

Shusheng Fu*, Department of Mathematics, Fuzhou University, Fuzhou
Fujian, China

ON SOME THEOREMS IN FRACTIONAL CALCULUS FOR SINGULAR FUNCTIONS

Abstract

In this paper we show that there are close relationships amongst Cantor bar totality, non-integer integral and Hausdorff integral. If a singular function f has zero Lipschitz $(1 - \nu)$ -numbers on a Cantor set C with $H\text{-dim } C = 1 - \nu$, $0 < \nu < 1$, then the $1 - \nu$ order fractional derivative of f exists almost everywhere on $[0, T]$. Moreover, under strong assumptions on the function f the $1 - \nu$ order derivative of f exists everywhere. Consequently, the ν order fractional integral of f equals $f(0)t^\nu$. Using the Concept of th Hausdorff derivative we prove that if a singular function has a Hausdorff derivative on C , then the fractional derivative of f exists almost everywhere on $[0, T]$. Finally, under some assumptions on f and C , we establish

$$\begin{aligned} (D^{-\nu} f)(t) &= \int_0^t \int_0^u v^{\nu-1} f'_H(u-v) dv^{1-\nu} du \\ &= \int_0^t u^{1-\nu} J_\nu(v^{\nu-1} f'_H(u-v))(u) du \end{aligned}$$

This identity is similar to the identity

$$(D^{-\nu} f)(t) = \frac{1}{B_\nu} t^\nu (J_\nu f)(t)$$

which professor R. R. Nigmatullin claimed.

Key Words: Fractional integral, Fractional derivative, Hausdorff integral, Hausdorff derivative, Lipschitz number, singular function, Cantor type set

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1 Introduction

In his paper R. R. Nigmatullin [7] pointed out that there is a relationship between Cantor bar totality and non-integer integral. He claimed that

$$(D^{-\nu} f)(t) = \frac{1}{B_\nu} t^\nu (J_\nu f)(t) \quad B_\nu \text{ constant} \tag{1}$$

where

$$(D^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du$$

$$(J_\nu f)(t) = \lim_{n \rightarrow \infty} \frac{1}{(2\xi)^n t} \int_0^t d\tau \sum_{m=1}^{2^n} \eta(t_{2m-1}^{(n)} < \tau < t_{2m}^{(n)}) f(\tau),$$

see [7, p. 248]. Some mathematicians doubt identity (1). R. S. Rutman [8] and S. Kempfle [4] showed that in fact (1) is not true. In my previous work [2] I have shown that the whole convolutional integral $(J_\nu f)(t)$ is equivalent to the Hausdorff integral. In this paper we show that there are close relationships amongst Cantor bar totality, non-integer integral and Hausdorff integral. Moreover, we will give an identity which is similar to the identity which R. R. Nigmatullin claimed. The identity is

$$(D^{-\nu} f)(t) = \int_0^t u^{1-\nu} J_\nu(v^{\nu-1} f'_H(u-v))(u) du.$$

B. B. Mandelbrot states “any nondecreasing function can be written as the sum of a singular function, of a function made of discrete jumps, and a differentiable function. The last two components are classical in mathematics and of wide use in physics. On the other hand, the singular component is widely regarded in physics as pathological and totally devoid of uses. A principal theme of this Essay is that this last opinion is totally devoid of merit”. Let us consider the evolution of some physical systems with losses. In the process there is no loss almost all the time, but there are losses in exceptional time. (The exceptional time subset has uncountable elements but its Lebesgue measure is zero.) Therefore, the functions which describe this phenomenon remain constant in most time and vary in exceptional time. The Cantor type functions (a kind of singular functions) possess this property; so they are suitable for studying the phenomenon mentioned above.

We begin with a theorem about the existence of the fractional derivatives. If a singular function f has zero Lipschitz $(1 - \nu)$ -numbers on a Cantor set C with $H\text{-dim } C = 1 - \nu$, then the $1 - \nu$ order fractional derivative of f exist

almost everywhere on $[0, T]$. This answers, in one way, an open problem posed by C. Tricot [10], “Are there any relationships between the Lipschitz η -number and η order fractional derivative of function f ?” Other answers to Tricot’s question were given by Hardy and Littlewood [3]. Moreover, under strong assumptions on the function f the ν order derivative of f exist everywhere on $[0, T]$. By elementary Calculus the $1 - \nu$ order fractional integral of f equals $f(0)t^\nu$. That is, if a Cantor type function f has bounded Lipschitz $(1 - \nu + \varepsilon)$ -numbers on C , then the ν order integral of f is equal to the power function $f(0)t^\nu$. Using the concept of Hausdorff derivative we prove that if a Cantor type function f has a Hausdorff derivative on C , then the sequence derivative of the Riemann-Liouville fractional integral of order ν of the function f exist almost everywhere on $[0, T]$. Finally, under additional conditions on f and C , we establish the identity

$$(D^{-\nu} f)(t) = \int_0^t u^{1-\nu} J_\nu(v^{\nu-1} f'_H(u-v))(u) du.$$

2 Notations and Terminology

The Cantor dust of dimension ν is constructed in the usual way. Let $[0, T]$ be an interval and let $0 < \xi < \frac{1}{2}$. A partition of $[0, T]$ is given by

$$a_1^1 = 0 < b_1^1 = \xi T < a_1^2 = (1 - \xi)T < b_1^2 = T,$$

write $U_1 = U_1^1 \cup U_1^2$, where $U_1^1 = [a_1^1, b_1^1]$, $U_1^2 = [a_1^2, b_1^2]$. A partition of U_1^j , $j = 1, 2$, is given by

$$a_2^1 = a_1^1 < b_2^1 < a_2^2 < b_2^2 = b_1^1 < a_2^3 = a_1^2 < b_2^3 < a_2^4 < b_2^4 = b_1^2,$$

where

$$b_2^1 = a_1^1 + \xi^2 T, \quad a_2^2 = b_1^1 - \xi^2 T, \quad b_2^3 = a_1^2 + \xi^2 T, \quad a_2^4 = b_1^2 - \xi^2 T.$$

