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C^1 SELECTIONS OF MULTIFUNCTIONS IN ONE DIMENSION

Abstract

Sufficient conditions are given for a multifunction (set-valued function) to admit a continuously differentiable selection in one dimension. These conditions are given in terms of Clarke generalized gradients of the Hamiltonian associated with the multifunction.

1 Introduction

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *selection* of a multifunction (i.e. a set-valued map) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ if $f(x) \in F(x)$ for each x. The function f is respectively a measurable, continuous, C^1, etc., selection if the function f is measurable, continuous, C^1, etc. Our main result, Theorem 3.1, provides sufficient conditions for a Lipschitz multifunction F with compact convex values to admit a C^1 selection. The result is only proven with the domain and range space to be of dimension one, where we write $\mathbb{R}$ for $\mathbb{R}^1$.

There is an extensive literature on selection theorems for multifunctions, but to our knowledge, there are no results giving conditions for a smooth (i.e. C^1) selection. On the other hand, the existing results are obtained in quite general spaces. For example, Michael's Continuous Selection Theorem [7] is valid for F defined on a complete metric space and mapping into the subsets of a Banach space. Recall, however, that here the values of F are required to be

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closed and convex. Many of the measurable-type selection theorems are also valid with very general hypotheses, as illustrated in the book by Castaing and Valadier [2], the survey by Wagner [13], and the many references contained therein. A highly readable treatment of the finite-dimensional case is given by Rockafellar [9]. There is also a Caratheodory-type selection result due to Lojasiewicz [6]. A Caratheodory function is a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\cdot, x)$ is measurable for each $x \in \mathbb{R}^n$ and $f(t, \cdot)$ is continuous for each $t \in \mathbb{R}$. The book of Aubin and Frankowska [1, chapter 9] provides an excellent exposition and reference to these results.

A common method used in obtaining the selections mentioned above is to construct a sequence of approximate selections that limits to a selection with the desired properties. Mere generalizations of this approach do not seem adequate in order to construct a smooth selection of a multifunction, for the C^1 conclusion is difficult to maintain in a limiting argument. Our proof is based upon a discretization procedure, followed by an elaborate patching together of the pieces. This helps to explain why the results are framed only in dimension one, where such discretizing and pasting is more readily realizable. Moreover, it is not a priori clear exactly which multifunctions should have C^1 selections, since there is no set-valued analogue of classical differentiability; obviously, if the values of the multifunction F should "collapse" to a *singleton* on some interval J (the term "singleton" refers to a set consisting of a single point), say $F(x) = \{f(x)\}$ for $x \in J$, then $f(\cdot)$ must be C^1 in the classical sense on the interior of J.

Recall that the Hamiltonian H of a multifunction F is given by

$$H(x, p) = \sup\{\langle v, p\rangle : v \in F(x)\}.$$

Let $(a, b) \subset \mathbb{R}$ be an open interval and $F : (a, b) \rightrightarrows \mathbb{R}$ be a Lipschitz multifunction with compact convex values. The main result Theorem 3.1 asserts that if the Clarke gradient of both of the Hamiltonian functions $x \mapsto H(\cdot, \pm 1)$ defined on (a, b) is submonotone (in the sense of Spingarn [12]), then F admits a C^1 selection on (a, b).

The issue of parametrizing a multifunction by smooth functions will be addressed by one of us in a forthcoming paper [4]; see [1, Sections 9.6, 9.7] for parametrization results under Caratheodory and measurability /Lipschitz assumptions. Both the selection and parametrization results in this paper appear in the first author's Ph.D. thesis [5], which was completed under the supervision of the second author.

The organization of the paper is as follows. Definitions and preliminaries are given in Section 2. The main result is stated in Section 3 along with some examples exemplifying the strength of the hypotheses. A further preparatory discussion is given in Section 4. The proof of Theorem 3.1 is finally provided in Section 5.

2 Definitions and Preliminary Discussion

Let $f : \mathbb{R}^n \to \mathbb{R}$. The function f is *locally Lipschitz* on the set $S \subseteq \mathbb{R}^n$ if for each $x_0 \in S$, there exists a neighborhood U of x_0 and a positive scalar K such that

$$|f(x) - f(y)| \leq K|x - y| \qquad \text{for all } x, y \in U.$$

If there is a global Lipschitz constant K for f on S, then f is simply called Lipschitz on S. The *directional derivative* of f at x in the direction $v \in \mathbb{R}^n$ is the quantity

$$f'(x; v) = \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

provided of course the limit exists.

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at x. The *generalized directional derivative* (in the sense of Clarke [3]) at x in the direction $v \in \mathbb{R}^n$ is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

We briefly recall some of the properties of $f^\circ(x; v)$ (see [3] for the complete exposition). The map $v \mapsto f^\circ(x; v)$ exists finitely at each $x \in \mathbb{R}^n$, is positively homogeneous, and is subadditive. The *Clarke gradient* of f at x is defined by

$$\partial f(x) = \{\xi \in \mathbb{R}^n \ : \ f^\circ(x; v) \geq \langle v, \xi \rangle \quad \text{for all } v \in \mathbb{R}^n\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$. The Clarke gradient can also be viewed as a multifunction, $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, and it has nonempty, compact, convex values. If f is C^1 on an open set U containing x (or more generally, strictly differentiable at x), then $\partial f(x)$ reduces to the singleton $\{\nabla f(x)\} = \partial f(x)$. Conversely, if $\partial f(x)$ is a singleton for each x belonging to an open set U, then f is C^1 on U.

A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is *subdifferentially regular* (introduced as *regular* in [3]) at $x \in \mathbb{R}^n$ if $f'(x; v)$ exists and $f'(x; v) = f^\circ(x; v)$ for all $v \in \mathbb{R}^n$. Subdifferential regularity allows "lower corners" but no "upper" ones. Note that any convex or C^1 function is subdifferentially regular. Many properties of subdifferentially regular functions are again collected in [3]. One notes that f is subdifferentially regular at $x \in \mathbb{R}^n$ if and only if $f'(x;v)$ exists for all v and

$$f'(x;v) = \sup_{\xi \in \partial F(x)} \langle \xi, v \rangle.$$

If both f and $-f$ are subdifferentially regular , then f is differentiable (in fact, strictly differentiable) at x and $\partial f(x) = \{\nabla f(x)\}$.

Recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *upper semi-continuous* (abbreviated as u.s.c.) at $x \in \mathbb{R}^n$ if

$$\limsup_{y \to x} \varphi(y) \le \varphi(x).$$

The function φ is u.s.c. on an open set U if φ is u.s.c. at each $x \in U$. The following characterization of subdifferentially regular is due to Rockafellar [10].

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then f is locally Lipschitz and subdifferentially regular on an open set U if and only if $f'(x;v)$ exists finitely for all $x, v \in \mathbb{R}^n$ and $x \mapsto f'(x;v)$ is upper semi-continuous on U.*

We now turn our attention to a discussion of multifunctions. Suppose $U \subset \mathbb{R}^n$ is open and $\Gamma : U \rightrightarrows \mathbb{R}^n$ is a multifunction, which we say has *nonempty*, *compact*, or *convex* values if at each $x \in U$, the set $\Gamma(x)$ is nonempty, compact, or convex, respectively. The multifunction Γ is *Lipschitz* on U if there exists a positive scalar K such that for all $x_1, x_2 \in U$ and all $\xi_1 \in \Gamma(x_1)$, there exists $\xi_2 \in \Gamma(x_2)$ such that

$$|\xi_1 - \xi_2| \le K|x_1 - x_2|.$$

A thorough exposition of basic properties of multifunctions is contained in [1].

