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Conditions for equality of Hausdorff and packing measures on $\mathbb{R}^n$

Abstract

This note answers the question, for which Hausdorff functions h may the h-Hausdorff and h-packing measures agree on some subset A of $\mathbb{R}^n$, and be positive and finite. We show that these conditions imply that h is a *regular density function*, in the sense of Preiss, and that for each such function there is a subset of $\mathbb{R}^n$ on which the h-Hausdorff and h-packing measures agree and are positive and finite.

In [8] and [10], Taylor and Tricot introduced a new family of measures, namely *packing measures*, which complement the well-known Hausdorff measures. For any *Hausdorff function*, that is, any non-decreasing function $h : (0,\infty) \to (0,\infty)$ with $h(0+) = 0$, we may define the Hausdorff and packing measures associated with this function.

In [7], Saint Raymond and Tricot considered the implications of equality on subsets of $\mathbb{R}^n$ of the Hausdorff and packing measures associated with some function $h(r) = r^s$. They showed that if the two measures are positive and finite on some subset A of $\mathbb{R}^n$, then they agree on A if and only if s is an integer and A is s-rectifiable.

In this note we extend the above result. We show that the Hausdorff functions h for which there may exist a subset of $\mathbb{R}^n$ on which Hausdorff and packing measures are equal, and positive and finite, are precisely those named *regular density functions* by D. Preiss, (see [6]). For this work we use theorems which adapt and extend the standard density-type theorems, and draw heavily on the concepts and results of [6] and [4].

We now review the definitions and results needed for what follows. For definitions of Hausdorff measure, $\mathcal{H}^h$, we refer the reader to [3, 4.9].

By a *packing* of a subset S of $\mathbb{R}^n$ we mean a finite or countable collection of closed balls $\{B(x_i, r_i) : x_i \in S\}$ such that, for each $i \neq j$,

$$B(x_i, r_i) \cap B(x_j, r_j) = \emptyset.$$

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A *δ-packing* is a packing such that for each i, diam $B(x_i, r_i) \le \delta$.

If h is a Hausdorff function then $\mathcal{P}^h(S)$, the h-packing measure of S, may be defined thus;

$$P^h_\delta(S) = \sup\left\{\sum h(\text{diam } B(x_i, r_i)) : \{B(x_i, r_i)\} \text{ a } \delta\text{-packing of } S\right\},$$
$$P^h_0(S) = \lim_{\delta\to 0} P^h_\delta(S),$$
$$\mathcal{P}^h(S) = \inf\left\{\sum_1^\infty P^h_0(S_i) : S \subset \bigcup_1^\infty S_i\right\}.$$

We shall write $\mathcal{H}^s$ and $\mathcal{P}^s$ for the measures constructed from the function $h(r) = r^s$.

By a *measure* μ we shall mean a non-zero Borel regular outer measure on $\mathbb{R}^n$, such that the Borel sets are μ measurable. If μ is also locally finite we shall call μ a *Radon measure*. We note that μ is locally finite if and only if every compact subset of $\mathbb{R}^n$ has finite μ-measure. By $B(x,r)$ and $U(x,r)$ we shall mean the closed and open balls, respectively, centered at x with radius r, by $\partial B(x,r)$ the boundary of these balls, and by $B(A,r)$ the set of all points at distance no greater than r from the set A. We write spt μ for the smallest closed subset C of $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n \setminus C) = 0$, and $\mu|_A$ for the measure on $\mathbb{R}^n$ defined by $\mu|_A(S) = \mu(A \cap S)$ for each subset S of $\mathbb{R}^n$. In the definitions and notation below we follow [3] and [6].

(i) If h is a Hausdorff function, μ measures $\mathbb{R}^n$, and $x \in \mathbb{R}^n$, we define $\overline{D}^h(\mu, x)$ and $\underline{D}^h(\mu, x)$, the *upper and lower h-densities of μ at x*, by the formulae

$$\overline{D}^h(\mu, x) = \limsup_{r\searrow 0} \mu B(x,r)/h(2r)$$

and

$$\underline{D}^h(\mu, x) = \liminf_{r\searrow 0} \mu B(x,r)/h(2r).$$

If the upper and lower h-densities of μ at x coincide and are positive and finite, we denote their common value by $D^h(\mu, x)$, and say that x is an *h-density point of μ*.