Continue this process to obtain a sequence of intervals $\{U_n^i\}_{i=1}^{2^n}$, $n = 1, 2, \dots$. Let

$$U_n = \bigcup_{i=1}^{2^n} U_n^i, \quad \text{and} \quad C = \bigcap_{n=1}^{\infty} U_n.$$

We call the closed set C a Cantor dust. It is well known that $mC = 0$, $\nu = H\text{-dim } C = \frac{\log 2}{\log \frac{1}{\xi}}$ and $0 < H^\nu(C) < \infty$, where $H\text{-dim}$ denotes Hausdorff dimension and H^ν Hausdorff measure. The points a_n^i or b_n^i , $i = 1, 2, \dots, 2^n$,

$n = 1, 2, \dots$ are end points of C ; The points $t \in C \setminus \{a_n^i, b_n^i\}$ are interior points of C in the sense of related topology. We call a continuous function f a Cantor type function on C if f is constant on each open interval of $[0, T] \setminus C$. Throughout this paper we will consider Cantor type functions.

The discussion will be related to the concepts of fractional calculus, Hausdorff calculus and others.

2.1 Fractional Integral and Fractional Derivative

Let ν be a real positive number and f be a continuous function. The expression

$$(D^{-\nu} f)(t) = \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} f(t-u) du$$

is the Riemann-Liouville fractional integral of order ν of the function f [10]. The ν order derivative of the function f is defined as

$$(D^\nu f)(t) = \frac{d}{dt} \frac{1}{\Gamma(1-\nu)} \int_0^t u^{-\nu} f(t-u) du$$

2.2 Hausdorff Integral and Hausdorff Derivative

For any singular function f on C [2], we define the net extreme Hausdorff derivative as

$$\begin{aligned} \overline{N-HDF}(t) &= \lim_{n \rightarrow \infty} \frac{f(b_n^{it}) - f(a_n^{it})}{H^\nu(C \cap [a_n^{it}, b_n^{it}])}, \\ \underline{N-HDF}(t) &= \lim_{n \rightarrow \infty} \frac{f(b_n^{it}) - f(a_n^{it})}{H^\nu(C \cap [a_n^{it}, b_n^{it}])}, \end{aligned} \quad t \in (a_n^{it}, b_n^{it}), n = 1, 2, \dots$$

If $\overline{N-HDF}(t) = \underline{N-HDF}(t)$, we say that the function $f(t)$ is net Hausdorff differentiable at t ; this limit is termed the net Hausdorff derivative of $f(t)$ at t and is denoted by $f'_H(t)$.

A real valued function f on C with $H\text{-dim } C = S$ is said to be Hausdorff integrable on $[a, b]$, if there exists r , such that for every $\epsilon > 0$ there is a gauge function $\delta(t) > 0$ on C , such that for any net δ -fine division $D = \{([u, v], t)\}$ of C , we have

$$\left| (D) \sum f(t) H^\nu(C \cap [u, v]) - r \right| < \epsilon.$$

As usual, we write $\int_a^b f(t) dt^s = r$.

2.3 Sequence Derivative

Let $\{h_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $h_n \downarrow 0$, let $f : [0, T] \rightarrow \mathbb{R}$ and let $t \in (0, T)$. The function f has a sequence derivative with $\{h_n\}_{n=1}^{\infty}$ at t if both the limits

$$\lim_{n \rightarrow \infty} \frac{f(t - h_n) - f(t)}{-h_n} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{f(t + h_n) - f(t)}{h_n}$$

exist and are equal. We will use $(q) \frac{df}{dt}$ to denote the sequence derivative of f at t .

2.4 Lipschitz Number

The derivative of the form

$$L^s(f, t) = \lim_{\substack{y, z \rightarrow x \\ y < x < z}} \frac{f(z) - f(y)}{(z - y)^s}$$

following Besicovich [9, p. 1] we call the Lipschitz s -number.

2.5 Self-Similar Property

For each $t \in C$, there exists a sequence of intervals $\{[a_n^{i_{t_n}}, b_n^{i_{t_n}}]\}$ such that $t \in [a_n^{i_{t_n}}, b_n^{i_{t_n}}]$. A Cantor type function f has the self-similar property if there exists a constant β such that

$$\lim_{n \rightarrow \infty} \frac{\int_{a_n^{i_{t_n}}}^{b_n^{i_{t_n}}} [f(u) - f(a_n^{i_{t_n}})] du}{\int_{a_n^{i_{t_n}}}^{b_n^{i_{t_n}}} f'_H(u) du^s} = \beta$$

where $t \in [a_n^{i_{t_n}}, b_n^{i_{t_n}}], t \in C; \int_{a_n^{i_{t_n}}}^{b_n^{i_{t_n}}} f'_H(u) du^s \neq 0$ for each n and $H\text{-dim } C = S$.

For example the Cantor ternary function has the self-similar property.

2.6 Notations from Henstock Integration Theory

Let I be the set of all closed intervals $[u, v]$ in $[a, b]$ having a nonempty interior. An element $([u, v], t) \in I \times [a, b]$ is called an interval-point pair. We write $D = \{[u, v], t\}$ where $([u, v], t)$ denotes a typical interval-point pair in D . Then D is said to be a partition of $[a, b]$ (C resp.) if $\{[u, v]; ([u, v], t) \in D\}$ is a finite collection of nonoverlapping subintervals of $[a, b]$ and $[a, b] =$

$\cup_{([\alpha, \beta], t) \in D} [\alpha, \beta](C = \cup_{([\alpha, \beta], t) \in D} [\alpha, \beta] \cap C)$. Let $\delta(t)$ be a positive function on $[a, b](E)$. The partition D is called a δ -fine division of $[a, b](E)$ if $([\alpha, \beta], t) \in D$, then $\beta - \alpha \leq \delta(t)$.

3 Main Result

We assume for simplicity and without loss of generality that in the following, the Cantor type functions f are nondecreasing.