The multifunction Γ is *submonotone* at $x_0 \in U$ provided

$$\liminf_{\substack{x \to x_0,\ x \ne x_0 \\ y \in \Gamma(x),\ y_0 \in \Gamma(x_0)}} \frac{\langle y - y_0,\ x - x_0 \rangle}{|x - x_0|} \ge 0. \tag{2.1}$$

Submonotonicity as such is defined by Spingarn [12]. Also defined in [12] is a strict submonotonicity condition, which is (2.1) with the added stipulation that the inequality be satisfied uniformly over x_0 in a compact set. Strict submonotonicity is used in [12] to characterize *lower-C^1* functions, which are those Lipschitz functions f that can be represented as the supremum of a compactly indexed family of C^1 functions. The characterization is that ∂f is strictly submonotone if and only if f is lower C^1. Strict submonotonicity will play a major role in the forthcoming paper [4] on smooth parametrizations of multifunctions, however, submonotonicity is sufficient for our purposes here. Spingarn [12] also characterized those functions f whose Clarke gradient ∂f

is submonotone. One ingredient of this characterization is the subdifferential regularity condition defined above; the other ingredient is defined next.

A sequence $\{x_n\}$ of points of $\mathbb{R}^n$ is said to *converge to x in the direction $v \in \mathbb{R}^n$*, written $x_n \overset{v}{\to}\to x$, provided $x_n \to x$, $x_n \neq x$ for all large n, and

$$\frac{x_n - x}{|x_n - x|} \to \frac{v}{|v|}$$

as $n \to \infty$. A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is *semismooth* at x, (defined originally by Mifflin [8]), provided that for all $v \in \mathbb{R}^n$, if $x_n \overset{v}{\to}\to x$ and $y_n \in \partial f(x_n)$, then $\langle v, y_n \rangle \to f'(x; v)$. In the case of differentiable functions, semismoothness at x just implies that the derivative is continuous at x. Notice that part of the definition requires that $f'(x; v)$ exists, although this actually can be deduced if in the definition one replaces $f'(x; v)$ by $\limsup_{h \to 0^+} \frac{1}{h}(f(x+hv) - f(x))$. If $U \subset \mathbb{R}^n$ is open, then f is semismooth on U if it is semismooth at each point of U. For nonsmooth Lipschitz functions, semismoothness on U implies that the function has directional derivatives in all directions, but moreover has the property that if points of nondifferentiability bunch up, then they must do so in a particular manner. The following theorem is due to Spingarn [12].

Theorem 2.2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz. Then ∂f is submonotone at x if and only if f is semismooth and subdifferentially regular at x.*

Theorems 2.1 and 2.2 will be of considerable help in proving our results. We close this section by setting up some notation and by illustrating the above concepts as they apply in dimension one, which is how they will be used in this paper.

Suppose $f : \mathbb{R} \to \mathbb{R}$. We set ${f_+}'(\cdot) = f'(\cdot; 1)$ and ${f_-}'(\cdot) = -f'(\cdot; -1)$. That is, f'_+ is the right derivative and f'_- is the left derivative:

$$f'_+(x) = \lim_{y \to x+} \frac{f(y) - f(x)}{y - x} \quad \text{and} \quad f'_-(x) = \lim_{y \to x-} \frac{f(y) - f(x)}{y - x}.$$

A locally Lipschitz function $f : (a, b) \to \mathbb{R}$ is regular at $x \in (a, b)$ if and only if both $f'_+(x)$ and $f'_-(x)$ exist and $f'_+(x) \geq f'_-(x)$. In this case, we have that $\partial f(x)$ is the compact interval $[f'_-(x), f'_+(x)]$. We also note that a Lipschitz function $f : (a, b) \to \mathbb{R}$ is semismooth on (a, b) provided that at each $x \in (a, b)$, we have

$$\{f'_+(x)\} = \lim_{y \to x+} \partial f(y) \text{ and } \{f'_-(x)\} = \lim_{y \to x-} \partial f(y) \tag{2.2}$$

both hold, where the limits are taken with respect to the Hausdorff metric.

3 Statement of the Main Result and Some Examples

Let $(a,b) \subset \mathbb{R}$ be a fixed bounded interval. Throughout the rest of this paper, we will consider a Lipschitz multifunction $F : (a,b) \rightrightarrows \mathbb{R}$ with nonempty, compact, and convex values. Specifically, then, the multifunction assigns to each $x \in (a,b)$ a nonempty (but possibly degenerate) closed bounded interval. Thus the values of the multifunction F can be described by writing $F(x) = [h(x), H(x)]$ for each $x \in (a,b)$, where h and H are real-valued functions satisfying $h(x) \leq H(x)$ for $x \in (a,b)$. The Lipschitz assumption on the multifunction F translates into nothing more than that each of the functions h and H are Lipschitz continuous in the ordinary sense of functions.

One of the many simplifications in working in one dimension is that only the two functions h and H are required to know everything about F. In higher dimensions, where convex sets can be quite complicated, one needs to know the support function to recapture the convex set. This is expressed in terms of the Hamiltonian function H as defined earlier. If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has nonempty, compact, and convex values, the relationship

$$F(x) = \{y \in \mathbb{R}^n : \langle y, p \rangle \leq H(x,p) \quad \text{for all} \quad p \in \mathbb{R}^n\}$$

holds. Since we shall be working exclusively in dimension $n = 1$, only the values $p = \pm 1$ in the Hamiltonian are needed to describe the multifunction; one has $H(x) := H(x,1)$ describing the "top" and $h(x) := -H(x,-1)$ describing the "bottom." In the statement of our result, we shall use only the functions h and H. It is reasonable to conjecture that our main theorem will remain true in higher dimensional spaces if the assumptions stated below for only the two functions $-h$ and H are rather assumed to hold for all of the functions $x \mapsto H(x,p)$, where p is a unit vector in $\mathbb{R}^n$.

The main result of this paper follows, which gives sufficient conditions for F to admit a C^1 selection.

Theorem 3.1. *If both of the subgradient multifunctions $\partial H(\cdot)$ and $\partial(-h)(\cdot)$ are submonotone on (a,b), then F admits a C^1 selection on (a,b).*

We next discuss under what kind of assumptions multifunctions might admit a C^1 selection. Suppose for example that F is Lipschitz, and each of its values is convex with nonempty interior. For such an F, there is a "tube" of values $(f(x) - \epsilon, f(x) + \epsilon) \subset F(x)$ where $f(x)$ is continuous. Such an F is uninteresting in this context because for every such tube, there is a C^1 function $g(\cdot)$ with $g(x) \in (f(x) - \epsilon, f(x) + \epsilon)$ for all $x \in (a,b)$, and hence the multifunction admits a C^1 selection trivially. It is much more intriguing to allow multifunctions to take on singleton values. A natural starting place is to re-examine Michael's Continuous Selection Theorem ([7]), which says that lower-semicontinuous multifunctions (with closed and convex values) admit continuous selections. Recall that a multifunction F is called *lower-semicontinuous* at x if for any $y \in F(x)$ and for any sequence x_n converging to x, there exists a sequence of elements $y_n \in F(x_n)$ which converge to y. The following simple example shows that this assumption is not sufficient to insure a C^1 selection, even though almost all of the values of F have interior.