A Hausdorff function h is said to be a *density function (in $\mathbb{R}^n$)* if there is a measure μ over $\mathbb{R}^n$ such that μ almost every $x \in \mathbb{R}^n$ is an h-density point of μ.

A density function h will be called *regular* (in the sense of Preiss) if $\lim_{r\searrow 0} h(tr)/h(r)$ exists for each $t > 0$. We refer the reader to [6, 6.5] for a complete characterization of regular density functions in terms of limiting conditions near zero.

(ii) We shall use the notation $\lim_{k\to\infty} \mu_k = \mu$, or $\mu_k \to \mu$ for the usual notion of *weak convergence of measures*, see for example [3, 1.21].

(iii) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is Borel measurable and μ measures $\mathbb{R}^n$, we define $T[\mu]$, the image of μ under T, by

$$T[\mu](E) = \mu(T^{-1}(E)) \text{ for every Borel set } E \subseteq \mathbb{R}^m.$$

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R} \setminus \{0\}$. We define the map $T_{x,r} : \mathbb{R}^n \to \mathbb{R}^n$ by $T_{x,r}(z) = (z - x)/r$.

(iv) Let μ measure $\mathbb{R}^n$ and $x \in \mathbb{R}^n$. We say that a locally finite measure ψ is a *tangent measure of* μ *at* x if there are sequences $r_k \searrow 0$ and $c_k > 0$ such that $\psi = \lim_{k\to\infty} c_k T_{x,r_k}[\mu]$, and write $\psi \in \mathrm{Tan}(\mu, x)$. (Tangent measures in this form were introduced by D. Preiss in [6].)

(v) A measure μ on $\mathbb{R}^n$ is said to be *uniformly distributed* if $\mu B(x, r) = \mu B(y, r) < \infty$ whenever $x, y \in \mathrm{spt}\ \mu$ and $0 < r < \infty$.

(vi) Let μ be a Radon measure on $\mathbb{R}^n$. Then $x \in \mathrm{spt}\ \mu$ is called a *symmetric point of* μ if for every $\rho > 0$

$$\int_{B(x,\rho)} z \, d\mu(z) = x\mu B(x, \rho).$$

(vii) A Radon measure μ on $\mathbb{R}^n$ is called *flat* if $\mu = c\mathcal{H}^m|_V$ for some $c > 0$ and some m-dimensional affine subspace V of $\mathbb{R}^n$, $(1 \le m \le n)$.

We now have all the concepts required to state both the theorem of Saint Raymond and Tricot, (see [7]), for functions $h(r) = r^s$, and the results from [6] and [4] that we will need.

Theorem 1 *If $A \subseteq \mathbb{R}^n$ satisfies $\mathcal{P}^s(A) < \infty$, then $\mathcal{H}^s(A) = \mathcal{P}^s(A)$ if and only if the density $D^s(\mathcal{H}^s|_A, x)$ exists and equals 1 for $\mathcal{P}^s$ almost all $x \in A$. (This in turn implies that s is an integer, and that A is s-rectifiable.)*

Lemma 1 *Let X be a separable metric space, and let h be a regular density function. Then $\mathcal{H}^h(A) \le \mathcal{P}^h(A)$ for all subsets A of X.*

A proof of the above lemma for the functions $h(r) = r^s$ may be found in [3, 5.12]. Since the proof given there also works for regular density functions, we omit the proof of Lemma 1. For the proofs of Lemma 2 and Theorem 2 we refer the reader to [6, 1.11(4), 2.12] respectively. Theorem 3 follows from the definition of a regular density function and [6, 4.11(1), 4.11(4)], and Theorem 4 from [6, 6.1(5), 4.7(1)]. The reader may wish to note that in [6], the notation $\mathcal{M}_n$ is used for the set of all flat measures on $\mathbb{R}^n$, see [6, 3.7(1)], and $\mathcal{U}_n$ for the set of all uniformly distributed measures on $\mathbb{R}^n$ which have 0 in their support, see [6, 3.1(2)]. Theorem 5 is proved in [4].