Theorem 1 *Let f be a Cantor type function on C with $H\text{-dim } C = 1 - \nu$, $0 < \nu < 1$. Suppose that the Lipschitz $(1 - \nu)$ -numbers $L^{1-\nu}(f, t) = 0$ uniformly on C . Then*

$$D^{1-\nu} f(t) = \frac{f(0)}{\Gamma(\nu)} t^{\nu-1}$$

for $t \in [0, T] \setminus C$. That is the $1 - \nu$ order derivative of the function f exists almost everywhere.

PROOF. We can write

$$\begin{aligned} (D^{-\nu} f(t+h) - D^{-\nu} f(t)) &= \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} [f(t+h-u) - f(t-u)] du \\ &\quad + \frac{1}{\Gamma(\nu)} \int_t^{t+h} u^{\nu-1} f(t+h-u) du \\ &= I_1(t, h) + I_2(t, h) \end{aligned}$$

where $h > 0$. Clearly

$$\frac{I_2(t, h)}{h} = \frac{(t+\theta h)^{\nu-1}}{\Gamma(\nu)} f(\overline{1-\theta h}) \rightarrow \frac{f(0)}{\Gamma(\nu)} t^{\nu-1}$$

for $h \rightarrow 0$.

We will show that $\frac{I_1(t, h)}{h} = o(1)(h \rightarrow 0)$ for $t \in [0, T] \setminus C$. Let n_t be an index for which $t \in (b_{n_t}^{i_t}, a_{n_t}^{i_t+1})$. Let $\varepsilon > 0$. Under the assumption on the Lipschitz numbers of f , there is a $\delta_1 > 0$ such that

$$\frac{|f(u+P_1) - f(u-P_2)|}{(P_1+P_2)^{1-\nu}} < \varepsilon_1$$

where $u \in [0, T]$, $\varepsilon_1 = \frac{(\frac{t-b_{n_t}^{i_t}}{2})^{1-\nu}}{bT^{1-\nu}} \varepsilon$, $P_1 \geq 0, P_2 \geq 0$ and $0 < P_1 + P_2 \leq \delta_1$.

Let $\delta = \min(\delta_1, \frac{t - b_{n_t}^{i_t}}{2})$. For any h , $0 < h < \delta$, choose an index n such that $h_n \leq h \leq h_{n-1}$ and $h_n = \xi^n T$. Then the integral $I'_1(t, h)(I'_1(t, h) = \frac{1}{\Gamma(\nu)} I_1(t, h))$ may be expressed as

$$\begin{aligned} & \int_0^{t-b_{n_t}^{i_t}} + \int_{t-b_{n_t}^{i_t}}^{t-b_{n_t}^{i_t}+h} + \cdots + \int_{t-b_{n_t}^{i_t}+mh}^t u^{\nu-1} [f(t+h-u) - f(t-u)] du \\ &= \int_{I_1} + \int_{I_2} + \cdots + \int_{I_{m+2}} u^{\nu-1} [f(t+h-u) - f(t-u)] du \\ &= \sum_{I_j \in D_1} \int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du \\ & \quad + \sum_{I_j \in D_2} \int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du \end{aligned}$$

where D_1 denotes the collection of all intervals for which the functions $f(t-u)$ and $f(t+h-u)$ are equal for $u \in I_j$; and D_2 the collection of the other terms of $\{I_j\}_{j=1}^{m+2}$. Clearly

$$\sum_{I_j \in D_1} \int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du = 0.$$

Observe that the number of the terms of D_2 is less than 3×2^n . Therefore

$$\left| \sum_{I_j \in D_2} \int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du \right| \leq 3 \times 2^n \frac{\varepsilon_1 h^{1-\nu}}{(\frac{x-b_{n_t}^{i_t}}{2})^{1-\nu}} h.$$

By our choice of the index n ,

$$2^n h^{1-\nu} \leq 2^n h_{n-1}^{1-\nu} \leq 2 \times 2^{n-1} \xi^{(n-1) \frac{\log 2}{\log \xi}} T^{1-\nu} \leq 2T^{1-\nu}.$$

Hence we have

$$\left| \frac{\sum_{I_j \in D_2} \int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du}{h} \right| \leq \varepsilon_1 \frac{6T^{1-\nu}}{(\frac{x-b_{n_t}^{i_t}}{2})^{1-\nu}} = \varepsilon$$

for $0 < h \leq \delta$ as required. The case for $h < 0$ is treated in a similar way. \square

Theorem 2 *Let f be a Cantor type function on C with $H\text{-dim } C = 1 - \nu$. Suppose that the Lipschitz r -numbers $L^r(f, t) \leq A$ $1 - \nu < r$ uniformly on C . Then*

$$(D^{-\nu} f)(t) = f(0)t^\nu.$$

PROOF. Under the assumption on the Lipschitz numbers of f , we have

$$L^{1-\nu}(f, t) = 0$$

uniformly on C . By Theorem 1

$$(D^{1-\nu}f)(t) = \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}$$

for $t \in [0, T] \setminus C$. We will show that $D^{1-\nu}f(t) = \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}$ for $t \in C$. Let $r = 1 - \nu + \eta$, $\eta > 0$, and let $t_0 = t - \frac{T^{\frac{\eta}{2(1-\nu)}}}{2}$. We may write

$$\begin{aligned} D^{-\nu}f(t+h) - D^{-\nu}f(t) &= \frac{1}{\Gamma(\nu)} \int_0^{\frac{T^{\frac{\eta}{2(1-\nu)}}}{2}} u^{\nu-1} [f(t+h-u) - f(t-u)] du \\ &\quad + \frac{1}{\Gamma(\nu)} \int_t^{t+h} u^{\nu-1} f(t+h-u) du \\ &= I_1(t, h) + I_2(t, h) + I_3(t, h). \end{aligned}$$

Arguing as in the proof of Theorem 1 shows that

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{I_2(t, h)}{h} &= 0, \\ \lim_{h \rightarrow 0} \frac{I_3(t, h)}{h} &= \frac{f(0)}{\Gamma(\nu)}t^{\nu-1}. \end{aligned}$$

We decompose $I_1(t, h)$ into two groups as

$$I_1(t, h) = I_{1.1}(t, h) + I_{1.2}(t, h)$$

where $I_{1.1}(t, h)$ is the sum of all integrals $\int_{I_j} u^{\nu-1} [f(t+h-u) - f(t-u)] du$ for which the functions $f(t-u)$ and $f(t+h-u)$ are equal for $u \in I_j$, and $I_{1.2}(t, h)$ the sum of the other integrals. Consequently, $I_{1.1}(t, h) = 0$. We will show that $\lim_{h \rightarrow 0} \frac{I_{1.2}(t, h)}{h} = 0$.

