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ON SOME WEAKER FORMS OF CONTINUITY FOR MULTIFUNCTIONS

Abstract

The aim of this paper is to improve some recent results of Noiri and Popa on multifunctions by utilizing the concepts of almost β-continuous and weakly α-continuous multifunctions. We give some more results on strong irresolvability.

1 Prelude

In topology, there has been recently significant interest in characterizing and investigating the properties of several weak forms of continuity for multifunctions. The development of such a theory is in fact very well motivated.

In economics, under major consideration is the so called parameterized maximization problem (see [25]). One of the very first obstacles an economist might face is that the solution of that problem might not in general be a function. Often, it is a correspondence, or what topologists call it, a multifunction. Of major interest is how this multifunction changes; in particular how and when it changes 'continuously'. That is why, extending and studying the different forms of generalized continuity for multifunctions is a 'real' problem also beyond the field of topology. The famous theorem of the maximum

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[25], for example, involves several topological concepts — in brief, it states that every continuous function with compact range and nonempty compact-valued continuous constraints has continuous maximum and the solution of the parameterized maximization problem is an upper semi-continuous multifunction.

2 Introduction

In a recent paper [4], Borsík and Doboš presented a decomposition of quasi continuity. They defined almost quasi continuous functions but soon after their paper appeared Popa and Noiri [20] showed that almost quasi continuity is in fact equivalent to β-continuity. Most of the results of the present paper will be about functions closely related to β-continuity and about strong irresolvability of topological spaces, a concept introduced in 1991 by Foran and Liebnitz [8].

In [12], Neubrunn introduced the concept of upper and lower α-continuous multifunctions and proved that every lower (resp. upper) α-continuous and upper (resp. lower) quasi continuous multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) weakly continuous. Some of the theorems of Neubrunn from [12] were recently corrected in [5].

In 1993, Popa and Noiri [21] improved the results of Neubrunn by proving that every lower (resp. upper) α-continuous and upper (resp. lower) β-continuous multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) weakly continuous. In another recent article, Noiri and Popa [15] improved some results of Popa [18, 19] by proving that if a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) almost weakly continuous and upper (resp. lower) almost continuous (or quasi continuous), then F is lower weakly continuous.

In 1996, Popa and Noiri [23] introduced the concept of upper and lower almost α-continuous multifunctions and proved that if a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) almost α-continuous and upper (resp. lower) β-continuous, then F is lower weakly continuous. Thus they strengthened their results from the 1993 paper. Moreover, in [23], Popa and Noiri proved that every upper almost α-continuous compact-valued multifunction into a Hausdorff space (Y,σ) has an α-closed graph in $X \times Y$.

The aim of this paper is to improve all of the results stated above (from [15], [21], [23]), by proving the following:

(A) Every lower (resp. upper) weakly α-continuous and upper (resp. lower) almost β-continuous multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) weakly continuous.

(B) If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (resp. upper) almost weakly continuous and upper (resp. lower) almost quasi continuous, then F is lower (resp. upper) weakly continuous.

(C) If $F\colon (X,\tau) \to (Y,\sigma)$ is an upper weakly (almost) α-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y,σ), then its graph $G(F)$ is α-closed in $X \times Y$.

Additionally we prove that three recent concepts of Noiri [14] coincide, namely the conditions (β), (β') and (p). They all are stronger forms of connectedness and strong irresolvability.

We assume that the reader is familiar with the concepts of generalized open sets: semi-open sets, α-closure, β-interior, etc. However, their definitions can be found in almost any paper listed in our references. Only one of the concepts we use is relatively new, so we recall its definition: A subset S of a topological space X is called *b-open* [1] or *sp-open* [7] if $A \subseteq \mathrm{Cl}\,(\mathrm{Int}\,(A)) \cup \mathrm{Int}\,(\mathrm{Cl}\,(A))$. Note that every preopen and every semi-open set is b-open and every b-open set is β-open. The family of all semi-open (resp. α-open, regular open, regular closed, semi-regular, θ-semi-open, preopen, sg-open, b-open, β-open) subsets of X is denoted by $SO(X)$ (resp. $\alpha(X)$, $RO(X)$, $RC(X)$, $SR(X)$, θ-$SO(X)$, $PO(X)$, $SGO(X)$, $BO(X)$, $\beta(X)$). Let $\mathcal{K}$ be a collection of sets of a topological space (X,τ). For a point $x \in X$, we set $\mathcal{K}(X,x) = \{K \in \mathcal{K}\colon x \in K\}$.