Lemma 2 *If $\mu_k \to \mu$ are measures on $\mathbb{R}^n$ then for each compact set $D \subset \mathbb{R}^n$ and each open set $G \subset \mathbb{R}^n$,*

$$\mu(D) \geq \limsup_{k\to\infty} \mu_k(D),$$
$$\mu(G) \leq \liminf_{k\to\infty} \mu_k(G).$$

Theorem 2 *Let μ measure $\mathbb{R}^n$. Then μ almost every $x \in \mathbb{R}^n$ is a* point of translational invariance of $\operatorname{Tan}(\mu, x)$*, that is, μ almost every $x \in \mathbb{R}^n$ has the following property: Whenever $\psi \in \operatorname{Tan}(\mu, x)$ and $u \in \operatorname{spt}\ \psi$ then*

$$T_{u,1}[\psi] \in \operatorname{Tan}(\mu, x).$$

Theorem 3 *If μ measures $\mathbb{R}^n$, h is a regular density function, and μ almost every point of $\mathbb{R}^n$ is an h-density point of μ, then at μ almost every point x of $\mathbb{R}^n$, every tangent measure to μ at x is flat. Conversely, if μ almost every point of $\mathbb{R}^n$ is an h-density point of μ, and at μ almost every point of $\mathbb{R}^n$, every tangent measure to μ at x is flat, then h is a regular density function.*

Theorem 4 *If μ is a locally finite measure on $\mathbb{R}^n$ and almost every point of $\mathbb{R}^n$ is an h-density point of μ, then at almost every point of $\mathbb{R}^n$, every tangent measure ψ to μ at x is uniformly distributed, with $0 \in \operatorname{spt}\ \psi$.*

Theorem 5 *Let μ be a Radon measure on $\mathbb{R}^n$. If for μ almost every point x in $\mathbb{R}^n$, every tangent measure to μ at x has 0 as a symmetric point, then at μ almost every point x in $\mathbb{R}^n$, every tangent measure to μ at x is flat.*

To prove our result for more general Hausdorff functions h we need two simple density lemmas for the measures $\mathcal{H}^h$ and $\mathcal{P}^h$, which replace the standard density lemmas for $\mathcal{H}^s$ and $\mathcal{P}^s$. We state Lemma 3 without proof, referring the reader to [3, 6.10]. The result proved there is again only for functions $h(r) = r^s$, but generalizes without trouble to all other Hausdorff functions.

Lemma 3 *If $A \subseteq \mathbb{R}^n$ satisfies $0 < \mathcal{P}^h(A) < \infty$, then for $\mathcal{P}^h|_A$ almost every $x \in \mathbb{R}^n$,*

$$\underline{D}^h\left(\mathcal{P}^h|_A, x\right) \geq 1.$$

The proof of Lemma 4 requires the following covering theorem, which is due to Morse, see [5].

Theorem 6 *Let μ be a Radon measure in $\mathbb{R}^n$, $A \subseteq \mathbb{R}^n$, $0 \leq \alpha < 1$, and let $\mathcal{B}$ be a family of closed balls in $\mathbb{R}^n$ such that for each point y of A and each $r > 0$ we may find a ball $B(x,s) \in \mathcal{B}$ with $s \leq r$ and $y \in B(x, \alpha s)$. Then there is a countable collection of disjoint balls $\{B_i\} \subseteq \mathcal{B}$ such that $\mu\left(A \setminus \bigcup_i B_i\right) = 0$.*

Lemma 4 *Let $A \subseteq \mathbb{R}^n$ satisfy $\mathcal{H}^h(A) < \infty$. Then, for $\mathcal{H}^h|_A$ almost every x, for any $0 < \alpha < 1$ and $t > 1$ there is $r > 0$ such that, for every $s \leq r$ and every $y \in B(x, \alpha s)$,*

$$\frac{\mathcal{H}^h\left(B\left(y,s\right) \cap A\right)}{h(2s)} \leq t.$$

In particular,

$$\overline{D}^h(\mathcal{H}^h|_A, x) \leq 1.$$

Proof. Since $\mathcal{H}^h$ is regular we may assume that A is $\mathcal{H}^h$ measurable. For $0 < \alpha < 1$ and $t > 1$ write

$$\begin{aligned} A_{\alpha,t} = \{x \in A : &\text{for each } r > 0, \text{ there are } s \leq r \text{ and } y \in U(x, \alpha s) \\ &\text{such that } \mathcal{H}^h\left(A \cap B\left(y,s\right)\right) > t\, h(2s)\}. \end{aligned}$$

It is sufficient to show that $\mathcal{H}^h\left(A_{\alpha,t}\right) = 0$ for any $0 < \alpha < 1$ and $t > 1$.

