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THE HAUSDORFF DIMENSION OF THE HYPERSPACE OF COMPACT SETS

Abstract

Let (X, ρ) be a separable metric space and let $(\mathcal{K}(X), \widetilde{\rho})$ denote the space of non-empty compact subsets of X with the Hausdorff metric. The purpose of this paper is to investigate the relationship of the Hausdorff dimension of a set $E \subset X$ to that of $\mathcal{K}(E) \subset \mathcal{K}(X)$.

1 Introduction and Notation.

1.1 The Hausdorff Metric

Given a separable metric space (X, ρ), let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X. Define a metric $\widetilde{\rho}$ on $\mathcal{K}(X)$ as follows: For $A, B \in \mathcal{K}(X)$ let

$$\widetilde{\rho}(A, B) = \max\{\sup_{x \in A}\{\text{dist}(x, B)\}, \sup_{y \in B}\{\text{dist}(y, A)\}\}.$$

The space $(\mathcal{K}(X), \widetilde{\rho})$ is called the Hausdorff metric space, or hyperspace, associated with X and inherits several nice geometrical properties from X. For example, $\mathcal{K}(X)$ is complete whenever X is complete and $\mathcal{K}(X)$ is compact whenever X is compact. A discussion of the Hausdorff metric including proofs of the above may be found in [Ed] section 2.4. To avoid confusion between metric spaces and their corresponding hyperspaces, tildes will be used to denote reference to the hyperspace. So for example, if $A \subset X$ is compact and $\varepsilon > 0$, then $\widetilde{B}_\varepsilon(A) \subset \mathcal{K}(X)$ denotes the closed ball of radius ε about the set A.

The Hausdorff metric has been studied extensively. Some early results on the Hausdorff dimension of $\mathcal{K}([0,1])$ may be found in [Boa], [Goo1], and [Goo2].

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1.2 Hausdorff Measure and Dimension

In this section, a definition of Hausdorff dimension valid for infinite dimensional sets will be developed following the very general approach of Rogers [Rog]. Let Φ denote the set of all non-decreasing, continuous functions φ defined on some interval $[0, \delta)$ so that $\varphi(0) = 0$ and $\varphi(t) > 0$ for $0 < t < \delta$. Such functions will be called *Hausdorff functions*. The asymptotic behavior near zero of two Hausdorff functions may be compared by writing:

- $\varphi \prec \psi$ if $\lim_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} = 0$
- $\varphi \preceq \psi$ if $\limsup_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} < \infty$
- $\varphi \asymp \psi$ if $0 < \liminf_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} \leq \limsup_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} < \infty$.

Given $\varphi \in \Phi$, define a measure $\mathcal{H}^\varphi$ on the separable metric space X as follows: For $\varepsilon > 0$ an ε-*cover* of $E \subset X$ will be a countable or finite collection of sets, $E_i \subset X$, so that $E \subset \cup_i E_i$ and $\operatorname{diam}(E_i) \leq \varepsilon$ for every i. Then let

$$\mathcal{H}^\varphi_\varepsilon(E) = \inf \left\{ \sum_i \varphi(\operatorname{diam}(E_i)) : \{E_i\}_i \text{ is an } \varepsilon\text{-cover of } E \right\},$$

$$\mathcal{H}^\varphi(E) = \lim_{\varepsilon \searrow 0} \mathcal{H}^\varphi_\varepsilon(E).$$

Note that $\mathcal{H}^\varphi_\varepsilon(E)$ increases as ε decreases so that $\mathcal{H}^\varphi(E)$ is well defined, though possibly infinite. In [Rog] it is proven that $\mathcal{H}^\varphi$ is a metric outer measure on X. A *metric outer measure* is an outer measure that satisfies $\mathcal{H}^\varphi(E \cup F) = \mathcal{H}^\varphi(E) + \mathcal{H}^\varphi(F)$, whenever $\operatorname{dist}(E, F) > 0$. This implies that all analytic (and in particular all Borel) subsets of X are $\mathcal{H}^\varphi$-measurable. Denote the restriction of $\mathcal{H}^\varphi$ to the $\mathcal{H}^\varphi$-measurable subsets of X also by $\mathcal{H}^\varphi$ and call this the *Hausdorff φ-measure* on X.