For a multifunction $F\colon (X,\tau) \to (Y,\sigma)$, following [3], we denote the upper and lower inverse of a subset V of Y by $F^+(V)$ and $F^-(V)$, respectively:

$$F^+(V) = \{x \in X\colon F(x) \subseteq V\} \text{ and } F^-(V) = \{x \in X\colon F(x) \cap V \neq \emptyset\}.$$

The reader can find undefined notions of some generalized continuities for multifunctions from the references.

3 Almost β-continuous and Weakly α-continuous Multifunctions

Definition 1. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is called:

(a) *upper almost β-continuous* if for each point x of X and for each open set V of Y with $F(x) \subseteq V$, there exists $U \in \beta(X,x)$ such that $F(U) \subseteq \mathrm{sCl}\,(V)$.

(b) *lower almost β-continuous* if for each point x of X and for each open set V of Y with $F(x) \cap V \neq \emptyset$, there exists $U \in \beta(X,x)$ such that $F(u) \cap \mathrm{sCl}\,(V) \neq \emptyset$ for each $u \in U$.

Theorem 1. *For a multifunction $F\colon (X,\tau) \to (Y,\sigma)$, the following conditions are equivalent:*

(0) *F is upper almost β-continuous.*

(1) *For each $x \in X$ and each $V \in RO(Y)$ containing $F(x)$, there exists $U \in \beta(X,x)$ such that $F(U) \subseteq V$.*

(2) *$F^+(V) \in \beta(X)$ for each $V \in RO(Y)$.*

(3) *$F^-(W)$ is β-closed for each $V \in RC(Y)$.*

(4) *$\beta\mathrm{Cl}\,(F^-(V)) \subseteq F^-(\mathrm{Cl}\,(V))$ for every $V \in \beta(Y)$.*

(5) *$\beta\mathrm{Cl}\,(F^-(V)) \subseteq F^-(\mathrm{Cl}\,(V))$ for every $V \in SO(Y)$.*

(6) *$\beta\mathrm{Cl}\,(F^-(V)) \subseteq F^-(\mathrm{Cl}\,(V))$ for every $V \in RC(Y)$.*

(7) *$F^+(V) \subseteq \beta\mathrm{Int}\,(F^+(\mathrm{Int}\,(\mathrm{Cl}\,(V))))$ for every $V \in PO(Y)$.*

(8) *$F^+(V) \subseteq \beta\mathrm{Int}\,(F^+(\mathrm{Int}\,(\mathrm{Cl}\,(V))))$ for every $V \in \sigma$.*

(9) *$F^+(V) \subseteq \beta\mathrm{Int}\,(F^+(\mathrm{Int}\,(\mathrm{Cl}\,(V))))$ for every $V \in RO(Y)$.*

PROOF. The proof is very similar to and based on the technique used in the proofs of [22, Theorem 3.3] and [16, Theorem 1]. □

Remark 2. (i) In statement (5) of Theorem 1, $SO(Y)$ can be replaced by θ-$SO(Y)$, $SR(Y)$, $SGO(Y)$ or $BO(Y)$. This follows from the observation that each one of those classes of sets contains $RC(Y)$ and is contained in $\beta(Y)$. Those relations are clear and well-known except probably the fact that every sg-closed set is β-closed, which on its behalf was recently proved in [6].