Fix $0 < \alpha < 1$ and $t > 1$, choose $\varepsilon > 0$, and let K be a compact subset of $A_{\alpha,t}$ satisfying $\mathcal{H}^h(K) \geq (1-\varepsilon)\mathcal{H}^h\left(A_{\alpha,t}\right)$. We may now choose $\delta_i \searrow 0$, and use Theorem 6 to choose disjoint balls $\{B_{i,j}\}_{j=1}^{\infty} = \{B\left(y_{i,j}, r_{i,j}\right)\}_{j=1}^{\infty}$ for each i, such that

(i) $K \cap B\left(y_{i,j}, \alpha r_{i,j}\right) \neq \emptyset$,

(ii) $r_{i,j} \leq \delta_i/2$,

(iii) $\mathcal{H}^h\left(A \cap B_{i,j}\right) > t\, h\left(2r_{i,j}\right)$,

(iv) $\mathcal{H}^h\left(K \setminus \cup_j B_{i,j}\right) = 0$.

If $y \in \bigcup_{i\geq k}\bigcup_j B_{i,j}$, then $\operatorname{dist}(K,y) < \delta_k$. So if $y \in \bigcap_{k\geq 1}\bigcup_{i\geq k}\bigcup_j B_{i,j}$, then $\operatorname{dist}(K,y) = 0$, so $y \in K$. Therefore

$$\begin{aligned} \mathcal{H}^h(A_{\alpha,t}) \geq &\mathcal{H}^h(K) \geq \mathcal{H}^h\left(\bigcap_{k\geq 1}\bigcup_{i\geq k}\bigcup_j B_{i,j}\right) \\ &= \lim_{k\to\infty} \mathcal{H}^h\left(A \cap \bigcup_{i\geq k}\bigcup_j B_{i,j}\right) \geq \limsup_{k\to\infty} \mathcal{H}^h\left(A \cap \bigcup_j B_{k,j}\right) \end{aligned}$$

$$
\begin{aligned}
&= \limsup_{k\to\infty} \sum_{j=1}^{\infty} \mathcal{H}^h \left(A \cap B_{k,j}\right) > \limsup_{k\to\infty}\ t \sum_{j=1}^{\infty} h\left(2r_{k,j}\right) \\
&\geq t \limsup_{k\to\infty} H^h_{\delta_k} \left(K \cap \bigcup_j B_{k,j} \right) = t\mathcal{H}^h\left(K\right) \\
&\geq t\left(1-\varepsilon\right)\mathcal{H}^h\left(A_{\alpha,t}\right).
\end{aligned}
$$

Letting $\varepsilon \searrow 0$, we see that $\mathcal{H}^h\left(A_{\alpha,t}\right) = 0$. The second statement of the lemma follows immediately.

Theorem 7 *Let $A \subseteq \mathbb{R}^n$, and $\mu = \mathcal{P}^h|_A = \mathcal{H}^h|_A$ be a positive finite measure. Then, for μ almost every $x \in A$, every tangent measure to μ at x is flat and h is a regular density function.*

PROOF. Write A^* for those points of A which are exceptional points of neither Lemma 3 nor Lemma 4, at which all tangent measures have 0 in their support and are uniformly distributed, and which are points of translational invariance of $\mathrm{Tan}(\mu, x)$. Then $\mu(A^*) = \mu(A)$, and every point of A^* is an h-density point of μ.

Fix $x \in A^*$. We now show that for each $z \in \mathbb{R}^n$, each $\rho > 0$, and each tangent measure ψ to μ at x,

$$\psi B(z,\rho) \leq \psi B(0,\rho). \tag{1}$$

Fix $\psi \in \mathrm{Tan}(\mu, x)$ and $\rho > 0$. Since ψ is uniformly distributed and $0 \in \mathrm{spt}\ \psi$, it only remains to show the required inequality for $z \notin \mathrm{spt}\ \psi$.