The phrase "E is of (non-)σ-finite $\mathcal{H}^\varphi$ measure", will be abbreviated by $\mathcal{H}^\varphi(E)$ is (non-)σ-finite. The *Hausdorff dimension* of a set $E \subset X$ is a partition of Φ. Specifically, $\dim(E) = (\Phi_\infty(E), \Phi_+(E), \Phi_0(E))$, where

$$\Phi_\infty(E) = \{\varphi \in \Phi : E \text{ is of non-}\sigma\text{-finite } \mathcal{H}^\varphi \text{ measure}\},$$

$$\Phi_+(E) = \{\varphi \in \Phi : \mathcal{H}^\varphi(E) > 0 \text{ and } E \text{ is of } \sigma\text{-finite } \mathcal{H}^\varphi \text{ measure}\},$$

$$\Phi_0(E) = \{\varphi \in \Phi : \mathcal{H}^\varphi(E) = 0\}.$$

The idea behind the Hausdorff dimension is that the value of $\mathcal{H}^\varphi(E)$ is governed by the asymptotic properties of $\varphi(t)$ as $t \searrow 0$ in a way indicative of the dimension of E. For example, if $X = \mathbb{R}^n$, μ_n is Lebesgue measure, and $\psi_\alpha(t) = t^\alpha$, then

$$\mathcal{H}^{\psi_\alpha} \begin{cases} \equiv & 0 & \text{if } \alpha > n \\ = & c_n\mu_n & \text{if } \alpha = n \ (c_n \text{ constant}) \\ \text{is} & \text{non-}\sigma\text{-finite} & \text{if } \alpha < n. \end{cases}$$

The following lemmas show that the $\prec$ relation places a partial order on Φ in which the faster the function disappears at the origin, the larger the dimension.

Lemma 1.1. *If $\mathcal{H}^\varphi(E)$ is σ-finite and $\varphi \prec \psi$, then $\mathcal{H}^\psi(E) = 0$.*

Lemma 1.2. *If $\mathcal{H}^\varphi(E) > 0$ and $\varphi \succ \psi$, then $\mathcal{H}^\psi(E)$ is non-σ-finite.*

For proofs see [Rog] theorem 40 and the corollary which follows it.

Ideally, one would like to completely describe the partition $\dim(E)$. Φ is a very rich set, however, and the ordering imposed by $\prec$ is by no means total. It is, consequently, not a tractable problem to understand how $\dim(E)$ compares with every $\varphi \in \Phi$. Thus, typically one defines an appropriate one parameter family, $\{\varphi_s\}_{s>0} \subset \Phi$, such that $s_1 < s_2$ implies $\varphi_{s_1} \prec \varphi_{s_2}$. Then, there is a critical value $s_0 \in [0, \infty]$ such that

$$\mathcal{H}^{\varphi_s}(E) = \begin{cases} 0 & \text{if } s > s_0 \\ \infty & \text{if } s < s_0. \end{cases}$$

For example, $\psi_s(t) = t^s$ leads to the standard numerical Hausdorff dimension. When working with the Hausdorff metric for subsets of a finite dimensional set, two useful families of Hausdorff functions are $\{\psi^s\}_{s>0}$ defined by $\psi^s(t) = 2^{-1/t^s}$ and $\{\varphi_M\}_{M>0}$ defined by $\varphi_M(t) = 2^{-M(1/t^s)}$ where $s > 0$ is fixed.

In [Fal] it is proven that the standard numerical Hausdorff dimension is preserved by bi-Lipschitz transformations. There, however, he is working with the family $\psi_s(t) = t^s$. One needs to be more careful when working with more general functions. The following lemma does hold.

Lemma 1.3. *Let $f : X \to X$ and let $0 < r_1 < r_2$.*
(a) If $\rho(f(x), f(y)) \leq r_2\rho(x, y)$ and $\mathcal{H}^\varphi(E)$ is σ-finite, then $\mathcal{H}^{\varphi(t/r_2)}(f(E))$ is σ-finite.
(b) If $r_1\rho(x, y) \leq \rho(f(x), f(y))$ and $\mathcal{H}^\varphi(E) > 0$, then $\mathcal{H}^{\varphi(t/r_1)}(f(E)) > 0$.

Proof. (a) Suppose first that $\mathcal{H}^\varphi(E) < \infty$. If $\{E_i\}$ is an ε-cover of E, then $\{f(E_i)\}$ is an $r_2\varepsilon$-cover of $f(E)$. The ε-cover $\{E_i\}$ may be chosen so that $\sum_i \varphi(\operatorname{diam}(E_i)) < 2\mathcal{H}^\varphi_\varepsilon(E)$. Then,

$$\sum_i \varphi\left(\frac{\operatorname{diam}(f(E_i))}{r_2}\right) \leq \sum_i \varphi\left(\frac{r_2 \operatorname{diam}(E_i)}{r_2}\right) < 2\mathcal{H}^\varphi_\varepsilon(E) \leq 2\mathcal{H}^\varphi(E).$$

So, $\mathcal{H}^{\varphi(t/r_2)}_{r_2\varepsilon}(f(E)) < 2\mathcal{H}^{\varphi}(E) < \infty$. This is true for arbitrarily small $\varepsilon > 0$ so, $\mathcal{H}^{\varphi(t/r_2)}(f(E)) < 2\mathcal{H}^{\varphi}(E) < \infty$. Next, if $E = \cup_i A_i$ and $\mathcal{H}^{\varphi}(A_i) < \infty$ for each i, then $\mathcal{H}^{\varphi(t/r_2)}(f(E))$ is seen to be σ-finite by applying the above logic to each A_i.

