \min\{r_0, r/2\}.$$

Let $\|x-x_0\| \geq r/2$. We set

$$x_1 = x + \frac{1}{2}r\ \frac{x_0 - x}{\|x_0 - x\|} \text{ and } r_1 = \min\Big\{\frac{r}{2},\quad \frac{rr_0}{2\|x-x_0\|}\Big\}.$$

Now, each point y_1 from $\overline{U}(x_1,r_1)$ is a convex combination of x and a point $y_0 \in \overline{U}(x_0,r_0)$ and hence belongs to K. Since $r_1 \leq r/2$, obviously $y_1 \in \overline{U}(x,r)$. □

In the next lemma, we recall some standard tricks from differentiation theory. If f is a differentiable function on K, we let

$$E_{i,m,k}(f) = \Big\{x \in K:\ \|x-x_0\| \leq 2^i r_0,$$
$$\big[y \in K, \|y-x\| < 2^{-m} \Rightarrow |f(y) - f(x) - f'(x)\cdot(y-x)| \leq 2^{-k}\|y-x\|\big]\Big\}.$$

Lemma 4 *Let f be a differentiable function on K and $i,m,k \in \mathbb{N}$. Then*

(a) $\limsup_{y\to x,\ y\in E_{i,m,k}} \|f'(y) - f'(x)\| \leq 2^{i-k+4}$ *for any nonisolated point x of $\overline{E}_{i,m,k}(f)$,*

(b) $\overline{E}_{i,m,k}(f) \subset E_{i,m,k-i-5}(f)$.

Proof. Choose $x \in \overline{E}_{i,m,k}(f)$. We find $r \in (0, \min\{r_0, 2^{-m-1}\})$ such that

$$|f(y) - f(x) - f'(x)\cdot(y-x)| < 2^{-k}\|y-x\|$$

for all $y \in \overline{U}(x,r) \cap K$. Notice that then

$$|f(y') - f(y) - f'(x)\cdot(y'-y)| < 2^{-k+1}r \tag{2}$$

for all $y, y' \in \overline{U}(x,r) \cap K$. If $z \in E_{i,m,k} \cap U(x,r)$ and $y, y' \in \overline{U}(x,r) \cap K$, then $\|y-z\| < 2r < 2^{-m}$, $\|y'-z\| < 2r < 2^{-m}$ and thus also

$$|f(y') - f(y) - f'(z)\cdot(y'-y)| < 2^{-k}(\|y'-z\| + \|y-z\|) \leq 2^{-k+2}r. \tag{3}$$

By Lemma 3, there are $x_1 \in K$ and $r_1 > 0$ such that

$$r_1 \geq 2^{-i-1} r \text{ and } \overline{U}(x_1, r_1) \subset K \cap \overline{U}(x, r).$$

Let $z \in E_{i,m,k} \cap U(x,r)$ and $h \in X$, $\|h\| = r_1$. Then x_1, $x_1 + h \in K \cap \overline{U}(x,r)$. Using (2) and (3) we obtain

$$\begin{aligned}
|(f'(z) - f'(x)) \cdot h| = &\Big|\Big(f(x_1+h) - f(x_1) - f'(x)\cdot h\Big) \\
&- \Big(f(x_1+h) - f(x_1) - f'(z)\cdot h\Big)\Big| \\
\leq &\big|f(x_1+h) - f(x_1) - f'(x)\cdot h\big| + \big|f(x_1+h) - f(x_1) - f'(z)\cdot h\big| \\
\leq &2^{-k+1} r + 2^{-k+2}\, r \leq 2^{i-k+4}\|h\|.
\end{aligned}$$

This proves (a).

Now, we choose $x \in \overline{E}_{i,m,k}$ and $y \in K$ with $\|y - x\| < 2^{-m}$. There is a sequence x_j of points from $E_{i,m,k}(f)$ converging to x. Then for j large enough, and with the aid of (a),

$$\begin{aligned}
|f(y) - &f(x) - f'(x)\cdot(y-x)| \\
&= \Big|\Big(f(y) - f(x_j) - f'(x_j)\cdot(y - x_j)\Big) \\
&\quad - \Big(f(x) - f(x_j) - f'(x_j)\cdot(x - x_j)\Big) \\
&\quad + \Big(f'(x_j)\cdot(y-x) - f'(x)\cdot(y-x)\Big)\Big| \\
&\leq \big|f(y) - f(x_j) - f'(x_j)\cdot(y-x_j)\big| \\
&\quad + \big|f(x) - f(x_j) - f'(x_j)\cdot(x-x_j)\big| \\
&\quad + \big|f'(x_j)\cdot(y-x) - f'(x)\cdot(y-x)\big| \\
&\leq 2^{-k}\|y - x_j\| + 2^{-k}\|x_j - x\| + 2^{i-k+4}\|y - x\|.
\end{aligned}$$