Example 3.1. Consider the following multifunction defined on $(-1, 1)$. Let

$$F(x) = \begin{cases} \{|x|\}, & \text{if } x = \pm\frac{1}{n} \text{ for } n \in \mathbb{N}, \\ [0, 1], & \text{if } x \neq \pm\frac{1}{n} \text{ for } n \in \mathbb{N}. \end{cases}$$

It is easy to see that F is lower semicontinuous. Moreover, any selection f of F must satisfy $f(\pm\frac{1}{n}) = \frac{1}{|n|}$ for each $n \in \mathbb{N}$, and thus clearly cannot be differentiable at 0.

We now further examine the strength of the hypotheses in Theorem 3.1. First recall (Theorem 2.2) that the submonotonicity condition on the subgradient multifunction $\partial H(\cdot)$ is equivalent to $H(\cdot)$ being semismooth and subdifferentially regular, and a similar statement holds with regard to $-h$. The first example provides a multifunction F for which $H(\cdot)$ and $-h(\cdot)$ are semismooth, but F does not admit a C^1 selection.

Example 3.2. Consider the following multifunction defined on $(-1, 1)$. For $x \in (-1, 1)$ let

$$F(x) = \{y \ : \ |x| \leq y \leq 2|x|\}.$$

Then F does not admit a C^1 selection, due to the nature of the corner at 0. Indeed, if $f(x) \in F(x)$ for all $x \in (-1, 1)$, then

$$\limsup_{x \to 0-} \frac{f(x) - f(0)}{x} \leq -1 < 1 \leq \liminf_{x \to 0+} \frac{f(x) - f(0)}{x},$$

and thus cannot be differentiable at 0. The only hypothesis not satisfied in Theorem 3.1 is that $-h(x) = -|x|$ is not subdifferentially regular at $x = 0$.

Next we show the existence of a multifunction F for which $H(\cdot)$ and $-h(\cdot)$ are both subdifferentially regular, but still F does not admit a C^1 selection due to the lack of semismoothness.

Example 3.3. In this example, the multifunction F will also be defined on the interval $(-1,1)$. We first let

$$F(x) := [x, -x] \quad \text{whenever} \quad x \in (-1, 0].$$

Whenever $x \in (0,1)$, the value $F(x)$ will be a singleton (say, $F(x) = \{f(x)\}$), but is specifically somewhat difficult to describe (we do so below). The properties we seek for f is that it be C^1 on $(0,1)$, with its derivative $f'(x)$ taking on both the values of 0 and 1 for arbitrarily small $x > 0$, but in such a manner that the limit $\lim_{x\to 0^+} \frac{f(x)}{x}$ exists and equals one. Suppose we have such an f, then it is clear that the multifunction F can have no C^1 selection, because any selection will have an oscillating derivative as $x \to 0+$. But it can be verified that both H and $-h$ are subdifferentially regular (when $x \neq 0$, this is trivial; the case of $x = 0$ is somewhat less obvious, but easy to verify in lieu of the the following representation:

$$\begin{aligned} \partial f(x) = \overline{\text{co}} \Big\{ \xi : \exists \{x_i\} \text{ so that } & x_i \to x, \nabla f(x_i) \text{ exists, and} \\ & \lim_{i\to\infty} \nabla f(x_i) = \xi \Big\}, \end{aligned} \tag{3.1}$$

where $\overline{\text{co}}\, S$ denotes the closed convex hull of the set S. The representation (3.1) is Theorem 2.5.1 in [3]. The hypotheses of Theorem 3.1 not satisfied here is that H and $-h$ are not semismooth. We now describe a function f as indicated above. First, we define $f(x)$ for some values of x. Let

$$f(x) = \frac{3}{16} \quad \text{for} \quad x \in \Big[\frac{1}{4}, 1\Big)$$
$$f(x) = x - x^2 \quad \text{for} \quad x = \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$$

We next describe the derivative $f'(x)$ of f on certain intervals. The function f itself is then found by integrating. The slightly complicating aspect of this procedure is that it must be ensured that these integrations agree with the above values already assigned to f. For $n = 4, 6, 8, \ldots,$ $f'(\cdot)$ is continuous and decreases from 1 to 0 on the interval $[\frac{1}{(n+1)}, \frac{1}{n}]$. For $n = 5, 7, 9, \ldots,$ $f'(x) = 1$ whenever $x \in [\frac{n^3+n^2+4n+2}{n^2(n+1)^2}, \frac{1}{n}]$, and f' increases linearly from 0 to 1 on the interval $[\frac{1}{n+1}, \frac{n^3+n^2+4n+2}{n^2(n+1)^2}]$. Note that $\frac{1}{n+1} < \frac{n^3+n^2+4n+2}{n^2(n+1)^2} < \frac{1}{n}$, so the latter is well defined.

To summarize the examples in this section, we have demonstrated that the conditions of subdifferential regularity and semismoothness are not by themselves sufficient to obtain a C^1 selection.

4 Further Discussion

In this section, we present the tools needed to prove Theorem 3.1.

As previously mentioned, it is a simple matter to find a C^1 selection when F has an interior, which happens when

$$H(x,-1) = \inf\{y : y \in F(x)\} < \sup\{y : y \in F(x)\} = H(x,1).$$

However, when F collapses to a single value (that is, when $-H(x,-1) = H(x,1)$), a selection must be chosen carefully so that it will pass through such a point while remaining smooth. If F remains a singleton on an entire closed interval $J \subseteq (a,b)$, say $F(x) = \{f(x)\}$, then it is clear that f must be C^1 on J in the sense of ordinary functions, since any selection must equal $f(x)$. The following lemma deals with this case. We say that a function f is C^1 on a closed interval $J = [c,d]$ provided that f is ordinarily C^1 on (c,d), both $f'_+(c)$ and $f'_-(d)$ exist, and the following limits hold:

$$\lim_{x\to c+} f'(x) = f'_+(c) \text{ and } \lim_{x\to d-} f'(x) = f'_-(d). \tag{4.1}$$

Lemma 4.1. *Suppose the multifunction F as in Theorem 3.1 collapses to a singleton on a closed interval $J = [c,d] \subset (a,b)$, which means that $F(x) = \{H(x)\} = \{h(x)\}$ for $x \in J$. Then $H = h$ is C^1 on J.*

Proof. Let $f(x) = H(x) = h(x)$ for $x \in J$, and recall Theorem 2.2. Both f and $-f$ are regular on the interior of J, and subsequently f is C^1 there. The semismoothness property (2.2) immediately implies that (4.1) holds, and hence f is C^1 on J. □

We now distinguish between points according to whether the values of F collapse. To accomplish this, we define the sets

$$V = \{x \in I : -H(x,-1) < H(x,1)\},$$
$$A = \{x \in I : -H(x,-1) = H(x,1)\}.$$

Now, since each of $H(x,1)$ and $H(x,-1)$ is continuous, each $x \in V$ lies in an open interval contained in V. So V is open and therefore V is the union of disjoint open intervals

$$V = \bigcup_{i=1}^{\infty} (c_i, d_i). \tag{4.2}$$

The main difficulty in obtaining a C^1 selection of F on (a,b) consists of constructing a C^1 selection of F defined on V that can be extended to a selection on the closure of V while maintaining smoothness. We begin by developing some tools. The following lemma will enable us to "splice" two C^1 functions together without losing any differentiability.