By the assumption on the Lipschitz numbers of f , there is a $\delta > 0$ such that

$$\frac{|f(t+P_1) - f(t-P_2)|}{(P_1 + P_2)^r} \leq A$$

$P_1 \geq 0, P_2 \geq 0, 0 < P_1 + P_2 \leq \delta$ and $t \in C$, then $t \in [0, T]$. For any $h > 0$, choose an index n such that $h_n \leq h < h_{n-1}$. By the symmetric distribution of

$\{[a_n^i, b_n^i]_{i=1}^{2^n}\}_{n=1,2,\dots}$ the number of the terms of $I_{1,2}(t, h)$ is less than $3[2^{n \frac{\eta}{2(1-\nu)}}]$. Consequently

$$|I_{1,2}(t, h)| \leq 3 \left[2^{n \frac{\eta}{2(1-\nu)}} \right] A \frac{h^r}{h^{1-\nu}} h.$$

Therefore

$$\begin{aligned} \left| \frac{I_{1,2}(t, h)}{h} \right| &\leq 3A2^{n \frac{\eta}{2(1-\nu)}} h^\eta \leq 3A2^{n \frac{\eta}{2(1-\nu)}} h_{n-1}^\eta \\ &\leq 3A2^{n \frac{\eta}{2(1-\nu)}} \frac{T^{\frac{\eta}{2}}}{2^{n \frac{\eta}{2(1-\nu)}}} h_{n-1}^{\frac{\eta}{2}} \leq 3AT^{\frac{\eta}{2}} h_{n-1}^{\frac{\eta}{2}}. \end{aligned}$$

The last inequality shows that $\lim_{h \rightarrow 0} \frac{I_{1,2}(t, h)}{h} = 0$ as required.

We have shown that $(D^{1-\nu} f)(t) = \frac{f(0)}{\Gamma(\nu)} t^{\nu-1}$ for each $t \in [0, T]$. By the elementary calculus we get $(D^{-\nu} f)(t) = f(0)t^\nu$. \square

Theorem 3 *Let f be a Cantor type function on C with $H\text{-dim } C = 1 - \nu$. Suppose that $f'_H(t)$ exists and is continuous on C and f has the self-similar property. Then*

$$(q) \frac{dD^{-\nu} f}{dt}(t) = \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu} + \frac{f(0)}{\Gamma(\nu)} t^{\nu-1}$$

for $t \in [0, T] \setminus C$.

PROOF. We may write

$$\begin{aligned} (D^{-\nu} f)(t+h_n) - D^{-\nu} f(t) &= \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} [f(t+h_n-u) - f(t-u)] du \\ &\quad + \frac{1}{\Gamma(\nu)} \int_t^{t+h_n} u^{\nu-1} f(t+h_n-u) du \\ &= I_1(t, h_n) + I_2(t, h_n) \end{aligned}$$

where $h_n > 0$. Clearly

$$\frac{I_2(t, h_n)}{h_n} = \frac{(t+\theta h_n)^{\nu-1}}{\Gamma(\nu)} f(1-\theta h_n) \rightarrow \frac{f(0)}{\Gamma(\nu)} t^{\nu-1}$$

as $n \rightarrow \infty$.

We show that

$$\lim_{n \rightarrow \infty} \frac{I_1(t, h_n)}{h_n} = \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu} \quad \text{for } t \in [0, T] \setminus C.$$

Let n_t be an index for which $t \in (b_{n_t}^{i_t}, a_{n_t}^{i_t+1})$. Let $\varepsilon > 0$. By the uniformly continuity of the function $u^{\nu-1}$ on $[t - b_{n_t}^{i_t}, T]$ there is a $\delta_1 > 0$, such that

$$|u^{\nu-1} - v^{\nu-1}| < \varepsilon \quad (2)$$

whenever $u, v \in [t - b_{n_t}^{i_t}, T]$ and $|u - v| < \delta_1$. By the self-similarity of f , there is a positive integer N_1 such that

$$\left| \int_{a_n^i}^{b_n^i} [f(u) - f(a)] du - \beta \int_{a_n^i}^{b_n^i} f'_H(u) du^{1-\nu} \right| < \varepsilon \int_{a_n^i}^{b_n^i} f'_H(u) du^{1-\nu} \quad (3)$$

for $n > N_1$. By the continuity of f and f'_H there is a positive integer N_2 such that

$$\left| \frac{1}{h_n} \int_{t-b_n}^t u^{\nu-1} [f(t+h_n-u) - f(t-u)] du \right| < \varepsilon \quad (4)$$

$$\left| \int_{t-b_n'}^{t-a_n'} u^{\nu-1} [f'_H(t-u)] du^{1-\nu} \right| < \varepsilon. \quad (5)$$

Take an integer N such that $h_N < \frac{t-b_{n_t}^{i_t}}{2}$, $2h_N < \delta$ and $N > \max(N_1, N_2)$. Let $n > N$. We decompose $I_1(t, h_n)$ as

$$\begin{aligned} & \left(\int_{t-b_n^m}^{t-b_n^m+h_n} + \int_{t-b_n^m+h_n}^{t-b_n^m+2h_n} \right) + \left(\int_{t-b_n^{m-1}}^{t-b_n^{m-1}+h_n} + \int_{t-b_n^{m-1}+h_n}^{t-b_n^{m-1}+2h_n} \right) + \dots \\ & + \left(\int_{t-b_n^2}^{t-b_n^2+h_n} + \int_{t-b_n^2+h_n}^{t-b_n^2+2h_n} \right) + \int_{t-b_n'}^t u^{\nu-1} [f(t+h_n-u) - f(t-u)] du \end{aligned}$$

where $b_{n_t}^{i_t} = b_n^m$.