Letting $j \to \infty$ we obtain

$$|f(y) - f(x) - f'(x)\cdot(y-x)| \leq 2^{i-k+5}\|y - x\|$$

which proves (b). □

PROOF OF THEOREM 1. Let $G^+, G^- \subset X^*$ be open sets such that $f'(K) \subset G^+ \cup G^-$ and $G^+ \cap G^- \cap f'(K) = \emptyset$. We write

$$F^+ = \{x \in K : f'(x) \in G^+\} \text{ and } F^- = \{x \in K : f'(x) \in G^-\}.$$

Suppose that both F^+ and F^- are nonempty. This will lead to a contradiction. Denote $H = \overline{F}^+ \cap \overline{F}^-$. Since K is connected, we deduce that $H \neq \emptyset$. Denote

$$F^+_{i,m,k} = \{y \in E_{i,m,k}(f) : \operatorname{dist}(f'(y), X^* \setminus G^+) \geq 2^{-k+2i+13}\}$$
$$F^-_{i,m,k} = \{y \in E_{i,m,k}(f) : \operatorname{dist}(f'(y), X^* \setminus G^-) \geq 2^{-k+2i+13}\}.$$

Then

$$K = \bigcup_{i,m,k} F^+_{i,m,k} \cup \bigcup_{i,m,k} F^-_{i,m,k}.$$

Using the Baire category theorem in the space H, we find $z \in H$, $\rho_0 > 0$ and i, m, $k \in \mathbb{N}$ such that

$$H \cap U(z, 2\rho_0) \subset \overline{F}^+_{i,m,k} \tag{4}$$

or

$$H \cap U(z, 2\rho_0) \subset \overline{F}^-_{i,m,k} \tag{5}$$

Assume e.g. that case (4) holds. Also, we may assume that $\rho_0 \leq r_0$ and $3\rho_0 < 2^{-m}$. From Lemma 4(a) it follows that

$$\operatorname{dist}(f'(y), X^* \setminus G^+) \geq 2^{-k+2i+13} - 2^{-k+i+4} > 0 \tag{6}$$

for each $y \in \overline{F}^+_{i,m,k}$. In particular

$$H \cap U(z, 2\rho_0) \subset F^+. \tag{7}$$

Since $H \subset \overline{F}^-$, there is a point $x \in F^- \cap U(z, \rho_0)$. By (7), $x \notin H$. Let V be the largest ball centered at x such that $V \cap H = \emptyset$. The radius r of V is less than ρ_0 as $z \notin V$. Hence $\overline{V} \subset U(z, 2\rho_0)$. Since $K \cap V$ is connected and $V \cap H = \emptyset$, we deduce that $K \cap V \subset F^-$. Maximality of V yields that there is a point $w \in U(x, 2r) \cap U(z, 2\rho_0) \cap H$. Then, by (4) and (6), $w \in \overline{F}^+_{i,m,k}$ and

$$\operatorname{dist}(f'(w), X^* \setminus G^+) \geq 2^{-k+2i+12}. \tag{8}$$

We write

$$g(y) = f(y) - f(w) - f'(w) \cdot (y - w), \quad y \in X.$$

We have

$$\|y - w\| \leq 3r \leq 3\rho_0 < 2^{-m}$$

for all $y \in \overline{V}$. Since $\|x - x_0\| \leq \|z - x_0\| + \|x - z\| \leq 2^{i+1} r_0$, we use Lemma 3 to find a ball

$$\overline{U}(x_1, r_1) \subset K \cap \overline{V}$$

such that
$$r_1 \geq 2^{-i-2} r.$$
By Lemma 4(b),
$$\overline{F}^+_{i,m,k} \subset E_{i,m,k-i-5}(f).$$
Thus
$$|g(y)| \leq 2^{-k+i+5} \|y-w\| \leq 2^{-k+i+7} r \leq 2^{-k+2i+9} r_1$$
for each $y \in \overline{V}$, which implies that
$$g(y) - g(x_1) \leq 2^{-k+2i+10} r_1$$
for each $y \in \overline{U}(x_1, r_1)$. By Lemma 2, there is a point $u \in \overline{U}(x_1, r_1/2)$ such that
$$g(y) \leq g(u) + 2^{-k+2i+11} \|y-u\|$$
for all $y \in U(x_1, r_1)$, so that
$$\|f'(u) - f'(w)\| = \|g'(u)\| \leq 2^{-k+2i+11}.$$
This contradicts (8) because $u \in K \cap V \subset F^-$. The proof is complete. □

Remark 5 It is not true that $f'(L)$ is connected when $L \subset D$ is a line segment. As an example, consider
$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2^4}{x_1^2 + x_2^4}, & [x_1, x_2] \neq 0 \\ 0, & [x_1, x_2] = 0. \end{cases}$$
Then f is differentiable,
$$\frac{\partial f}{\partial x_1}(0, x_2) = \begin{cases} 1, & x_2 \neq 0 \\ 0, & x_2 = 0. \end{cases}$$

Remark 6 An easy example on $\mathbb{R}$ shows that the Darboux property fails for vector-valued functions; the counterexample is given by f which is defined as the antiderivative of
$$f'(x) = \begin{cases} (\cos \frac{1}{x},\ \sin \frac{1}{x}), & x \neq 0 \\ 0, & x = 0. \end{cases}$$
cf. [D], Problem 8.5.4.

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