Lemma 4.2. *Suppose $[y,z] \subset \mathbb{R}$ and f, g are C^1 on $[y,z]$. Then there exists a C^1 function v on $[y,z]$ such that*

1. *$v(y) = f(y)$, $v'_+(y) = f'_+(y)$, $v(z) = g(z)$, and $v'_-(z) = g'_-(z)$.*
2. *$v(x)$ lies between or equal to $f(x)$ and $g(x)$ for each $x \in [y,z]$.*
3. *Let M and N be such that $M \le f'(x), g'(x) \le N$ for each $x \in [y,z]$ and $S = \sup_{x\in[y,z]} |f(x) - g(x)|$. Then $M - \frac{2S}{z-y} \le v'(x) \le N + \frac{2S}{z-y}$ for each $x \in [y,z]$, where we use the convention $v'(y) = v'_+(y)$ and $v'(z) = v'_-(z)$.*

PROOF. Let $\delta = \frac{z-y}{2}$. Define, for $x \in [y,z]$,

$$\begin{aligned} a'(x) &= \begin{cases} \frac{y-x}{\delta^2}, & x \in \left[y, \frac{y+z}{2}\right] \\ \frac{x-z}{\delta^2}, & x \in \left[\frac{y+z}{2}, z\right] \end{cases} \\ b'(x) &= \quad -a'(x) \end{aligned}$$

For $x \in [y,z]$, let

$$\begin{aligned} a(x) &= 1 + \int_y^x a'(t)\,dt \qquad \text{and} \\ b(x) &= \int_y^x b'(t)\,dt\,. \end{aligned}$$

The following properties can be easily verified.

$$a(x),\, b(x) \in [0,1] \;\text{ and }\; a(x) + b(x) = 1 \;\text{ for each }\; x \in [y,z], \tag{4.3}$$

$$\begin{gathered} a'(y) = a'(z) = b'(y) = b'(z) = 0, \\ a(z) = b(y) = 0, \quad a(y) = b(z) = 1, \end{gathered} \tag{4.4}$$

$$|a'(x)| = |b'(x)| \le \frac{2}{z-y} \;\text{ for each }\; x \in [y,z]. \tag{4.5}$$

Now, we set $v(x) = a(x)f(x) + b(x)g(x)$ for $x \in [y,z]$. Then assertions (1) and (2) of the lemma follow easily from (4.3) and (4.4). We also see that

$$\begin{aligned} v'(x) &= a'(x)f(x) + a(x)f'(x) + b'(x)g(x) + b(x)g'(x) \\ &\le b'(x)\left(g(x) - f(x)\right) + a(x)N + b(x)N \qquad (\text{since } a'(x) = -b'(x)) \\ &\le \frac{1}{\delta}\,|f(x) - g(x)| + N(a(x) + b(x)) \qquad (\text{by (4.5) and since } b' \ge 0) \\ &\le N + \frac{S}{\delta}, \end{aligned}$$

where $S = \sup_{x\in[y,z]} |f(x) - g(x)|$. A similar argument yields the lower bound thereby establishing assertion (3). $\square$

Remark 4.3. *1. Consider the functions $a(\cdot)$ and $b(\cdot)$ above as functions of the endpoints of the interval $[y, z]$ as well as of x (i.e. $a(x, y, z) = 1 + \int_y^x a'(t)\, dt$ for $x \in [y, z]$ and $b(\cdot,\cdot,\cdot)$ similarly). Then each of $a(\cdot,\cdot,\cdot)$, $a_x(\cdot,\cdot,\cdot)$, $b(\cdot,\cdot,\cdot)$, and $b_x(\cdot,\cdot,\cdot)$ is jointly continuous from $\mathbb{R}^3 \to \mathbb{R}$. So the functions $v(\cdot,\cdot,\cdot)$ and $v_x(\cdot,\cdot,\cdot)$ above are also jointly continuous for $x \in [y, z]$. Note that $v(x, y, z)$ may blow up to infinity if $y \to z$.*

2. The estimate in Lemma 4.2(3) suffices nicely for our purposes in this paper. However, in the companion paper [4] of parametrizing multifunctions by smooth functions, a somewhat more pointwise version is required. Suppose $f(x) \geq g(x)$ for all $x \in [y, z]$. Then the estimate can be stated by replacing S with $(f(x) - g(x))$.

The following lemma will enable us to guide our selection through points of A; that is, through those spots where F collapses to a singleton.

Lemma 4.4. *Suppose each of $H(\cdot)$ and $-h(\cdot)$ is subdifferentially regular at $x \in (a, b)$, and $H(x, 1) = -H(x, -1)$. Then*

$$-\partial H(x, -1) \cap \partial H(x, 1) \neq \emptyset.$$

Furthermore, if either $H'_-(x) = h'_-(x)$ or $H'_+(x) = h'_+(x)$, then the intersection consists of a single point.

Proof. Suppose $H(x, 1) = -H(x, -1)$. We have by regularity (and since $\partial(-h)(x) = -\partial h(x)$) that

$$\begin{aligned} \partial H(x, 1) &= [H'_-(x), H'_+(x)] \quad \text{and} \\ -\partial H(x, -1) &= [h'_+(x), h'_-(x)]. \end{aligned} \tag{4.6}$$

Also, since $h(\cdot) \leq H(\cdot)$ with equality at x, we have

$$h'_+(x) \leq H'_+(x) \quad \text{and} \quad H'_-(x) \leq h'_-(x). \tag{4.7}$$

If either $h'_+(x) \in \partial H(x, 1)$ or $h'_-(x) \in \partial H(x, 1)$, then we are done. If neither of these are true, then from (4.6) and (4.7) we see that $h'_+(x) < H'_-(x)$ and $h'_-(x) > H'_+(x)$, in which case the inclusion $\emptyset \neq \partial H(x, 1) \subset -\partial H(x, -1)$ holds by (4.6).

The furthermore assertion follows immediately from (4.6) and (4.7). $\square$

Our construction of a selection of F on V will use Theorem 2.1, the result saying that the directional derivatives of subdifferentially regular functions are u.s.c. The following proposition from advanced calculus facilitates our exploitation of this property.

Lemma 4.5. *Let J be a compact interval, and suppose $\varphi : J \to \mathbb{R}$ is u.s.c. Then there is a sequence of continuous functions $\Phi_n : J \to \mathbb{R}$ such that*

1. $\Phi_n(x) \searrow \varphi(x)$ *as* $n \to \infty$ *for each* $x \in J$.
2. $\inf_{y\in J} \varphi(y) \le \Phi_n(x) \le \sup_{y\in J} \varphi(y)$ *for each* $x \in J$ *and* $n \in \mathbb{N}$.

