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ON APPROXIMATE UNSMOOTHING OF FUNCTIONS

Abstract

The smoothing $T_a f$, for $a > 0$, of a locally-integrable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$(T_a f)(x) := \frac{1}{2a} \int_{-a}^{+a} f(x+y)dy, \quad x \in \mathbb{R}.$$

For a given $g : \mathbb{R} \longrightarrow \mathbb{R}$, any solution f of the equation $T_a f = g$ is called an unsmoothing of g. In this note we analyse the problem of constructing a function $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $(T_a \tilde{f})(x_i) = g(x_i)$ for a given set of points $x_1, x_2, \ldots, x_n \in \mathbb{R}$. We give an iterative process of constructing such an $\tilde{f}$ under the assumption $f \in L_2(\mathbb{R})$.

1 Introduction

Let $L_{loc}(\mathbb{R})$ denote the space of all locally-integrable functions on the real line and for $a > 0, f \in L_{loc}(\mathbb{R})$, let

$$(T_a f)(x) := \frac{1}{2a} \int_{-a}^{+a} f(x+y)dy, x \in \mathbb{R}.$$

The function $T_a f$ is called the **smoothing** of f and T_a is called the smoothing operator. In [2], the range and kernel of the smoothing operator were discussed and a right-inverse for T_a was constructed which preserved the differentiability properties optimally. In practical problems, $T_a f$ represents the smoothing (moving average or sliding mean) of the raw data f. The problem of constructing some function f such that $T_a f = g$, g given, is called the

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unsmoothing problem. In the case when f is an integrable function with compact support, reconstruction formulas using two-sided Laplace transform were obtained by Van der Pol in [3].

We consider the following situation : let a function g be the smoothing of some function f, i.e., $T_a f = g$. One knows g at only a finite number of points, i.e., one knows the values $g(x_i) = c_i$, for some $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Of course, one cannot hope to recover the function f exactly from this data. However, it is meaningful to ask the question : Can one find some $\tilde{f}$ such that $(T_a \tilde{f}) = c_i, 1 \leq i \leq n$? Since, in practical situations, the raw data $f(x)$ is a bounded function (at least on finite time intervals), it is not unreasonable to assume that there exists some f which is bounded and has compact support such that $(T_a f)(x_i) = c_i, 1 \leq i \leq n$. So, we may assume that there exists some $f \in L_2(\mathbb{R})$ such that $(T_a f)(x_i) = c_i, 1 \leq i \leq n$. The problem is to find some $\tilde{f} \in L_2(\mathbb{R})$ such that $(T_a \tilde{f})(x_i) = c_i, 1 \leq i \leq n$. We call this the **approximate unsmoothing** problem.

2 Construction of 'Approximate Unsmoothing'

We are given $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. We also have the knowledge that there exists some $f \in L_2(\mathbb{R})$ such that $(T_a f)(x_i) = c_i, 1 \leq i \leq n$. We want to construct some $\tilde{f} \in L_2(\mathbb{R})$ such that $(T_a \tilde{f})(x_i) = c_i, 1 \leq i \leq n$. In case the intervals $[x_i - a, x_i + a], 1 \leq i \leq n$ are pairwise disjoint, an obvious choice for $\tilde{f}$ is given by

$$\tilde{f} = \sum_{i=1}^{n} c_i \chi_{[x_i - a, x_i + a]}.$$

It is easy to see that $(T_a \tilde{f})(x_i) = c_i, 1 \leq i \leq n$. In the general case, we proceed as follows: We choose arbitrary functions $\phi_i \in L_2(\mathbb{R})$, such that $(T_a \phi_i)(x_i) = 1, 1 \leq i \leq n$ (for example $\phi_i = \chi_{[x_i - a, x_i + a]}$). Let $\langle . , . \rangle$ denote the inner-product on $L_2(\mathbb{R})$ and

$$N_i = \{g \in L_2(\mathbb{R}) | \langle g, \, \chi_{[x_i - a, x_i + a]} \rangle = 0\}, 1 \leq i \leq n.$$

Then each N_i is a closed subspace of $L_2(\mathbb{R})$ of co-dimension one. Let

$$Q_i : L_2(\mathbb{R}) \longrightarrow N_i \tag{1}$$

denote the orthogonal projection onto N_i. Let $P_i : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R})$ be defined by

$$P_i(g) := Q_i(g) + (I - Q_i)(f), \quad \forall \, g \in L_2(\mathbb{R}).$$

We note that $(I - Q_i)(f) \in \text{span}\{\phi_i\}$ and hence

$$(I - Q_i)(f) = \alpha_i \phi_i, \quad \text{for some } \alpha_i \in \mathbb{R}.$$

In fact,

$$\begin{aligned}
\alpha_i &= \alpha_i \left[(T_a \phi_i)(x_i)\right] = \left[T_a(\alpha_i \phi_i)\right](x_i) \\
&= \left[T_a((I - Q_i)(f))\right](x_i) = (T_a f)(x_i) - \left[(T_a Q_i)(f)\right](x_i) \\
&= c_i - \langle Q_i(f), \chi_{[x_i - a, x_i + a]} \rangle = c_i.
\end{aligned}$$