First we compute

$$\begin{aligned} I_{1,j} &= \int_{t-b_n^j}^{t-b_n^j+h_n} u^{\nu-1} [f(t+h_n-u) - f(t-u)] du \\ &+ \int_{t-b_n^j+h_n}^{t-b_n^j+2h_n} u^{\nu-1} [f(t+h_n-u) - f(t-u)] du, \quad 2 \leq j \leq m \end{aligned}$$

$$\begin{aligned}
&= \int_{t-b_n^j}^{t-b_n^j+h_n} u^{\nu-1} [f(b_n^j) - f(t-u)] du \\
&\quad + \int_{t-b_n^j+h_n}^{t-b_n^j+2h_n} u^{\nu-1} [f(t+h_n-u) - f(a_n^j)] du \\
&= (t-b_n^j + \theta_j h_n)^{\nu-1} \int_{t-b_n^j}^{t-b_n^j+h_n} [f(b_n^j) - f(u)] du \\
&\quad + (t-b_n^j + \overline{1 + \theta_j} h_n)^{\nu-1} \int_{a_n^j}^{b_n^j} [f(u) - f(a_n^j)] du
\end{aligned}$$

Note that

$$\int_{a_n^j}^{b_n^j} [f(b_n^j) - f(u)] du + \int_{a_n^j}^{b_n^j} [f(u) - f(a_n^j)] du = [f(b_n^j) - f(a_n^j)](b_n^j - a_n^j).$$

We have

$$\begin{aligned}
I_{1,j} &= (t-b_n^j + \theta_j h_n)^{\nu-1} [f(b_n^j) - f(a_n^j)](b_n^j - a_n^j) \\
&\quad + [(t-b_n^j + \overline{1 + \theta_j} h_n)^{\nu-1} - (t-b_n^j + \theta_j h_n)^{\nu-1}] \int_{a_n^j}^{b_n^j} [f(u) - f(a_n^j)] du.
\end{aligned}$$

Using the above identity we show that

$$\frac{I_1(t, h_n)}{h_n} - \frac{1}{\Gamma(\nu)} \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu} = o(1).$$

Because

$$\int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu} = \sum_{j=1}^m \int_{t-b_n^j}^{t-a_n^j} u^{\nu-1} f'_H(t-u) du^{1-\nu},$$

we may write

$$\frac{I_1(t, h_n)}{h_n} - \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu}$$

as

$$\begin{aligned}
&\sum_{j=2}^m \left[(t-b_n^j + \theta_j h_n)^{\nu-1} \int_{a_n^j}^{b_n^j} f'_H(u) du^{1-\nu} - \int_{t-b_n^j}^{t-a_n^j} u^{\nu-1} f'_H(t-u) du^{1-\nu} \right] \\
&\quad + \sum_{j=2}^m [(t-b_n^j + \overline{1 + \theta_j} h_n)^{\nu-1} - (t-b_n^j + \theta_j h_n)^{\nu-1}]
\end{aligned}$$

$$\begin{aligned}
 & \times \left[\frac{1}{h_n} \int_{a_n^j}^{b_n^j} [f(u) - f(a_n^j)] du - \beta \int_{a_n^j}^{b_n^j} f'_H(u) du^{1-\nu} \right] \\
 & + \beta \sum_{j=2}^m [(t - b_n^j + \overline{1 + \theta_j h_n})^{\nu-1} - (t - b_n^j + \theta_j h_n)^{\nu-1}] \int_{a_n^j}^{b_n^j} f'_H(u) du^{1-\nu} \\
 & + \frac{1}{h_n} \int_{t-b_n^1}^t u^{\nu-1} [f(t + h_n - u) - f(t - u)] du - \int_{t-b_n^1}^{t-a_n^1} u^{\nu-1} f'_H(t - u) du^{1-\nu}.
 \end{aligned}$$

This expression combining (2), (3), (4), (5) yields

$$\begin{aligned}
 & \left| \frac{I_1(t, h_n)}{h_n} - \int_0^t u^{\nu-1} f'_H(t - u) du^{1-\nu} \right| \\
 & \leq \sum_{j=2}^m \int_{a_n^j}^{b_n^j} |(t - b_n^j + \theta_j h_n)^{\nu-1} - (t - u)^{\nu-1}| f'_H(u) du^{1-\nu} \\
 & \quad + \varepsilon \sum_{j=2}^m |(t - b_n^j + \overline{1 + \theta_j h_n})^{\nu-1} - (t - b_n^j + \theta_j h_n)^{\nu-1}| \int_{a_n^j}^{b_n^j} f'_H(u) du \\
 & \quad + \beta \sum_{j=2}^m |(t - b_n^j + \overline{1 + \theta_j h_n})^{\nu-1} - (t - b_n^j + \theta_j h_n)^{\nu-1}| \int_{a_n^j}^{b_n^j} f'_H(u) du \\
 & \quad + \frac{1}{h_n} \int_{t-b_n^1}^t u^{\nu-1} |f(t + h_n - u) - f(t - u)| du \\
 & \quad + \int_{t-b_n^1}^{t-a_n^1} u^{\nu-1} f'_H(t - u) du^{1-\nu} \\
 & \leq \varepsilon \int_0^t f'_H(u) du^{1-\nu} + \varepsilon^2 \int_0^t f'_H(u) du^{1-\nu} + \beta \varepsilon \int_0^t f'_H(u) du^{1-\nu} + 2\varepsilon \\
 & = \varepsilon \left[(1 + \varepsilon + \beta) \int_0^t f'_H(u) du^{1-\nu} + 2 \right]
 \end{aligned}$$

as required. \square

Remark 1 The Hausdorff integral $\int_0^t u^{\nu-1} f'_H(t - u) du^{1-\nu}$ exists for $t \in [0, T] \setminus C$, since the integrand function $u^{\nu-1} f'_H(t - u)$ is a continuous function.