PROOF. See [11, page 50] for an outline of the proof. □

We next apply the last lemma to one-sided derivatives of certain continuous selections of F. Suppose $J = [c, d] \subseteq (a, b)$ is a closed interval and that $\beta : (a, b) \to \mathbb{R}$ is either linear or is a continuous function so that the restrictions of β to $[c, \frac{c+d}{2}]$ and $[\frac{c+d}{2}, d]$ are both linear. We also assume that $\beta(x) \in [0, 1]$ for $x \in J$. For these types of β functions, we define

$$K_\beta(x) = \beta(x)H(x, 1) - (1 - \beta(x))H(x, -1) \quad \text{for each } x \in J. \tag{4.8}$$

The function K_β stays between $H(x, -1)$ and $H(x, 1)$ on J since $\beta(\cdot)$ has values in $[0, 1]$. Moreover, $(K_\beta)'_\pm(x)$ exists for each $x \in (a, b)$ since $h'_\pm(x)$, $H'_\pm(x)$, and $\beta'_\pm(x)$ all exist.

Proposition 4.6. *Suppose each of $H(\cdot)$ and $-h(\cdot)$ is subdifferentially regular. Let $J \subset (a, b)$ be a closed interval and suppose $\beta : (a, b) \to \mathbb{R}$ is one of the functions as described in the last paragraph. Then there exists a sequence of continuous functions $C_n : J \to \mathbb{R}$ such that*

1. $\lim_{n\to\infty} C_n(x) = (K_\beta)'_+(x)$ *for each* $x \in J$;
2. $M^x_J \le C_n(x) \le N^x_J$ *for each* $x \in J$ *and* $n \in \mathbb{N}$, *where*

$$\begin{aligned} N^x_J &= \beta'_+(x)H(x, 1) + \beta(x) \sup_{y\in J}\{\xi : \xi \in \partial H(y)\} \\ &\quad + \beta'_+(x)H(x, -1) + (1 - \beta(x)) \sup_{y\in J}\{\xi : \xi \in \partial h(y)\}, \qquad \textit{and} \\ M^x_J &= \beta'_+(x)H(x, 1) + \beta(x) \inf_{y\in J}\{\xi : \xi \in \partial H(y)\} \\ &\quad + \beta'_+(x)H(x, -1) + (1 - \beta(x)) \inf_{y\in J}\{\xi : \xi \in \partial h(y)\}. \end{aligned}$$

PROOF. Recall Rockafellar's Theorem 2.1, from which we have that each of $H'_+(\cdot)$ and $-h'_+(\cdot)$ is u.s.c. Applying Lemma 4.5 twice, once each to $\varphi = H'_+, -h'_+$, we obtain two sequences of continuous functions, $\Phi_n,\ \Psi_n : J \to \mathbb{R}$, such that

$$\lim_{n\to\infty} \Phi_n(x) = H'_+(x) \qquad \text{and} \qquad \lim_{n\to\infty} \Psi_n(x) = -h'_+(x) \tag{4.9}$$

for each $x \in J$. For each $n \in \mathbb{N}$ and $x \in J$, let

$$\begin{aligned} C_n(x) = {} & \beta'_+(x)H(x,1) + \beta(x)\Phi_n(x) \\ & + \beta'_+(x)H(x,-1) - (1-\beta(x))\Psi_n(x). \end{aligned} \tag{4.10}$$

Using the product rule in differentiating (4.8), we have that

$$\begin{aligned} (K_\beta)'_+(x) = {} & \beta'_+(x)H(x,1) + \beta(x)H'_+(x) \\ & + \beta'_+(x)H(x,-1) + (1-\beta(x))h'_+(x). \end{aligned} \tag{4.11}$$

Hence the convergences in (4.9) immediately imply that (4.10) converges to (4.11), or that assertion (1) holds.

By Lemma 4.5(2) and (4.6), we have

$$\begin{aligned} &\Phi_n(x) \le \sup_{y\in J} H'_+(y) \le \sup_{y\in J}\{\xi : \xi \in \partial H(y)\} \qquad \text{and} \\ &\Psi_n(x) \ge \inf_{y\in J} -h'_+(y) \ge -\sup_{y\in J}\{\xi : \xi \in \partial h(y)\}. \end{aligned} \tag{4.12}$$

Then since $0 \le \beta(x) \le 1$ on J, for each $x \in J$ and $n \in \mathbb{N}$ we deduce from (4.12) that

$$\begin{aligned} C_n(x) &= \beta'_+(x)H(x,1) + \beta(x)\Phi_n(x) + \beta'_+(x)H(x,-1) - (1-\beta(x))\Psi_n(x) \\ &\le \beta'_+(x)H(x,1) + \beta(x)\sup_{y\in J}\{\xi : \xi \in \partial H(y)\} \\ &\qquad + \beta'_+(x)H(x,-1) + (1-\beta(x))\sup_{y\in J}\{\xi : \xi \in \partial h(y)\} \\ &= N_J^x. \end{aligned}$$

A similar argument shows that the lower estimate involving M_J^x also holds, thereby yielding assertion (2). □

5 Proof of Theorem 3.1

We are now ready to prove the selection theorem. Lemma 4.4 guarantees that for each $x \in A$, we may choose a point $\xi_x \in -\partial H(x,-1) \cap \partial H(x,1)$. Our selection f will have the property that $f'(x) = \xi_x$ for each $x \in A$. We fix the choice of ξ's in advance because we want to handle the following situation. Suppose (c_i, d_i) and (c_j, d_j) are distinct intervals in V with $d_i = c_j$ (recall (4.2)). It must here be the case that $d_i = c_j \in A$. Suppose C^1 selections f_i and f_j as above are defined on $[c_i, d_i]$ and $[c_j, d_j]$ respectively. Since it is predetermined that $f'_i(d_i) = f'_j(c_j)$, then we can paste f_i and f_j together at $d_i = c_j$ and have a C^1 selection on the entire interval $[c_i, d_j]$.

Now we fix one of the disjoint intervals $(c,d) = (c_i, d_i)$ that comprise V. If either $c = a$ or $d = b$, then we need not worry about that particular endpoint, so we assume the closed interval $[c,d]$ lies in (a,b). The most difficult part of the proof is to construct a C^1 selection f on $[c,d]$ with $f'_+(c) = \xi_c$ and $f'_-(d) = \xi_d$, which is what we do next.