(ii) For similar reasons, in statement (8) of Theorem 1, σ can be replaced by $\alpha(Y)$, $\tau_s(Y)$, or $GA(Y)$. Here τ_s denotes the semi-regularization topology, while $GA(Y)$ denotes the family of all generalized α-open subsets of Y (see [11] for details on $g\alpha$-closed sets). Only recently it was shown in [6], that generalized α-closed sets are preclosed.

A similar result to Theorem 1, holds for lower almost β-continuous multifunctions.

In [14], Noiri gave several new characterizations of hyperconnected spaces (= nonempty open sets are dense) and studied the preservation of hyperconnectedness via different kinds of mappings. In the same paper, Noiri considered the conditions (β), (β') and (p), which all are stronger forms of connectedness. By definition, a space has the property:

(β) [14] if $\beta\mathrm{Cl}\,(W) = X$ for every nonempty $W \in \beta(X)$;

(β') [14] if $\beta\mathrm{Cl}\,(W) = X$ for every nonempty $W \in PO(X)$;

(p) [14] if $\mathrm{pCl}\,(W) = X$ for every nonempty $W \in PO(X)$.

Our next result shows that those three properties are equivalent, in fact equivalent to irresolvability and hyperconnectedness taken together. Recall that a topological space (X, τ) is called *resolvable* [10] if X is the disjoint union of two dense subsets. In the opposite case (X, τ) is called *irresolvable*. Recall additionally that a space (X, τ) is *strongly irresolvable* [8] if no nonempty open set is resolvable, where a set is resolvable if it is resolvable as a subspace. The following characterization of strongly irresolvable spaces will be useful.

Theorem 3. *A topological space (X, τ) is strongly irresolvable if and only if $\beta(X) = SO(X)$.*

PROOF. (Necessity) Assume that X is strongly irresolvable. Let $S \in \beta(X)$. By definition, $S \subseteq \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(S)))$. Without loss of generality, we assume $S \neq \emptyset$. Let $U = \mathrm{Int}\,(\mathrm{Cl}\,(S))$. Then U is a nonempty open set with $\mathrm{Cl}\,(U) = \mathrm{Cl}\,(S)$. We show that $S \subseteq \mathrm{Cl}\,(\mathrm{Int}\,(S))$. If not, there exists a point $x \in S \setminus \mathrm{Cl}\,(\mathrm{Int}\,(S))$. Let V be an open neighborhood of x such that $V \cap \mathrm{Int}\,(S) = \emptyset$. Then we have $U \cap V \neq \emptyset$ and $V \subseteq X \setminus \mathrm{Int}\,(S)$. On one hand, $U \cap V \subseteq \mathrm{Cl}\,(U \cap V \cap S)$. Moreover, we have $U \cap V = U \cap V \cap (X \setminus \mathrm{Int}\,(S)) \subseteq \mathrm{Cl}\,((U \cap V) \setminus S)$. It follows that $U \cap V$ is a nonempty resolvable open subspace. This is a contradiction. Therefore $S \in SO(X)$.

(Sufficiency) Let U be a nonempty open resolvable subspace. Let (A, B) be a resolution of U. Observe that both A and B are preopen and hence β-open. Due to the assumption, they are semi-open and hence α-open (both in X and in U). Since A is both α-open and α-closed in U, then A is clopen and thus A is not dense in U. By contradiction, X is strongly irresolvable. □

Theorem 4. *For a topological space (X, τ) the following conditions are equivalent:*

(1) *X has the property (β).*

(2) *X has the property (β').*

(3) *X has the property (p).*

(4) *X is (strongly) irresolvable and hyperconnected.*

(5) *The dense subsets of X form an ultrafilter on X.*

PROOF. (1) ⇒ (2) and (2) ⇒ (3) are obvious.

(3) ⇒ (4) Let U be open and nonempty. If (A, B) is a resolution of U, then both A and B are preopen in X. Hence $\mathrm{pCl}\,(A) \neq X$. By contradiction

X is strongly irresolvable, since X has the property (p). For the second part, note that if X is not hyperconnected, then there exist two disjoint, nonempty, open (and hence preopen) subsets of X. Since X has the property (p), then X must be, again by contradiction, hyperconnected.