Remark 2 Let $G(t) = \int_0^t u^{\nu-1} f'_H(t - u) du^{1-\nu}$. Then G is defined on each interval $(b_n^i, a_n^{i+1}]$, $i = 1, 2, \dots, 2^n$, $n = 1, 2, \dots$. Since $|\int_0^t u^{\nu-1} f'_H(t - u) du^{1-\nu}| \leq$

At $^{\nu-1}$, $t \in (b_n^i, a_n^{i+1}]$, the integral $\int_{b_n^i}^{a_n^{i+1}} G(t) dt$ exists at least in the improper sense.

Lemma 1 Let f be a Cantor type function on C with $H\text{-dim } C = 1 - \nu$. Suppose that

1. $f'_H(t)$ exists uniformly and is continuous for $t \in C$.
2. $\int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu}$ exists and is bounded for $t \in C$.
3. f has self-similar property.
4. $f(0) = 0$ and $\xi \leq \frac{1}{3}$.

Then there is a constant A such that

$$\frac{|(D^{-\nu} f)(b_n^{i_{t_n}}) - (D^{-\nu} f)(a_n^{i_{t_n}})|}{(b_n^{i_{t_n}}) - (a_n^{i_{t_n}})} \leq A$$

where $a_n^{i_{t_n}} \leq t \leq b_n^{i_{t_n}}$, $t \in C$.

PROOF. We may write $(D^{-\nu} f)(b_n^{i_{t_n}}) - (D^{-\nu} f)(a_n^{i_{t_n}})$ as

$$\begin{aligned} & \frac{1}{\Gamma(\nu)} \int_0^{a_n^{i_{t_n}}} u^{\nu-1} [f(b_n^{i_{t_n}} - u) - f(a_n^{i_{t_n}} - u)] du \\ & + \frac{1}{\Gamma(\nu)} \int_{a_n^{i_{t_n}}}^{b_n^{i_{t_n}}} u^{\nu-1} f(b_n^{i_{t_n}} - u) du = I_1(t, h_n) + I_2(t, h_n). \end{aligned}$$

Note that as n increases the ratio $\frac{I_2(t, h_n)}{h_n}$ tends to zero. Therefore, there is a constant A_1 such that

$$\left| \frac{I_2(t, h_n)}{h_n} \right| \leq A_1 \quad (6)$$

$\frac{I_1(t, h_n)}{h_n}$ may be expressed as

$$\begin{aligned} & \frac{1}{h_n} \left[\int_0^{h_n} + \int_{b_n}^{b_n^{i_{t_n}} - b_n^{i_{t_n}-1}} + \int_{b_n^{i_{t_n}} - b_n^{i_{t_n}-1} + 2h}^{b_n^{i_{t_n}} - b_n^{i_{t_n}-1} + 2h} + \int_{b_n^{i_{t_n}} - b_n^{i_{t_n}-1} + 2h}^{b_n^{i_{t_n}} - b_n^{i_{t_n}-2}} + \dots \right. \\ & \left. + \int_{b_n^{i_{t_n}} - b_n^2}^{b_n^{i_{t_n}} - b_n^2 + 2h} + \int_{b_n^{i_{t_n}} - b_n^2 + 2h}^{b_n^{i_{t_n}} - b_n^1} u^{\nu-1} [f(b_n^{i_{t_n}} - u) - f(a_n^{i_{t_n}} - u)] du \right] \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{h_n} \left[\int_0^{h_n} + \int_{b_n^{i_{t_n}} - b_n^{i_{t_n}-1}}^{b_n^{i_{t_n}} - b_n^{i_{t_n}-1} + 2h} + \int_{b_n^{i_{t_n}} - b_n^{i_{t_n}-1} + 2h}^{b_n^{i_{t_n}} - b_n^{i_{t_n}-2}} + \dots \right. \\
 &\quad \left. + \int_{b_n^{i_{t_n}} - b_n^2}^{b_n^{i_{t_n}} - b_n^2 + 2h} + \int_{b_n^{i_{t_n}} - b_n^2 + 2h}^{b_n^{i_{t_n}} - b_n^1} u^{\nu-1} [f(b_n^{i_{t_n}} - u) - f(a_n^{i_{t_n}} - u)] du \right].
 \end{aligned}$$

Using the identity in the proof of Theorem 3, we have

$$\begin{aligned}
 &\sum_{j=1}^{i_{t_n}-2} \frac{1}{h_n} \left(\int_{a_n^{i_{t_n}} - b_n^{i_{t_n}-j}}^{a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + h_n} + \int_{a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + h_n}^{a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + 2h_n} \right) \\
 &\quad \times u^{\nu-1} [f(b_n^{i_{t_n}} - u) - f(a_n^{i_{t_n}} - u)] du \\
 &= \sum_{j=1}^{i_{t_n}-2} (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} [f(b_n^{i_{t_n}-j}) - f(a_n^{i_{t_n}-j})] \\
 &\quad + \sum_{j=1}^{i_{t_n}-2} \left[(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \overline{1 + \theta_j h_n})^{\nu-1} - (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} \right] \\
 &\quad \times \frac{1}{h_n} \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} [f(u) - f(a_n^{i_{t_n}-j})] du.
 \end{aligned}$$

Now $\frac{I_1(t, h_n)}{h_n}$ may be displayed as

$$\begin{aligned}
 &\frac{1}{h_n} \int_0^{h_n} u^{\nu-1} [f(b_n^{i_{t_n}} - u) - f(a_n^{i_{t_n}} - u)] du \\
 &\quad + \sum_{j=1}^{i_{t_n}-2} (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} [f(b_n^{i_{t_n}-j}) - f(a_n^{i_{t_n}-j})] \\
 &\quad + \left[\sum_{j=1}^{i_{t_n}-2} (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \overline{1 + \theta_j h_n})^{\nu-1} - (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} \right] \\
 &\quad \times \frac{1}{h_n} \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} [f(u) - f(a_n^{i_{t_n}-j})] du = I_{1.1}(t, h_n) + I_{1.2}(t, h_n) + I_{1.3}(t, h_n).
 \end{aligned}$$

Clearly

$$|I_{1.1}(t, h_n)| \leq \frac{C_1}{h_n} \int_0^{h_n} u^{\nu-1} h_n^{1-\nu} du = \frac{C_1}{h_n} \frac{h_n^\nu}{\Gamma(\nu)} h_n^{1-\nu} \leq A_2 \quad (7)$$

where A_2 is a constant.

