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ON A MEASURE WHICH MEASURES AT LEAST ONE SELECTOR FOR EVERY UNCOUNTABLE SUBGROUP

Abstract

We show that there exists in ZFC an invariant extension of Lebesgue measure on $\mathbb{R}$ such that for every uncountable subgroup H of $\mathbb{R}$ there exists at least one selector of H measurable with respect to this measure. This answers a question of Sławomir Solecki in [S].

Notation

Let H be some additive subgroup of $\mathbb{R}$. We denote this fact by $H \leq \mathbb{R}$.

Symbol $\mathrm{Sel}(H)$ denotes the class of all selectors of the subgroup H, i.e., selectors from the class $\{x + H : x \in \mathbb{R}\}$ of cosets of H.

Let $B = \{h_\alpha : \alpha < 2^\omega\}$ be a Hamel base of $\mathbb{R}$. Every $r \in \mathbb{R}$ can be written in the exactly one way as $r = \sum_{\alpha<2^\omega} x(\alpha)\, h_\alpha$, where $x(\alpha) \in \mathbb{Q}$ and all $x(\alpha)$ but finitely many are equal to zero. Thus, we can treat $r \in \mathbb{R}$ as such a function x. If $x \in \mathbb{R}$ then we put $\mathrm{supp}(x) = \{\alpha < 2^\omega : x(\alpha) \neq 0\}$. If $X \subseteq \mathbb{R}$ then $\mathrm{supp}(X)$ denotes $\bigcup_{x\in X} \mathrm{supp}(x)$.

If $X \subseteq \mathbb{R}$ then $\langle X\rangle_{\mathbb{Q}}$ denotes the linear span of X over the field $\mathbb{Q}$. We say that a set $X \subseteq \mathbb{R}$ is *totally imperfect set* if it does not contain any nonvoid perfect subset of $\mathbb{R}$.

Main Result

In [S] the author asked a following question.

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Question 1 *Does there exist in ZFC an invariant extension of Lebesgue measure on $\mathbb{R}$ such that for every uncountable subgroup H of $\mathbb{R}$ there exists at least one selector of H measurable with respect to this measure?*

In [S] the author proved, that assuming CH the answer is yes. We will show, that the answer is also positive in ZFC.

Theorem 1 *There exists an invariant extension of the Lebesgue measure on $\mathbb{R}$ such that for every uncountable subgroup H of $\mathbb{R}$ there exists at least one selector of H measurable with respect to this measure.*

Proof. From [S] we know, that CH implies Theorem 1. So we consider only the case $\neg$CH.

Lemma 1 *Assume* $\neg$CH. *For every subgroup $H \leq \mathbb{R}$ of cardinality ω_1 we can find at least one selector $S_H \in \mathrm{Sel}(H)$ such that the following condition holds.*

(*) If $\{H_i\}_{i\in\omega}$ is a sequence of subgroups each of power ω_1, and $\{r_i\}_{i<\omega}$ are real numbers then $\bigcup_{i=0}^{\infty} S_{H_i} + r_i$ is not of full (Lebesgue) measure. In fact, this set is totally imperfect.

Proof. Let $F : 2^\omega \to \omega_1 \times 2^\omega \times 2^\omega$ be any bijection. We put

$$F(\alpha) = \langle F_1(\alpha), F_2(\alpha), F_3(\alpha)\rangle.$$

Let $\{P_\alpha\}_{\alpha\in 2^\omega}$ be an enumeration of all perfect subsets of $\mathbb{R}$. By transfinite induction one can easily construct a Hamel base $B = \{h_\alpha : \alpha \in 2^\omega\}$ such that

$$(\diamond) \quad h_\alpha \in P_{F_2(\alpha)} \text{ for every } \alpha \in 2^\omega.$$

Take any $H \leq \mathbb{R}$ with $|H| = \omega_1$. We construct S_H in the following way.

Let $H = \{g^H_\beta : \beta < \omega_1\}$ be an enumeration, without repetitions, of elements of H. We take also any selector T_H from the class of quotient group $\langle\{h_\alpha : \alpha \in \mathrm{supp}(H)\}\rangle_{\mathbb{Q}}/H$.

For every $s \in \mathbb{R}$ such that $\mathrm{supp}(s) \cap \mathrm{supp}(H) = \emptyset$ and every $t \in T_H$ we put

$$y_{s,t} := s + t + g^H_{F_1(\alpha_s)}$$

where $\alpha_s = \max \mathrm{supp}(s)$. We define S_H by:

$$S_H = \{y_{s,t} : s \in \mathbb{R} \;\&\; \mathrm{supp}(s) \cap \mathrm{supp}(H) = \emptyset \;\&\; t \in T_H\}.$$

It is easy to see that $S_H \in \mathrm{Sel}(H)$.