(4) $\Rightarrow$ (1) Let $W \in \beta(X)$. Observe that in hyperconnected spaces irresolvability and strong irresolvability coincide. By Theorem 3, W is semi-open. By [14, Theorem 3.1 (e)], $\beta\mathrm{Cl}\,(W) = X$. Thus X has the property (β).

(4) $\Leftrightarrow$ (5) is proved in [9]. □

One can easily find an example showing that almost β-continuous images of hyperconnected spaces fail to be hyperconnected (the indiscrete topology has the property (β) and the discrete topology on every set with at least two points is not hyperconnected). However, the following results holds. Recall first that a space X is called *nearly compact* (resp. *semi-compact*) if every cover of X consisting of regular open (resp. semi-open) sets has a finite subcover.

Theorem 5. *Let $f\colon (X,\tau) \to (Y,\sigma)$ be an almost β-continuous surjection.*

(i) *If (X,τ) has the property (β), then (Y,σ) is hyperconnected.*

(ii) *If (X,τ) is semi-compact and strongly irresolvable, then (Y,σ) is nearly compact.*

Proof. (i) Assume that U and V are two nonempty disjoint open sets of Y. Then $\mathrm{Int}\,(\mathrm{Cl}\,(U))$ and $\mathrm{Int}\,(\mathrm{Cl}\,(V))$ are nonempty regular open disjoint sets of Y. Since f is an almost β-continuous surjection, then by Theorem 1, $U' = f^{-1}(\mathrm{Int}\,(\mathrm{Cl}\,(U)))$ and $V' = f^{-1}(\mathrm{Int}\,(\mathrm{Cl}\,(V)))$ are disjoint β-open subsets of X. Clearly $\beta\mathrm{Cl}\,(U') \neq X$. Thus, by contradiction, Y is hyperconnected.

(ii) Let $\{V_i \colon i \in I\}$ be a cover of Y consisting of regular open sets. Since f is almost β-continuous, then by Theorem 1, each one of the sets $U_i = f^{-1}(V_i)$ is β-open and by Theorem 3 semi-open. Since $\{U_i \colon i \in I\}$ is a semi-open cover of X and since X is semi-compact, then for some finite $F \subseteq I$, we have $X = \cup_{i\in F} U_i$. Thus $Y = \cup_{i\in I} V_i$, which shows that Y is nearly compact. □

Remark 6. A set-valued version of Theorem 5 (ii) exists as follows: Let $F\colon (X,\tau) \to (Y,\sigma)$ be an upper almost β-continuous compact-valued multifunction. If (X,τ) is semi-compact and strongly irresolvable, then (Y,σ) is nearly compact.

Definition 2. A multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is called:

(a) *upper weakly α-continuous* [23] if for each point x of X and for each open set V of Y with $F(x) \subseteq V$, there exists $U \in \alpha(X,x)$ such that $F(U) \subseteq \mathrm{Cl}\,(V)$.

(b) *lower weakly α-continuous* [23] if for each point x of X and for each open set V of Y with $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha(X, x)$ such that $F(u) \cap \mathrm{Cl}\,(V) \neq \emptyset$ for each $u \in U$.

Theorem 7. *For a multifunction $F\colon (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(0) *F is upper weakly α-continuous.*

(1) *For each $x \in X$ and each $V \in \sigma$ containing $F(x)$,* $x \in \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(F^+(\mathrm{Cl}\,(V)))))$.