Note from (4.6) that

$$h'_+(c) \le \xi_c \le H'_+(c) \quad \text{and} \quad H'_-(d) \le \xi_d \le h'_-(d).$$

If $h'_+(c) = H'_+(c)$ then set $\beta(c) = 0$, and similarly if $H'_-(d) = h'_-(d)$ then set $\beta(d) = 0$. Otherwise, we set

$$\beta(c) = \frac{\xi_c - h'_+(c)}{H'_+(c) - h'_+(c)} \quad \text{and} \quad \beta(d) = \frac{h'_-(d) - \xi_d}{h'_-(d) - H'_-(d)},$$

which are quantities lying $[0,1]$. We now want to define $\beta : \mathbb{R} \to \mathbb{R}$ so that it is in agreement with the above quantities $\beta(c)$ and $\beta(d)$ at the points c and d respectively, and such that $0 < \beta(x) < 1$ for all $x \in (c,d)$. To do this we simply define β as

$$\beta(x) = \frac{1}{d-c}\{(x-c)\beta(d) + (d-x)\beta(c)\}$$

unless $\beta(c) = \beta(d) = 0$ or $\beta(c) = \beta(d) = 1$. In the special cases, if $\beta(c) = \beta(d) = 0$, then define

$$\beta(x) = \begin{cases} \dfrac{x-c}{d-c}, & x \in [c, \frac{c+d}{2}] \\ \dfrac{d-x}{d-c}, & x \in [\frac{c+d}{2}, d]. \end{cases}$$

Similarly if $\beta(c) = \beta(d) = 1$, then define

$$\beta(x) = \begin{cases} \dfrac{d-x}{d-c}, & x \in [c, \frac{c+d}{2}] \\ \dfrac{x-c}{d-c}, & x \in [\frac{c+d}{2}, d]. \end{cases}$$

Note that in all cases, we have

$$|\beta'_+(x)| \le \frac{1}{d-c} \qquad \text{for each } x \in [c,d], \tag{5.1}$$

and that β is a function of the type introduced earlier in Section 4 when used to define K_β. It can be readily calculated that the one-sided derivatives at the endpoints c and d of K_β exist and equal

$$(K_\beta)'_+(c) = \xi_c \quad \text{and} \quad (K_\beta)'_-(d) = \xi_d. \tag{5.2}$$

The strategy now is to construct a C^1 function on $[c,d]$ so that it closely resembles the function $K_\beta(\cdot)$. We first work on the half interval $[\frac{c+d}{2},d]$; the other half $[c,\frac{c+d}{2}]$ will be handled similarly.

We fix two interwoven strictly increasing sequences $\{y_i\}_{i=0}^\infty$ and $\{z_i\}_{i=0}^\infty$ so that

$$y_0 = z_0 < \frac{c+d}{2} < y_1 < z_1 < y_2 < z_2 < \cdots < y_i < z_i < \cdots < d$$

such that $y_i, z_i \to d$ as $i \to \infty$. For each $i \in \mathbb{N}$, we set

$$J_i = [y_{i-1}, z_i].$$

Now, since $(c,d) \subseteq V$ and $\beta(x) \in (0,1)$ for all $x \in (c,d)$, the value $K_\beta(x)$ lies strictly between $h(x)$ and $H(x)$ on J_i, and so there exists an $\epsilon_i > 0$ such that $K_\beta(x) \pm \epsilon_i$ lies in the interior of $F(x)$ for each $x \in J_i$. For further technical reasons, we also stipulate that these choices satisfy

$$\epsilon_i \le \frac{\min\{(d-z_i),(z_i-y_i)\}}{2i}(d-c) \tag{5.3}$$

and $\epsilon_{i+1} \le \epsilon_i$ for all i.

Next, for each $i \in \mathbb{N}$, we use Proposition 4.6 on the interval $J = J_i$ to obtain a sequence of continuous functions $\{C_n^i\}_{n=1}^\infty$ that converge pointwise on each J_i to $(K_\beta)'_+$ and satisfy the bound

$$M_{J_i}^x \le C_n^i(x) \le N_{J_i}^x \tag{5.4}$$

for all $i, n \in \mathbb{N}$ and $x \in J_i$. By (5.4), we see that

$$\sup_{\substack{n\in\mathbb{N}\\ x\in J_i}} C_n^i(x) < \infty,$$

and so for each $i \in \mathbb{N}$ and for any measurable set $I \subseteq J_i$ we have that

$$\lim_{n\to\infty} \int_I C_n^i(t)\,dt = \int_I (K_\beta)'_+(t)\,dt \tag{5.5}$$

by the Dominated Convergence Theorem. Define $f_n^i : J_i \to \mathbb{R}$ by

$$f_n^i(x) = K_\beta(y_{i-1}) + \int_{y_{i-1}}^x C_n^i(t)\,dt.$$

Note that the derivative K'_β exists almost everywhere since K_β is Lipschitz, and so by integration we can recover K_β from $(K_\beta)'_+$. Thus by (5.5) we have for each $x \in J_i$ that

$$\begin{aligned} K_\beta(x) &= K_\beta(y_{i-1}) + \int_{y_{i-1}}^{x} (K_\beta)'_+(t)\,dt \\ &= K_\beta(y_{i-1}) + \lim_{n\to\infty} \int_{y_{i-1}}^{x} C_n^i(t)\,dt \\ &= \lim_{n\to\infty} f_n^i(x). \end{aligned} \tag{5.6}$$

Hence $\{f_n^i\}_{n=1}^\infty$ is a sequence of C^1 functions converging pointwise on J_i to the continuous function K_β. Furthermore, it is clear from (5.4) that $\{f_n^i\}_{n=1}^\infty$ is equicontinuous for each $i \in \mathbb{N}$, and consequently the convergence in (5.6) is uniform over $x \in J_i$. Therefore for each $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ large enough so that

$$\sup_{x \in J_i} \left| f_{n_i}^i(x) - K_\beta(x) \right| < \epsilon_i. \tag{5.7}$$

To simplify notation, we set $f^i(x) = f_{n_i}^i(x)$. Note from the estimate (5.4) that

$$M_{J_i}^x \le \left(f^i\right)'(x) \le N_{J_i}^x \qquad \text{for each } x \in J_i. \tag{5.8}$$

The domains of the functions f^i and f^{i+1} overlap on the interval $[y_i, z_i]$, and it is here that we use our smooth splicing lemma. For each $i \in \mathbb{N}$, we obtain a C^1 function v^i by invoking Lemma 4.2 to the functions $f = f^i$ and $g = f^{i+1}$ on the interval $[y, z] = [y_i, z_i]$. Then v^i and its derivative match up with f^i at y_i and with f^{i+1} at z_i as described in Lemma 4.2(1). We can now define the function f on $[y_0, d)$ by

$$f(x) = \begin{cases} f^i(x), & x \in [z_{i-1}, y_i] \\ v^i(x), & x \in [y_i, z_i]. \end{cases}$$

By construction, and in particular by the properties in Lemma 4.2(1), the function f is C^1 on $[y_0, d)$.

Recall from Lemma 4.2(2) that $v^i(x)$ lies between $f^i(x)$ and $f^{i+1}(x)$ whenever $x \in [y_i, z_i]$. Then from (5.7) it follows that

$$|f(x) - K_\beta(x)| < \epsilon_i \quad \text{for all} \quad x \in J_i. \tag{5.9}$$

By the choice of ϵ_i, it follows from (5.9) that f is a selection of F on $[y_0, d)$. Recall that K_β is continuous on $[c, d]$, and thus (5.9) also implies that f can be extended continuously to $[y_0, d]$ by setting $f(d) = K_\beta(d)$, which is the unique element in $F(d)$. We still need to further analyze the behavior of f near d. The left derivative of f at d is calculated next.

Note from (5.9) that for each $x \in J_i$, the following inequality holds:

$$\left| \frac{f(x) - K_\beta(x)}{x - d} \right| \leq \frac{\epsilon_i}{d - z_i} \tag{5.10}$$

We thus have by (5.10) that

$$\begin{aligned} |f_-(d) - \xi_d| &= |f_-(d) - (K_\beta)'_-(d)| \\ &= \lim_{x \to d^-} \left| \frac{f(x) - f(d)}{x - d} - \frac{K_\beta(x) - K_\beta(d)}{x - d} \right| \\ &= \lim_{x \to d^-} \left| \frac{f(x) - K_\beta(x)}{x - d} \right| \\ &\leq \lim_{i \to \infty} \frac{\epsilon_i}{d - z_i} \\ &= 0, \end{aligned}$$

where (5.3) is used to deduce the final equality. We have shown f has the desired left derivative $f'_-(d) = \xi_d$.