By condition (ii) there is a constant C_3 such that

$$\left| \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu} \right| \leq C_3.$$

Let $K = I_{1,2}(t, h_n) - \int_0^t u^{\nu-1} f'_H(t-u) du^{1-\nu}$. To show that K is bounded, we may write K as

$$\begin{aligned} & \sum_{j=1}^{i_{t_n}-2} \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} [(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} - (t-u)^{\nu-1}] f'_H(u) du^{1-\nu} \\ & - \int_0^{t-a_n^{i_{t_n}}} u^{\nu-1} f'_H(t-u) du^{1-\nu} - \int_{t-b_n^1}^{t-a_n^1} u^{\nu-1} f'_H(t-u) du^{1-\nu}. \end{aligned}$$

Note that

$$\begin{aligned} (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} - (t-u)^{\nu-1} &= \frac{u_j h_n}{(\nu-1)(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \lambda_j h_n)^{2-\nu}} \\ u &\in [a_n^{i_{t_n}-j} - b_n^{i_{t_n}-j}], \quad 0 \leq u_j, \lambda_j \leq 2 \end{aligned}$$

and

$$\int_{a_n^j}^{b_n^j} u^{\nu-1} f'_H(t-u) du^{1-\nu} \leq D_3 h_n^{\nu-1}, \quad D_3 \text{ constant.}$$

Since $\xi \leq \frac{1}{3}$,

$$(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n) \geq j h_n.$$

We have

$$\begin{aligned} & \left| \sum_{j=1}^{i_{t_n}-2} \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} [(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} - (t-u)^{\nu-1}] f'_H(u) du \right| \\ & \leq \sum_{j=1}^{i-2} \frac{2D_3 h_n^{1-\nu} h_n}{j^{2-\nu} h_n^{2-\nu}} = E_3. \end{aligned}$$

Therefore,

$$|K| \leq E_3 + C_3 = A_3.$$

Final, let us estimate $I_3(t, h_N)$. As in the proof of Theorem 3, $I_3(t, h_n)$ may be expressed as

$$\sum_{j=1}^{i_{t_n}-2} \left[(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \overline{1 + \theta_j h_n})^{\nu-1} - (a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \theta_j h_n)^{\nu-1} \right]$$

$$\begin{aligned} & \times \left[\frac{1}{h_n} \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} [f(u) - f(a_n^{i_{t_n}-j})] du - \beta \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} f'_H(u) du^{1-\nu} \right] \\ & + \beta \sum_{j=1}^{i_{t_n}-2} [(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \overline{1 + \theta_j h_n})^{\nu-1} - (a_n^{i_{t_n}} - b_n^{i_{t_n}} - j + \theta_j h_n)^{\nu-1}] \\ & \times \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} f'_H(u) du^{1-\nu}. \end{aligned}$$

By the self-similarity of f , we have

$$\begin{aligned} |I_3(t, h_n)| \leq & F_3 \sum_{j=1}^{i_{t_n}-2} [(a_n^{i_{t_n}} - b_n^{i_{t_n}-j} + \overline{1 + \theta_j h_n})^{\nu-1} \\ & - (a_n^{i_{t_n}} - b_n^{i_{t_n}} - j + \theta_j h_n)^{\nu-1}] \int_{a_n^{i_{t_n}-j}}^{b_n^{i_{t_n}-j}} f'_H(u) du^{1-\nu}. \end{aligned}$$

By an argument similar to that in the above paragraph

$$|I_3(t, h_n)| \leq A_4, \tag{8}$$

combining (3), (6), (7), (8) we have

$$\frac{|(D^{-\nu} f)(b_n^{i_{t_n}}) - (D^{-\nu} f)(a_n^{i_{t_n}})|}{b_n^{i_{t_n}} - a_n^{i_{t_n}}} \leq A_1 + A_2 + A_3 + A_4 + C_3 = A. \quad \square$$

We need one more concept to finish this article. Let $[a, b]$ be a closed interval, let $\{h_n\}_{n=1}^\infty$ be a sequence of positive numbers with $h_n \downarrow 0$. A division D of $[a, b]$ is called an admissible division if $u = t - \alpha, v = t + \beta, \alpha = h_n$ or $0, \beta = h_m$ or 0 in the interval point pairs for all the $([u, v], t) \in D$ but associated point $t = b$.

Theorem 4 *Let f be a Cantor type function on C with $H\text{-dim } C = 1 - \nu$, suppose that f satisfies the conditions 1, 2, 3, 4 in Lemma 1. Then*

$$(D^{-\nu} f)(t) = \int_0^t \int_0^u v^{\nu-1} f'_H(u-v) dv^{1-\nu} du.$$

PROOF. First let us show that for each interval $[b_n^i, a_n^{i+1}]$, $i = 1, 2, \dots, 2^n$, $n = 1, 2, \dots$ and given $\varepsilon > 0$, there is an admissible division D such that

$$\left| [(D^{-\nu} f)(a_n^{i+1}) - (D^{-\nu} f)(b_n^i)] - \int_{b_n^i}^{a_n^{i+1}} M(t) dt \right|$$

$$= \left| [(D) \sum [(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha) - \int_{\alpha}^{\beta} M(t) dt] \right| < \varepsilon$$

where $M(t) = \int_0^t v^{\nu-1} f'_H(t-v) dv^{1-\nu}$.

Since $M(t)$ is a gauge integrable function in the interval $[b_n^i, a_n^{i+1}]$ [6, p. 112], for $\frac{\varepsilon}{2} > 0$ there is a $\delta_1(t)$ such that

$$\left| (D) \sum M(t)(\beta - \alpha) - \int_{b_n^i}^{a_n^{i+1}} M(t) dt \right| < \frac{\varepsilon}{2} \quad (9)$$

For every δ_1 -fine division D .