Thus,

$$P_i(g) = Q_i(g) + c_i \phi_i. \tag{2}$$

Hence, the operators Q_i and $P_i, 1 \leq i \leq n$ are completely known once the functions ϕ_i's are chosen. For further arguments, we need the following :

Theorem 2.1 *Let $\mathcal{H}$ be a Hilbert space and Q_i be the orthogonal projection onto a closed subspace $N_i \subset \mathcal{H}, 1 \leq i \leq n$. Let $N_0 = \bigcap_{i=1}^{n} N_i$ and Q_0 be the orthogonal projection onto N_0. Let $Q = Q_n Q_{n-1} \ldots Q_1$. Then $Q^m(g) \to Q_0(g)$ as $m \to \infty$ for every $g \in \mathcal{H}$.*

PROOF. We refer to [1].

Lemma 2.2 *Let $Q_i, P_i, 1 \leq i \leq n$ be as constructed in* (1) *and* (2). *Let Q_0 denote the orthogonal projection on the subspace $N_0 = \cap_{i=1}^{n} N_i$ and let $Q = Q_n Q_{n-1} \ldots Q_1, P = P_n P_{n-1} \ldots P_1$ and $P_0(g) = Q_0(g - f) + f, g \in L_2(\mathbb{R})$. Then $\lim\limits_{m \to \infty} P^m(g) = P_0(g) \; \forall \, g \in L_2(\mathbb{R})$.*

PROOF. We first show that $\forall m$ and $\forall g \in L_2(\mathbb{R})$, $P^m(g) = Q^m(g - f) + f$. For $m = 1$ and $g \in L_2(\mathbb{R})$,

$$\begin{aligned}
P(g) &= P_n(P_{n-1} \ldots (P_1(g))) \\
&= Q_n(P_{n-1} \ldots (P_1(g))) + (I - Q_n)(f) \\
&= Q_n[Q_{n-1}(P_{n-2} \ldots (P_1(g))) + (I - Q_{n-1})(f)] + (I - Q_n)(f) \\
&= Q_n[Q_{n-1}(P_{n-2} \ldots (P_1(g)))] + (I - Q_n Q_{n-1})(f) \\
&\quad \ldots\ldots\ldots \\
&= Q_n(Q_{n-1} \ldots (Q_1(g))) + (I - Q_n Q_{n-1} \ldots Q_1)(f) \\
&= Q(g - f) + f.
\end{aligned}$$

Suppose the claim is true for each $m \leq k-1$. Then $\forall g \in L_2(\mathbb{R})$

$$\begin{aligned} P^k(g) &= P^{k-1}(P(g)) = Q^{k-1}(P(g)-f)+f \\ &= Q^{k-1}(Q(g-f))+f = Q^k(g-f)+f. \end{aligned}$$

Thus, by induction, $\forall\, m \geq 1$ and $g \in L_2(\mathbb{R})$

$$P^m(g) = Q^m(g-f)+f.$$

By Theorem 2.1, $Q^m(g-f) \longrightarrow Q_0(g-f)$, where Q_0 is the orthogonal projection onto $\cap_{i=1}^n N_j$. Hence $P^m(g) \longrightarrow Q_0(g-f)+f = P_0(g)\ \ \forall\, g \in L_2(\mathbb{R})$.

Theorem 2.3 *Let $g_0 \in L_2(\mathbb{R})$ be arbitrary and let P be as in Lemma 2.2. Then $\{P^m(g_0)\}_{m\geq 1}$ is convergent and if $\tilde{f} := \lim\limits_{m\longrightarrow\infty} P^m(g_0)$, then*

$$(T_a\tilde{f})(x_i) = c_i, 1 \leq i \leq n.$$

PROOF. By Lemma 2.2, $\{P^m(g_0)\}_{m\geq 1}$ converges to $Q_0(g_0-f)+f$. Thus, if $\tilde{f} := Q_0(g_0-f)+f$, then $\tilde{f}-f = Q_0(g_0-f) \in N_0$. Hence $\forall\, i = 1,2,\ldots,n$,

$$0 = \langle \tilde{f}-f,\, \chi_{[x_i-a,x_i+a]}\rangle = \int_{x_i-a}^{x_i+a} (\tilde{f}-f)(x)dx = 2a(T_a(\tilde{f}-f)(x_i).$$

Thus $T_a(\tilde{f}(x_i)) = T_a(f)(x_i) = c_i\ \forall\, 1 \leq i \leq n$.

This completes the proof of the theorem.

Note 2.4 It is easy to see that the approximate unsmoothing can also be constructed in the case $\mathbb{R}$ is replaced by $\mathbb{R}^n$ and intervals $[x_i-a, x_i+a]$ are replaced by bounded subsets of $\mathbb{R}^n$ of positive Lebesgue measure.

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