We next show that $f'(x) \to \xi_d$ as $x \nearrow d$. The following lemma contains the crucial facts needed to prove this.

Lemma 5.1. *Suppose $\{x_k\}_{k=1}^\infty \subset [y_0, d)$ satisfies $x_k \nearrow d$, and let $i(k) \in \mathbb{N}$ be such that $x_k \in J_{i(k)}$. Then*

$$\xi_d = \lim_{k \to \infty} N_{J_{i(k)}}^{x_k} = \lim_{k \to \infty} M_{J_{i(k)}}^{x_k},$$

where N_J^x and M_J^x are defined as in Proposition 4.6.

PROOF. Recalling the definition of N_J^x and the semismoothness property (2.2), we see that

$$\begin{aligned} \lim_{k \to \infty} N_{J_{i(k)}}^{x_k} = \lim_{k \to \infty} \Big(& \beta'(x_k) H(x_k) + \beta(x_k) \sup_{y \in J_{i(k)}} \{\xi : \xi \in \partial H(y)\} \\ & - \beta'(x_k) h(x_k) + (1 - \beta(x_k)) \sup_{y \in J_{i(k)}} \{\xi : \xi \in \partial h(y)\} \Big) \\ = \beta(d) H'_-(d) &+ (1 - \beta(d)) h'_-(d). \end{aligned}$$

If $H'_-(d) = h'_-(d)$, then Lemma 4.4 implies that $\xi_d = H'_-(d) = h'_-(d)$, and the first limit in the lemma holds. If $H'_-(d) \neq h'_-(d)$, then the above string of equalities can be extended by inserting the value of $\beta(d)$, and we conclude that

$$\begin{aligned}\lim_{k\to\infty} N^{x_k}_{I_{i(k)}} &= \left(\frac{h'_-(d)-\xi_d}{h'_-(d)-H'_-(d)}\right)H'_-(d) + \left(\frac{\xi_d - H'_-(d)}{h'_-(d)-H'_-(d)}\right)h'_-(d)\\ &= \xi_d.\end{aligned}$$

The second limit in the lemma, the one involving $M^{x_k}_{J_{i(k)}}$, is proven in an analogous manner. □

Armed with the preceding lemma, we are now ready to prove $\lim_{x\to d-} f'(x) = \xi_d$. Let $\{x_k\}_{k=1}^{\infty} \subset [y_0, d)$ be an arbitrary sequence converging monotonically up to d. For each $k \in \mathbb{N}$ there exists $i(k) \in \mathbb{N}$ with $x_k \in J_{i(k)} = [y_{i(k)-1}, z_{i(k)}]$. Three possibilities arise; either $x_k \in (z_{i(k)-1}, y_{i(k)})$, $x_k \in [y_{i(k)}, z_{i(k)}]$, or $x_k \in [y_{i(k)-1}, z_{i(k)-1}]$. In each case we shall obtain a suitable estimate on $f'(x_k)$.

First consider the possibility that $x_k \in (z_{i(k)-1}, y_{i(k)})$. Then $f(\cdot) = f^{i(k)}(\cdot)$ in a neighborhood of x_k, in which case $f'(x_k) = \left(f^{i(k)}\right)'(x_k)$. Combining this with (5.8) we have

$$M^{x_k}_{J_{i(k)}} \le f'(x_k) \le N^{x_k}_{J_{i(k)}}. \tag{5.11}$$

Next, suppose $x_k \in [y_{i(k)}, z_{i(k)}]$. Then $f(x_k) = v^{i(k)}(x_k)$ and $f'(x_k) = (v^{i(k)})'(x_k)$, and so by Lemma 4.2(3), we have

$$M^{x_k}_{J_{i(k)}} - \frac{2S_k}{z_{i(k)} - y_{i(k)}} \le f'(x_k) \le N^{x_k}_{J_{i(k)}} + \frac{2S_k}{z_{i(k)} - y_{i(k)}}, \tag{5.12}$$

where S_k is defined by

$$S_k = \sup_{x\in[y_{i(k)}, z_{i(k)}]} |f^{i(k)}(x) - f^{i(k)+1}(x)|.$$

We have by (5.9) and (5.3) that

$$S_k \le 2\epsilon_{i(k)} \le \frac{d-c}{i(k)}.$$

Inserting this estimate on S_k into (5.12) yields

$$M^{x_k}_{J_{i(k)}} - \frac{(d-c)}{i(k)} \le f'(x_k) \le N^{x_k}_{J_{i(k)}} + \frac{(d-c)}{i(k)}. \tag{5.13}$$

The third possibility, that is $x_k \in [y_{i(k)-1}, z_{i(k)-1}]$, can be handled like the previous one, where we can obtain the estimate (5.13) x_k by replacing $i(k)$ by $i(k) - 1$.

We now can let $k \to \infty$, which in turn means that $i(k) \to \infty$. Using the appropriate estimate (5.11) or (5.13) for each k, we conclude from Lemma 5.1 that $f'(x_k) \to \xi_d$.

We have produced a C^1 selection of F on the interval $[y_0, d]$. A similar argument can be employed to obtain a C^1 selection on $[c, \hat{y}_0]$, where $\hat{y}_0 > \frac{c+d}{2}$, and with right derivative at c equal to ξ_c. Using Lemma 4.2, it is then a simple matter to splice these two selections on the interval $[y_0, \hat{y}_0]$ to obtain a C^1 selection $f_{[c,d]}$ of F on $[c, d]$ so that $(f_{[c,d]})'_+(c) = \xi_c$ and $(f_{[c,d]})'_-(d) = \xi_d$.

A selection f on (a, b) can now be defined by setting $f(x) = f_{[c_i,d_i]}(x)$ if $x \in V$ with $x \in (c_i, d_i)$, and letting $f(x)$ be the unique element in $F(x)$ if $x \in A$. It can be readily deduced from (5.9) that f is continuous on (a, b), but the C^1 property must still be verified, although this is also clear for at least some points.

The selection is obviously C^1 near any point $x \in V$. If $x \in A$ is an isolated point of A, then $x = c_i = d_j$ for some choices of i and j, and the above construction also gives that f is C^1 near x (the derivative f' is continuous at x because ξ_x was chosen a priori; see the remarks at the beginning of this section). Hence we need only verify that f is continuously differentiable at points of accumulation of A. The next lemma gives some information about such a point.