(b) If $\mathcal{H}^{\varphi}(E) > c > 0$, then we may choose an $\varepsilon > 0$ such that, $\mathcal{H}^{\varphi}_{\varepsilon}(E) > c$. Then, for any ε-cover $\{E_i\}_i$ of E, we have $\sum_i \varphi(\operatorname{diam}(E_i)) > c > 0$. Since f is bi-Lipschitz, any ε/r_1-cover of $f(E)$ may be written $\{f(E_i)\}_i$, where $\{E_i\}_i$ is an ε-cover of E. Then,

$$\sum_i \varphi\left(\frac{\operatorname{diam}(f(E_i))}{r_1}\right) \geq \sum_i \varphi\left(\frac{r_1 \operatorname{diam}(E_i)}{r_1}\right) > c.$$

So, $\mathcal{H}^{\varphi(t/r_1)}(f(E)) \geq \mathcal{H}^{\varphi(t/r_1)}_{\varepsilon/r_1}(f(E)) \geq c$. □

Consider, for example, the two parameter family of functions

$$\varphi_{M,s}(t) = 2^{-M(1/t)^s}.$$

For a fixed $s > 0$, a bi-Lipschitz map with ratios r_1 and r_2 as above can affect the critical value of M. But it can't be raised by more than a factor $1/r_2$ and it cannot be lowered by more than a factor of $1/r_1$. For $s_1 < s_2$, however, $\varphi_{M_1,s_1} \prec \varphi_{M_2,s_2}$ for any M_1 and M_2. So a bi-Lipschitz map won't affect the critical value of s.

1.3 Related Notions of Dimension

Although this paper is primarily concerned with the Hausdorff dimension, there are two other notions of dimension which will be useful. The first is the upper entropy index $\Delta(E)$ defined for totally bounded $E \subset X$ as follows: For $\varepsilon > 0$, let

$$N_\varepsilon(E) = \max \# \text{ of disjoint closed balls centered in } E \text{ with radius } \varepsilon/2.$$

Then, let

$$\Delta(E) = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(E)}{\log(1/\varepsilon)}.$$

This assigns a nonnegative number or infinity to $\Delta(E)$. It is possible to generalize this definition to a partition of Φ, by comparing the asymptotic behavior of N_ε to that of Hausdorff functions (see [Mcc]). This level of generality will not be needed here, however. For more information on the upper entropy index, see [Ed] section 6.5.

The next useful notion of dimension is the similarity dimension, which is valid only for self-similar sets. Self-similar sets are obtained as follows: Let $m \in \mathbb{N}$ and for $i = 1, \ldots, m$ let $f_i : X \to X$ be a similarity with ratio $r_i \in (0, 1)$. This means that for every $x, y \in X$ we have $\rho(f_i(x), f_i(y)) = r_i\rho(x, y)$. In this situation there exists a unique non-empty compact set $E \subset X$ such that $E = \cup_{i=1}^m f_i(E)$. The set E obtained this way is said to be *self-similar*. The similarity dimension of the set E is defined to be the unique positive number s_0 such that $\sum_{i=1}^m r_i^{s_0} = 1$. For more information on self-similar sets, see [Ed] chapter 4.

The standard numerical Hausdorff dimension $\dim_N(E)$, upper entropy index $\Delta(E)$, and similarity dimension s_0 are related as follows:

$$\dim_N(E) \leq \Delta(E) \leq s_0.$$

In Euclidean space, this relationship may be strengthened, assuming the set of contractions $\{f_i\}_{i=1}^m$ satisfies the *open set condition*. This means that there is an open set U, such that $U \supset \cup_{i=1}^m f_i(U)$ with this union disjoint. Assuming the open set condition is satisfied, the above inequalities may be replaced with equalities. For more information on the relationships between these dimensions, see [Ed] sections 6.3 and 6.5.

A corollary to the main theorems (3.3 and 3.4) of this paper can now be stated, to provide the gist of those results in a more concrete setting.

Corollary 1.1. *Suppose $E \subset \mathbb{R}^n$ is a self-similar set satisfying the open set condition. Let s_0 be the similarity dimension of E and let $\varphi_s(t) = 2^{-(1/t)^s}$. Then,*

$$\mathcal{H}^{\varphi_s}(\mathcal{K}(E)) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0. \end{cases}$$

Thus the Hausdorff dimension of E is clearly reflected in the Hausdorff dimension of $\mathcal{K}(E)$. Analogous statements for the upper and lower entropy indices and dimensions are proven in [Mcc].