(2) $F^+(V) \subseteq \mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,(F^+(\mathrm{Cl}\,(V)))))$ *for each $V \in \sigma$.*

(3) $\mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(F^-(V)))) \subseteq F^-(\mathrm{Cl}\,(V))$ *for each $V \in \sigma$.*

(4) $\alpha\mathrm{Cl}\,(F^-(V)) \subseteq F^-(\mathrm{Cl}\,(V))$ *for each $V \in \sigma$.*

(5) $F^+(V) \subseteq \alpha\mathrm{Int}\,(F^+(\mathrm{Cl}\,(V)))$ *for each $V \in \sigma$.*

(6) $\mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(F^-(\mathrm{Int}\,(W))))) \subseteq F^-(W)$ *for every closed subset W of Y.*

(7) $\alpha\mathrm{Cl}\,(F^-(\mathrm{Int}\,(W))) \subseteq F^-(W)$ *for every closed subset W of Y.*

(8) $\alpha\mathrm{Cl}\,(F^-(\mathrm{Int}\,(\mathrm{Cl}\,(B)))) \subseteq F^-(\mathrm{Cl}\,(B))$ *for every $B \subseteq Y$.*

(9) $F^+(\mathrm{Int}\,(B)) \subseteq \alpha\mathrm{Int}\,(F^+(\mathrm{Cl}\,(\mathrm{Int}\,(B))))$ *for every $B \in Y$.*

PROOF. The proof is very similar to the one of [15, Theorem 3.1 and Theorem 3.4]; hence we omit it. □

Remark 8. (i) For a singled valued function $f\colon (X, \tau) \to (Y, \sigma)$, the concept of weak α-continuity was introduced and studied for the first time in 1987 by Noiri [13]. In 1988, Rose [24] gave a functional tridecomposition of continuity by proving that a function $f\colon (X, \tau) \to (Y, \sigma)$ is continuous iff f is almost continuous, (sub)weakly α-continuous and locally weak*-continuous.

(ii) In statements (2) – (5) of Theorem 7, σ can be replaced by $\alpha(Y)$, $PO(Y)$, $GA(Y)$, $\tau_s(Y)$ or RO(Y).

(iii) Note that every upper almost α-continuous multifunction is always upper weakly α-continuous but not vice versa.

A similar result to Theorem 7, holds for lower weakly α-continuous multifunctions.

4 Some Miscellaneous Results

Theorem 9. *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower weakly α-continuous and upper almost β-continuous, then F is lower weakly continuous.*

Proof. Let $V \in \sigma$. Since F is lower weakly α-continuous, then observe that $F^-(V) \subseteq \operatorname{Int}(\operatorname{Cl}(\operatorname{Int}(F^-(\operatorname{Cl}(V)))) \subseteq \operatorname{Int}\beta\operatorname{Cl}(F^-(\operatorname{Cl}(V)))$. Since F is upper almost β-continuous, by Theorem 1 (6), we have $\beta\operatorname{Cl}(F^-(\operatorname{Cl}(V))) \subseteq F^-(\operatorname{Cl}(\operatorname{Cl}(V))) = F^-(\operatorname{Cl}(V))$. Thus $F^-(V) \subseteq \operatorname{Int}\beta\operatorname{Cl}(F^-(\operatorname{Cl}(V))) \subseteq \operatorname{Int} F^-(\operatorname{Cl}(V))$. From [17, Theorem 4], it follows that F is lower weakly continuous. □

Corollary 10. (Popa and Noiri [21], [23]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower (almost) α-continuous and upper β-continuous, then F is lower weakly continuous.*

Theorem 11. *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper weakly α-continuous and lower almost β-continuous, then F is upper weakly continuous.*

Proof. The proof is very similar to that of Theorem 9 and is thus omitted. □

Corollary 12. (Popa and Noiri [21], [23]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper (almost) α-continuous and lower β-continuous, then F is upper weakly continuous.* □

Theorem 13. *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower almost weakly continuous and upper almost quasi continuous, then F is lower weakly continuous.*