Take now any sequence $\{H_i\}_{i\in\omega}$ of subgroups of $\mathbb{R}$ each of power ω_1 and a sequence $\{x_i\}_{i\in\omega}$ of real numbers. For each $i \in \omega$ define a number $\beta_i < \omega_1$ in the following way. Consider $-x_i|\mathrm{supp}(H_i)$. Obviously $-x_i|\mathrm{supp}(H_i) = t + g_\beta^{H_i}$ for some $\beta < \omega_1$ and $t \in T_{H_i}$ and we define β_i as this β.

Let $\beta' := \big(\sup_{i\in\omega} \beta_i\big) + 1$. Because

$$\Big|\bigcup_{i\in\omega} \mathrm{supp}(H_i) \cup \mathrm{supp}(x_i)\Big| \le \omega_1 < 2^\omega$$

there exists $\alpha \in 2^\omega$ such that

$$(\triangle) \quad \alpha \notin \bigcup_{i\in\omega} \mathrm{supp}(H_i) \quad \text{and} \quad \alpha > \sup_{i\in\omega} \max\ \mathrm{supp}(x_i)$$

Let $\gamma' = F_3(\alpha)$. It is now easy to show, that

$$h_\alpha \notin \bigcup_{i\in\omega} S_{H_i} + x_i \text{ for every } \alpha \text{ such that } F_1(\alpha) = \beta' \text{ and } F_3(\alpha) = \gamma'. \tag{1}$$

Indeed, if $h_\alpha \in S_{H_i} + x_i$ for some $i \in \omega$ then

$$h_\alpha - x_i|\mathrm{supp}(H_i) = -x_i|\mathrm{supp}(H_i),$$

because $\alpha \notin \bigcup_{i\in\omega} \mathrm{supp}(H_i)$. So there exists $t \in T_H$ such that

$$-x_i|\mathrm{supp}(H_i) = t + g_{\beta_i}^{H_i}.$$

From $(\triangle)$ we know that $\max\ \mathrm{supp}(h_\alpha - x_i) = \alpha$. Therefore if $s = (h_\alpha - x_i)|(2^\omega \setminus \mathrm{supp}(H_i))$ then $y_{s,t} = s + t + g_{\beta'}^{H_i}$ from the definition of $y_{s,t}$. But $h_\alpha - x_i = (h_\alpha - x_i)|(2^\omega \setminus \mathrm{supp}(H_i)) + t + g_{\beta_i}^{H_i}$, since $\alpha > \max\ \mathrm{supp}(x_i)$ and $\beta' = F(\alpha)$. So $g_{\beta_i}^{H_i} = g_{\beta'}^{H_i}$, a contradiction with the definition of β', finishing the proof of (1).

Now $h_\alpha \notin \bigcup_{i\in\omega} S_{H_i} + x_i$ for every $\alpha \in 2^\omega$ such that $F_1(\alpha) = \beta'$ and $F_3(\alpha) = \gamma'$, and because $h_\alpha \in P_{F_2(\alpha)}$, we obtain that for every $\delta \in 2^\omega$,

$$P_\delta \setminus \bigcup_{i\in\omega} \big(S_{H_i} + x_i\big) \neq \emptyset.$$

Therefore the set

$$\bigcup_{i\in\omega} S_{H_i} + x_i$$

is not of full measure. In fact, this set is totally imperfect. $\square$

In the next part of the proof of Theorem 1 we will use the following Lemma from [S].

Lemma 2 *[Sz] Let μ be a translation invariant measure on $\mathbb{R}$ and let J be an invariant σ-ideal on $\mathbb{R}$ such that $\mu_*(A) = 0$ for $A \in J$ (where μ_* denotes the inner measure induced by μ). Then there exists an invariant extension of μ defined on the σ-field generated by the union of the σ-field of μ measurable subsets of $\mathbb{R}$ and J.*

Using now Lemma 1 and Lemma 2 we easily obtain our extension.

In fact, let J be the $\sigma-$ideal generated by all $S_H + x$, where $x \in \mathbb{R}$, $H \leq \mathbb{R}$, $|H| = \omega_1$. Assume by way of contradiction, that $\mu_*(A) > 0$ for some $A \in J$, where μ is Lebesgue measure. Using the Steinhaus theorem for the Lebesgue measure we obtain that $\mathbb{Q} + A$ is of full measure. But $\mathbb{Q} + A \in J$, which gives contradiction with Lemma 1.

Using Lemma 2 we obtain, that there exists an invariant extension $\overline{\mu}$ of Lebesgue measure such that every set from J is measurable with respect to $\overline{\mu}$, so for every uncountable subgroup of $\mathbb{R}$ there exists at least one selector measurable with respect to $\overline{\mu}$. $\square$

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