We first suppose that $z \in U(0,\rho)$. Since

$$\psi = \lim_{k\to\infty} c_k T_{x,r_k}[\mu],$$

we have by Lemma 2 that

$$\psi U(z,\rho) \leq \liminf_{k\to\infty} c_k \mu U(x + r_k z, r_k\rho).$$

Since $z \in B(0, \alpha\rho)$ for some $\alpha < 1$, since x is an exceptional point of neither Lemma 3 nor Lemma 4, and since $x + r_k z \in U(x, r_k\rho)$, we see that for each $t > 1$ there is a number k_1 such that if $k > k_1$ then

$$\mu U(x + r_k z, r_k\rho) \leq \mu B(x + r_k z, r_k\rho) \leq t\mu B(x, r_k\rho).$$

Therefore

$$\psi U(z,\rho) \leq \liminf_{k\to\infty} c_k \mu U(x + r_k z, r_k\rho) \leq \limsup_{k\to\infty} c_k \mu B(x, r_k\rho) \leq \psi B(0,\rho).$$

Replacing ρ by $\rho + \delta/2$ in the above calculations, we see that

$$\psi U(z, \rho + \delta/2) \le \psi B(0, \rho + \delta/2).$$

The measure ψ is Radon, so for each $\varepsilon > 0$ we may find $\delta > 0$ such that

$$\psi U(0, \rho + \delta) \le \psi B(0, \rho) + \varepsilon,$$

so

$$\psi B(z, \rho) \le \psi U(z, \rho + \delta/2) \le \psi B(0, \rho + \delta/2) \le \psi U(0, \rho + \delta) \le \psi B(0, \rho) + \varepsilon.$$

The choice of $\varepsilon > 0$ was arbitrary, so

$$\psi B(z, \rho) \le \psi B(0, \rho).$$

Now suppose that $z \in \partial B(0, \rho)$; then for each $\rho_1 > \rho$ we have $z \in U(0, \rho_1)$, and $\psi B(z, \rho_1) \le \psi B(0, \rho_1)$. Therefore

$$\psi B(z, \rho) \le \psi B(0, \rho_1) \text{ for each } \rho_1 > \rho,$$

and

$$\psi B(z, \rho) \le \lim_{\rho_1 \to \rho} \psi B(0, \rho_1) = \psi \left(\bigcap_{\rho_1 > \rho} B(0, \rho_1) \right) = \psi B(0, \rho).$$

The third case we must consider is that where $z \notin B(0, \rho)$. If $B(z, \rho) \cap \mathrm{spt}\ \psi = \emptyset$, the inequality $\psi B(z, \rho) \le \psi B(0, \rho)$ is obvious. If $B(z, \rho) \cap \mathrm{spt}\ \psi \ne \emptyset$, we may choose $w \in B(z, \rho) \cap \mathrm{spt}\ \psi$ and use the fact that x is a point of translational invariance of $\mathrm{Tan}(\mu, x)$ to see that

$$T_{w,1}[\psi] B(z - w, \rho) \le T_{w,1}[\psi] B(0, \rho),$$

and so

$$\psi B(z, \rho) \le \psi B(w, \rho) \le \psi B(0, \rho).$$

So every measure ψ in $\mathrm{Tan}(\mu, x)$ indeed satisfies inequality (1).

It is now not hard to show that 0 is a symmetric point of each measure $\psi \in \mathrm{Tan}(\mu, x)$.

Fix ψ in $\mathrm{Tan}(\mu, x)$ and let $\rho > 0$. For $y \in \mathbb{R}^n$, define

$$F(y) = \int \left(\rho^2 - \|z - y\|^2 \right) \chi_{B(y,\rho)}(z) d\psi(z).$$

Then by Fubini's Theorem and equation (1),

$$
\begin{aligned}
F(y) &= \int_0^\infty \psi \left\{ z : \left(\rho^2 - \|z-y\|^2\right) \chi_{B(y,\rho)}(z) > t \right\} dt \\
&= \int_0^{\rho^2} \psi B\left(y, \sqrt{\rho^2 - t}\right) dt \\
&\leq \int_0^{\rho^2} \psi B\left(0, \sqrt{\rho^2 - t}\right) dt = F(0).
\end{aligned}
$$

Since F is easily seen to be differentiable on $B(y,\rho)$, this implies that 0 is a maximum for F and that $\nabla F(0) = -2\int_{B(0,\rho)} z\, d\psi(z) = 0$, that is, 0 is a symmetric point of ψ. The result then follows from Theorem 5 and the second implication of Theorem 3.