By Theorem 3 for each $t \in (b_n^i, a_n^{i+1}]$ there is a $\delta_2(t) > 0$ such that

$$\left| (D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha) - \int_0^t u^{1-\nu} f'_H(t-u) du^{1-\nu} \right| < \frac{\varepsilon}{3(a_n^{i+1} - b_n^i)}(\beta - \alpha) \quad (10)$$

where $\alpha = t - h_a, \beta = t + h_m, 0 < \beta - \alpha < \delta_2(t)$. Evidently there is a $\delta_2(b_n^i)$ such that

$$\left| \int_{b_n^i}^t M(u) du \right| < \frac{\varepsilon}{6} \quad (11)$$

and

$$|(D^{-\nu} f)(t) - (D^{-\nu} f)(b_n^i)| < \frac{\varepsilon}{6} \quad (12)$$

where $0 < t - b_n^i < \delta_2(b_n^i)$.

Let $\delta(t) = \min(\delta_1(t), \delta_2(t))$. It is a routine argument to check there is a δ -fine admissible division D of $[b_n^i, a_n^{i+1}]$. We may write

$$[(D^{-\nu} f)(a_n^{i+1}) - (D^{-\nu} f)(b_n^i) - \int_{b_n^i}^{a_n^{i+1}} M(t) dt$$

as

$$\begin{aligned} & (D) \sum [(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha)] - (D) \sum \int_{\alpha}^{\beta} M(t) dt \\ &= (D) \sum^I [(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha)] + (D^{-\nu} f)(\beta_n^i) - (D^{-\nu} f)(b_n^i) \\ & \quad - (D) \sum^I \int_{\alpha}^{\beta} M(t) dt - \int_{b_n^i}^{\beta_n^i} M(t) dt \end{aligned}$$

where $([b_n^i, \beta_n^i], b_n^i) \in D$ and $(D) \Sigma' = (D) \Sigma \setminus ([b_n^i, \beta_n^i], b_n^i)$. Then

$$\begin{aligned} & (D) \sum [(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha)] - (D) \sum \int_{\alpha}^{\beta} M(t) dt \\ &= (D) \sum' \left[(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha) - \int_0^t u^{\nu-1} f'_H(t-u) du^{\nu-1}(\beta - \alpha) \right] \\ & \quad + (D) \sum' \left[\int_0^t u^{\nu-1} f'_H(t-u) du^{\nu-1}(\beta - \alpha) - \int_{\alpha}^{\beta} M(t) dt \right] \\ & \quad + (D^{-\nu} f)(\beta_n^i) - (D^{-\nu} f)(b_n^i) - \int_{b_n^i}^{\beta_n^i} M(t) dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Compute I_1 by (2)

$$|I_1| \leq (D) \sum' \frac{\varepsilon}{b(a_n^{i_n} - b_n^i)}(\beta - \alpha) < \frac{\varepsilon}{3}.$$

By Henstock's lemma, we have $|I_2| < \frac{\varepsilon}{2}$. Combining (3), (4), gives $|I_2| < \frac{\varepsilon}{3}$ as required.

By changing the index we may write $\{[b_n^i, a_n^{i+1}]_{i=1}^{2^n-1}\}_{n=1}^{\infty}$ as $\{[\alpha_m, \beta_m]_{m=1}^{\infty}\}$. For each $[\alpha_m, \beta_m]$ there is a $\delta_m(t)$ defined on $[\alpha_m, \beta_m]$ such that

$$\left| (D) \sum M(t)(\beta - \alpha) - \int_{\alpha_m}^{\beta_m} M(t) dt \right| < \frac{\varepsilon}{2^{m+1}}, \quad m = 1, 2, \dots$$

for every δ_m -fine division D .

Let A be the constant in Lemma 1 and let $\delta_0(t) = \eta$ be small enough. Let

$$\delta(t) = \begin{cases} \delta_0, & t \in C \\ \delta_m(t), & t \in [\alpha_m, \beta_m]. \end{cases}$$

By a covering lemma [2, p. 4], there is a net δ -fine division D of $[0, T]$. The $(D^{-\nu} f)(t) - \int_0^t M(u) du$ may be displayed as

$$\begin{aligned} & (D) \sum \left[(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\beta) - \int_{\alpha}^{\beta} M(u) du \right] \\ &= [(D_1) \sum + (D_2) \sum] \left[(D^{-\nu} f)(\beta) - (D^{-\nu} f)(\alpha) - \int_{\alpha}^{\beta} M(u) du \right] \end{aligned}$$

$$= J_1 + J_2$$

where D_1 denotes the collection of elements with associated point $t \in C$ and D_2 the collection of elements with associated point $t \in [0, T] \setminus C$. Clearly

$$|J_1| \leq (D_1) \sum A(\beta - \alpha) + (D_1) \sum C_3(\beta - \alpha) \leq (A + C_3)(2\xi)^n,$$

where C_3 denotes the upper bounded of $M(t)$.

$$|J_2| \leq (D_2) \sum \frac{\varepsilon}{2^m} \leq \frac{\varepsilon}{2}. \quad (13)$$

Take η small enough such that

$$(A + C_3)(2\xi)^n < \frac{\varepsilon}{2}. \quad (14)$$

Combining (13), (14)

$$\left| (D^{-\nu} f)(t) - \int_0^t M(u) du \right| < \varepsilon.$$

Since ε arbitrary, we have

$$(D^{-\nu} f)(t) = \int_0^t \int_0^u v^{\nu-1} f'_H(u-v) dv^{1-\nu} du. \quad \square$$

Corollary 1 *Let f be the function in Theorem 4. Then*

$$(D^{-\nu} f)(t) = \int_0^t u^{1-\nu} J_\nu(v f'_H(u-v)) du.$$

PROOF. Using the identity [2, p. 7]

$$\int_0^u v^{\nu-1} f'_H(u-v) dv^{1-\nu} = u^{1-\nu} J_\nu(v f'_H(u-v)). \quad \square$$

Remark 3 *Let $f \in C^1$, $f(0) = 0$, It is easy to show*

$$(D^{-\nu} f)(t) = \int_0^t \int_0^u v^{\nu-1} f'(u-v) dv du.$$

This identity is similar to the above identity, but the function f is smooth. Note that in our discussion the function f is singular.

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