Proof. Let $V \in \sigma$. Since F is lower almost weakly continuous, then by [15, Theorem 3.2 (b)], $F^-(V) \subseteq \operatorname{Int}(\operatorname{Cl}(F^-(\operatorname{Cl}(V))))$. Note that $\operatorname{Cl}(V)$ is regular closed in Y. Thus by [16, Lemma 2(d)], $F^-(\operatorname{Cl}(V))$ is semi-closed in X. Hence $F^-(V) \subseteq \operatorname{Int}(\operatorname{Cl}(F^-(\operatorname{Cl}(V)))) = \operatorname{Int}(F^-(\operatorname{Cl}(V)))$. From [17, Theorem 4], it follows that F is lower weakly continuous. □

Corollary 14. (Noiri and Popa [15]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower almost weakly continuous and upper almost continuous, then F is lower weakly continuous.* □

Corollary 15. (Noiri and Popa [15]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is lower quasi continuous and upper almost continuous, then F is lower weakly continuous.* □

Theorem 16. *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper almost weakly continuous and lower almost quasi continuous, then F is upper weakly continuous.*

PROOF. The proof is very similar to that of Theorem 13 and is thus omitted. □

Corollary 17. (Noiri and Popa [15]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper almost weakly continuous and lower almost continuous, then F is upper weakly continuous.*

Corollary 18. (Noiri and Popa [15]) *If a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is upper almost weakly continuous and lower quasi continuous, then F is upper weakly continuous.*

Recall first that a subset A of a space (X,τ) is called *α-paracompact* [2] if for every open cover $\mathcal{V}$ of A in (X,τ), there exists a locally finite open cover $\mathcal{W}$ of A which refines $\mathcal{V}$. Furthermore, a multifunction $F\colon (X,\tau) \to (Y,\sigma)$ is called *punctually α-paracompact* [21] if $F(x)$ is α-paracompact for each point $x \in X$. Theorem 4.6 from [21] and Theorem 21 from [23] can be generalized as follows.

Theorem 19. *Let $F\colon (X,\tau) \to (Y,\sigma)$ be an upper weakly (almost) α-continuous and punctually α-paracompact multifunction into a Hausdorff space (Y,σ). Then the graph $G(F)$ of F is α-closed in $X \times Y$.*

PROOF. Suppose that $(x_0, y_0) \notin G(F)$. Then $y_0 \notin F(x_0)$. Since (Y,σ) is a Hausdorff space, then for each $y \in F(x_0)$ there exist open sets $V(y)$ and $W(y)$ containing y and y_0 respectively such that $V(y) \cap W(y) = \emptyset$. The family $\{V(y):\ y \in F(x_0)\}$ is an open cover of $F(x_0)$ which is α-paracompact. Thus, it has a locally finite open refinement $\mathcal{U} = \{U_\beta:\ \beta \in I\}$ which covers $F(x_0)$. Let W_0 be an open neighborhood of y_0 such that W_0 intersects only finitely many members $U_{\beta_1}, U_{\beta_2}, \ldots, U_{\beta_n}$ of $\mathcal{U}$. Choose $y_1, y_2, \ldots, y_n$ in $F(x_0)$ such that $U_{\beta_i} \subseteq V(y_i)$ for each $1 \le i \le n$, and set $W = W_0 \cap (\cap_{i=1}^n W(y_i))$. Then W is an open neighborhood of y_0 with $W \cap (\cup_{\beta\in I} U_\beta) = \emptyset$, which implies that $W \cap \mathrm{Cl}\,(\cup_{\beta\in I} U_\beta) = \emptyset$. By the upper weak α-continuity of F, there is a $U \in \alpha(X, x_0)$ such that $F(U) \subseteq \mathrm{Cl}\,(\cup_{\beta\in I} U_\beta)$. It follows that $(U \times W) \cap G(F) = \emptyset$. Therefore the graph $G(F)$ is α-closed in $X \times Y$. □

Corollary 20. (Noiri and Popa [21] and [23]) *If $F\colon (X,\tau) \to (Y,\sigma)$ is an upper (almost) α-continuous multifunction into a Hausdorff space (Y,σ) such that $F(x)$ is compact for each $x \in X$, then the graph $G(F)$ is α-closed in $X \times Y$.*

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