Lemma 5 *If μ is a Radon measure on $\mathbb{R}^n$, A is a compact subset of $\mathbb{R}^n$, h is a regular density function, and $\underline{D}^h(\mu, x) \geq 1$ for all $x \in A$, then $\mu(A) \geq \mathcal{P}^h(A)$.*

Proof. Since μ is Radon, $\mu(A) < \infty$. For $t < 1$ and $\delta > 0$ write

$$A_{t,\delta} = \{x \in A : \mu B(x,r) \geq t\, h(2r) \text{ whenever } r \leq \delta/2\}.$$

Fix $t < 1$ and $\delta > 0$, then for every $0 < \eta \leq \delta$,

$$
\begin{aligned}
\mu B(A_{t,\delta}, \eta) &\geq t P_\eta^h(A_{t,\delta}), \\
\mu(\text{Clos } A_{t,\delta}) &\geq t P_0^h(A_{t,\delta}) = t P_0^h(\text{Clos } A_{t,\delta}),
\end{aligned}
$$

since if $\{B(x_i, r_i)\}$ is an η-packing of $A_{t,\delta}$, then $\mu B(x_i, r_i) \geq t\, h(2r_i)$, and the compact set $B(A_{t,\delta}, \eta)$ contains the disjoint union $\bigcup B(x_i, r_i)$. (The last equality just uses the well-known fact, see for example [3, 5.10], that if h is a regular density function, then $P_0^h(S) = P_0^h(\text{Clos } S)$ for each subset S of $\mathbb{R}^n$.)

By assumption, $A = \bigcup_{\delta>0} \text{Clos}(A_{t,\delta})$ for each $t < 1$. The measures μ and $\mathcal{P}^h$ are Borel regular and $A_{t,1/n} \subseteq A_{t,1/(n+1)}$ for each n, so for each $t < 1$,

$$
\begin{aligned}
\mathcal{P}^h(A) =& \mathcal{P}^h\Big(\bigcup_{n=1}^\infty \text{Clos } A_{t,1/n}\Big) = \lim_{n\to\infty} \mathcal{P}^h(\text{Clos } A_{t,1/n}) \\
\leq& \lim_{n\to\infty} P_0^h(\text{Clos } A_{t,1/n}) \leq t^{-1} \lim_{n\to\infty} \mu\left(\text{Clos } A_{t,1/n}\right) \\
=& t^{-1}\mu\Big(\bigcup_{n=1}^\infty \text{Clos } A_{t,1/n}\Big) = t^{-1}\mu(A).
\end{aligned}
$$

Since $t < 1$ was arbitrary, we have $\mu(A) \geq \mathcal{P}^h(A)$, as required.

Lemma 6 *If μ is a Radon measure on $\mathbb{R}^n$ and for each $x \in A$, $\overline{D}^h(\mu, x) \leq 1$ and all tangent measures to μ at x are flat, then for each $x \in A$,*

$$\lim_{\varepsilon \searrow 0} \left(\sup \left\{ \frac{\mu(D)}{h(\operatorname{diam}\ D)} : x \in D,\ \operatorname{diam}(D) < \varepsilon,\ D \textit{ compact, convex} \right\} \right) \leq 1.$$

Proof. Suppose not, then for some $x \in A$, without loss of generality $x = 0$, we may find numbers $c_k > 0,\ t > 1,\ r_k \searrow 0$, compact convex sets D_k of diameter 1 and containing x, $m \in \{1, \ldots, n\}$ and an m-dimensional linear subspace V of $\mathbb{R}^n$, such that

(i) $D_k \to D$ (a nonempty compact convex set with $\operatorname{diam}(D) \leq 1$) in the Hausdorff metric,

(ii) $c_k T_{x,r_k}[\mu] \to \mathcal{H}^m|_V \in \operatorname{Tan}(\mu, x)$,

(iii) $\mu(r_k D_k)/h(r_k) \geq t$ for each k.