Lemma 5.2. *Suppose $x \in A$ and there exists a sequence $\{x_k\} \subseteq A$ so that $x_k \nearrow x$ as $k \to \infty$. Then ξ_x is uniquely determined satisfying $\xi_x = H'_-(x) = h'_-(x)$, and $\xi_{x_k} \to \xi_x$ as $k \to \infty$. Similarly if $\{x_k\} \subseteq A$ is such that $x_k \searrow x$, then $\xi_x = H'_+(x) = h'_+(x)$ and $\xi_{x_k} \to \xi_x$ as $k \to \infty$.*

Proof. Suppose $x_k \nearrow x$ for all k. Since $\xi_{x_k} \in \partial H(x_k) \cap \partial h(x_k)$ and H and $-h$ are semismooth, we have by (2.2) that $\xi_{x_k} \to H'_-(x)$ and $\xi_{x_k} \to h'_-(x)$. Thus $H'_-(x) = h'_-(x)$, and by Lemma 4.4, we have that ξ_x is uniquely determined as the element $H'_-(x) = h'_-(x)$.

A similar argument shows that if $x_k \searrow x$, then $\xi_{x_k} \to \xi_x = H'_+(x) = h'_+(x)$. $\square$

We can now show that f is differentiable at a point of accumulation of A, and therefore at each point of A. Let us suppose first that x is a limit point of $(c, x) \cap A$. From Lemma 5.2, we have that $\xi_x = h'_-(x) = H'_-(x)$. Since $f(\cdot)$ is a selection it satisfies $h(y) \le f(y) \le H(y)$ for all y, and subsequently it follows that $f'_-(x)$ exists and equals ξ_x. If x is not a limit point of $(c, x) \cap A$, then x is equal to one of the right end points d_i used in (4.2) and by construction we also have that $f'_-(x)$ exists and equals ξ_x. By the same reasoning applied to the right side, we have that $f'_+(x) = h'_+(x) = H'_+(x) = \xi_x$. Thus the differentiability of f at all points has now been verified.

If x belongs to the interior of A, then the selection f is C^1 near x by Lemma 4.1. We are left only with verifying that f' is continuous at points of accumulation of A that are not in the interior of A. The difficult work in showing this is isolated in the following lemma.

Lemma 5.3. *Suppose there exists a sequence of intervals $\{(c_{i(k)}, d_{i(k)})\}_{k=1}^{\infty}$, which are among those that occur in the union* (4.2) *making up V, that satisfy*

$$\lim_{k\to\infty} c_{i(k)} = \lim_{k\to\infty} d_{i(k)} = x.$$

Then for any choice of $x_k \in (c_{i(k)}, d_{i(k)})$, we have $f'(x_k) \to \xi_x$ as $k \to \infty$.

PROOF. We assume for definiteness that each interval lies to the left of x. The case where an infinite collection of the intervals lies to the right is handled similarly.

Suppose $x_k \in (c_{i(k)}, d_{i(k)})$ for each k. Note that if $y \in I \subseteq J$, where I and J are intervals and J is one of those intervals in the union (4.2), then $M_J^x \le M_I^x$ and $N_J^x \ge N_I^x$. This is because the β function only depends on J. For each k, we can thus deduce the bounds

$$M^{x_k}_{[c_{i(k)},d_{i(k)}]} - \big(d_{i(k)} - c_{i(k)}\big) \le f'(x_k) \le N^{x_k}_{[c_{i(k)},d_{i(k)})]} + \big(d_{i(k)} - c_{i(k)}\big) \tag{5.14}$$

from (5.11) and (5.13). Thus the conclusion of the lemma follows from (5.14) provided it can be shown that

$$\lim_{k\to\infty} M^{x_k}_{[c_{i(k)},d_{i(k)}]} = \lim_{k\to\infty} N^{x_k}_{[c_{i(k)},d_{i(k)}]} = \xi_x.$$

We prove in detail only the limit involving $N^{x_k}_{[c_{i(k)},d_{i(k)}]}$; the other limit is proven the same way.

We write β_k for the chosen β function associated to the interval $[c_{i(k)}, d_{i(k)}]$, and recall that

$$\begin{aligned}
N^{x_k}_{[c_{i(k)},d_{i(k)}]} &= (\beta_k)'_+(x_k)H(x_k) + \beta_k(x_k) \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial H(y)\} \\
&\quad - (\beta_k)'_+(x_k)h(x_k) + \big(1 - \beta_k(x_k)\big) \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\} \\
&= (\beta_k)'_+(x_k)\big(H(x_k) - h(x_k)\big) \\
&\quad + \beta_k(x_k)\left(\sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial H(y)\} - \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\}\right) \\
&\quad + \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\}. \\
&= (I) + (II) + (III),
\end{aligned}$$

where (I), (II), (III) denote the respective terms of the previous line. By Lemma 5.2, we have that $h'_-(x) = H'_-(x) = \xi_x$. The semismoothness property (2.2) implies that

$$\begin{aligned} \lim_{k\to\infty} \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial H(y)\} &= H'_-(x) \\ = h'_-(x) &= \lim_{k\to\infty} \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\} \end{aligned} \tag{5.15}$$

Since $0 \le \beta_k(x_k) \le 1$ holds for all k, it follows from (5.15) that the term (II) approaches zero as $k \to \infty$. Also from (5.15) we deduce that the term (III) limits to $h'_-(x) = \xi_x$ as $k \to \infty$. So to finish the proof of the lemma, it suffices to show that the term (I) goes to zero as $k \to \infty$.

Since $h(d_{i(k)}) = H(d_{i(k)})$, we use (5.1) and the Lebourg mean value theorem (see [3, Theorem 2.3.7]) to obtain the estimate

$$\begin{aligned} \Big|\beta'(x_k)\big(H(x_k) - h(x_k)\big)\Big| &\le \frac{H(x_k) - h(x_k)}{d_k - c_k} \\ &\le \left(\frac{H(x_k) - H(d_k)}{d_k - x_k} - \frac{h(x_k) - h(d_k)}{d_k - x_k}\right) \\ &\le \sup_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial H(y)\} - \inf_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\}. \end{aligned} \tag{5.16}$$

Since (2.2) also implies that $h'_-(x) = \lim_{k\to\infty} \inf_{y\in[c_{i(k)},d_{i(k)}]} \{\xi : \xi \in \partial h(y)\}$, it follows from (5.15) and (5.16) that the terms (I) go to zero as $k \to \infty$. $\square$

Again let x be a point of accumulation of A, and suppose first that x is a limit point of $(c, x) \cap A$. We have $\xi_x = h'_-(x) = H'_-(x) = f'(x)$ by Lemma 5.2. Suppose x_k is any sequence satisfying $x_k \nearrow x$. For those k having $x_k \in A$, we have $f'(x_k) = \xi_{x_k} \to \xi_x$ by (2.2). On the other hand, if there is a subsequence (which we do not relabel) satisfying $x_k \notin A$, then there exists a sequence of intervals $\{(d_{i(k)}, c_{i(k)})\}_{k=1}^{\infty}$ that appear in the union (4.2) and that satisfy $x_k \in (d_{i(k)}, c_{i(k)})$ and $\lim_{k\to\infty} d_{i(k)} = x = \lim_{k\to\infty} c_{i(k)}$. Lemma 5.3 was designed to allow us also to conclude that $\lim_{k\to\infty} f'(x_k) = \xi_x = f'(x)$. If x is not a limit point of $(c, x) \cap A$, then we have already seen that f' is continuous from the left at x. Hence f' is continuous from the left at x.

Since the same arguments can be made when considering the continuity of $f'(\cdot)$ from the right at x, we conclude that $f'(\cdot)$ is continuous on all of (a, b). Theorem 3.1 is now completely proven. $\square$

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