Write $E_k = \operatorname{Clos}\left(\bigcup_{l \geq k} D_l\right)$, then $E_{k+1} \subseteq E_k$ and $\mathcal{H}^m|_V(E_k) \to \mathcal{H}^m|_V(D)$. Also,

$$\mu(r_l E_l) \geq \mu(r_l D_l) \geq t\, h(r_l),\ \text{for each } l.$$

Since

$$\mathcal{H}^m|_V = \lim_{l \to \infty} c_l T_{x,r_l}[\mu],$$

we may use Lemma 2 to see that

$$\mathcal{H}^m|_V(E_k) \geq \limsup_{l \to \infty} c_l \mu(r_l E_k) \text{ for each } k.$$

Choose k_1 so large that whenever $k \geq k_1$,

$$\mathcal{H}^m|_V(E_k) \leq \frac{(1+t)}{2} \mathcal{H}^m|_V(D).$$

Then, using the isodiametric inequality, for each $k \geq k_1$ we have

$$\frac{(1+t)}{2} \mathcal{H}^m|_V B(0, 1/2) \geq \frac{(1+t)}{2} \mathcal{H}^m|_V(D) \geq \limsup_{l \to \infty} c_l \mu(r_l E_k).$$

Since $E_{k+1} \subseteq E_k$ for each k,

$$\limsup_{l \to \infty} c_l \mu(r_l E_k) \geq \limsup_{l \to \infty} c_l \mu(r_l E_l).$$

Therefore

$$\frac{(1+t)}{2} \mathcal{H}^m|_V B(0, 1/2) \geq \limsup_{l \to \infty} c_l \mu(r_l E_l) \geq \limsup_{l \to \infty} c_l t\, h(r_l).$$

Since by assumption $\overline{D}^h(\mu, x) \leq 1$ for each $x \in A$,

$$\limsup_{l\to\infty} \mu\left(B(x, r_l/2)\right) h(r_l)^{-1} \leq 1.$$

Therefore

$$\begin{aligned}\frac{(1+t)}{2}\mathcal{H}^m|_V B(0,1/2) \geq & t \limsup_{l\to\infty} c_l \mu B(x, r_l/2) \\ \geq & t \liminf_{l\to\infty} c_l \mu U(x, r_l/2) \\ \geq & t\mathcal{H}^m|_V \left(U\left(0, 1/2\right)\right).\end{aligned}$$

So $t \leq (1+t)/2$, and $t \leq 1$, which is a contradiction.

Lemma 7 *If μ is a Radon measure on $\mathbb{R}^n$ and $A \subseteq \mathbb{R}^n$ such that for every point of A*

$$\lim_{\varepsilon\searrow 0}\left(\sup\left\{\frac{\mu(D)}{h(\mathrm{diam}\ \ D)} : x \in D,\ \mathrm{diam}(D) < \varepsilon,\ D \textit{ compact, convex}\right\}\right) \leq 1,$$

then

$$\mu(A) \leq \mathcal{H}^h(A).$$

For a proof of this lemma we refer the reader to [2, 2.10.17(2)].

We are now in a position to prove our main result.

Theorem 8 *If $A \subseteq \mathbb{R}^n$ and $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$ is a positive finite measure, then h is a regular density function and μ has h-density 1 almost everywhere. Conversely, for each regular density function h, there is a positive finite measure μ on $\mathbb{R}^n$ with h-density 1 almost everywhere, such that $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$ for some $A \subseteq \mathbb{R}^n$.*

Proof. Lemmas 3 and 4 together imply that if $\mu = \mathcal{H}^h|_A = \mathcal{P}^h|_A$, and μ is positive and finite, then μ has density 1 almost everywhere. Theorem 7 implies that h is regular.

In [6, 6.5], for each regular density function h there is given a construction of a Radon measure μ on $\mathbb{R}^n$ which has positive finite constant h-density μ almost everywhere in $\mathbb{R}^n$. We normalize μ to have h-density 1 almost everywhere and write D for the set where the h-density of μ is 1. Now D is a G_δ set with $\mu(D) > 0$, so we may find a compact subset C of D with $\mu(C) > 0$. Then Lemma 5 tells us that $\mu|_C(S) \geq \mathcal{P}^h|_C(S)$ for all closed subsets S of $\mathbb{R}^n$.

Theorem 3 ensures that, for $\mu|_C$ almost every x, every tangent measure to $\mu|_C$ at x is flat, and so we may use Lemmas 6 and 7 to show that $\mu|_C(S) \leq \mathcal{H}^h|_C(S)$

for all measurable subsets S of $\mathbb{R}^n$. Since h is regular Lemma 1 implies that $\mathcal{H}^h(A) \leq \mathcal{P}^h(A)$ for all $A \subseteq \mathbb{R}^n$. Therefore $\mathcal{P}^h|_C$, $\mu|_C$ and $\mathcal{H}^h|_C$ agree on closed, and therefore on all, subsets of $\mathbb{R}^n$.

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