1.4 Sequence Spaces

The main results will first be proven for some specific metric spaces called sequence spaces and then transferred to a more general setting. Let $m \in \mathbb{N}^+$ and let the *sequence space* Ω be defined by $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$. A family of metrics will be defined on Ω each inducing the product topology. First, some useful terminology will be developed. An initial segment α of length n is an element of $\{1, \ldots, m\}^n$. There is, by definition, one initial segment of length zero namely the empty segment denoted Λ. If α is an initial segment, write

$|\alpha|$ to denote the length of α. For $n \in \mathbb{N}$, let Ω^n denote the set of all initial segments of length n. Let $\Omega^* = \cup_{j=0}^{\infty} \Omega^j$ be the set of all initial segments. If $\sigma \in \Omega$ write $\sigma|_n$ for the initial segment $(\sigma_1, \ldots, \sigma_n) \in \Omega^n$. A partial order may be imposed on Ω^* as follows: For $\alpha, \beta \in \Omega^*$, say $\alpha = (\alpha_1, \ldots, \alpha_j)$ and $\beta = (\beta_1, \ldots, \beta_k)$, write $\alpha < \beta$ if $j < k$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, j$. If $\alpha < \beta$, then β is said to be a descendant of α. If $\alpha \in \Omega^n$, then let α^- denote the unique element of Ω^{n-1} such that $\alpha^- < \alpha$. Call α^- the *parent* and α the *child*. Also, if $\alpha \in \Omega^n$ is an initial segment, then let

$$[\alpha] = \{\sigma \in \Omega : \sigma_i = \alpha_i \text{ for } i = 1, \ldots, n\}.$$

For $i = 1, \ldots, m$, let $0 < r_i < 1$. The list $(r_1, \ldots, r_m)$ is called a *contraction ratio list* for Ω. Given $\alpha = (i_1, \ldots, i_n) \in \Omega^*$, let $r(\alpha) = r_{i_1} \cdots r_{i_n}$ and let $r(\Lambda) = 1$. Define a metric d as follows:

1. $d(\sigma, \sigma) = 0$ for all $\sigma \in \Omega$;
2. If $\sigma, \tau \in \Omega$ have α as their longest common initial segment, then $d(\sigma, \tau) = r(\alpha)$.

In [Ed] it is shown that the numerical Hausdorff dimension of this sequence space is given by $\psi_s(t) = t^s$, where $\sum_{i=1}^m r_i^s = 1$. In fact it is shown that

$$\mathcal{H}^s([\alpha]) \equiv \mathcal{H}^{\psi_s}([\alpha]) = r(\alpha)^s \tag{1}$$

for every $\alpha \in \Omega^*$.

2 Computational Tools

2.1 The Density Lemma

Lower bounds for Hausdorff measures are frequently obtained by using a density lemma. See for example, [RayTr] theorem 1. In [RayTr] and other sources it is assumed that Hausdorff functions are blanketed. That is, there is some constant $M > 0$, such that $\varphi(2t) < M\varphi(t)$. This is unsuitable for the purposes here so the following generalization is needed.

Lemma 2.1. *For a separable metric space X with a positive analytic measure μ, $x \in X$, and $\delta > 0$ let*

$$\mu_\delta(x) = \sup\{\mu(U) : x \in U \text{ and } U \text{ is an analytic set with } \operatorname{diam}(U) \leq \delta\}.$$

Let $\delta_k \searrow 0$. Suppose that $\varphi, \psi \in \Phi$ satisfy $\varphi(\delta_k) \leq A\psi(\delta_{k+1})$ for all $k \in \mathbb{N}$. Let $E \subset X$ be a Borel set which satisfies $\mu(E) > 0$ and

$$\overline{D}_\mu^\varphi(x, (\delta_k)_k) \equiv \limsup_{k \to 0} \frac{\mu_{\delta_k}(x)}{\varphi(\delta_k)} < M < \infty,$$

for every $x \in E$. Then $\mathcal{H}^{\psi}(E) \geq \frac{\mu(E)}{MA} > 0$.

PROOF. Let $k_0 \in \mathbb{N}$ and choose $0 < \varepsilon < \delta_{k_0}$. Let

$$E_{k_0} = \{x \in E : \mu_{\delta_k}(x) < M\varphi(\delta_k) \text{ for every } k \geq k_0\}.$$

Note that $\cup_{k_0=1}^{\infty} E_{k_0} = E$. Suppose that $\mathcal{C}$ is an ε-cover of E and so of E_{k_0}. For $k \geq k_0$ write

$$\mathcal{C}_k = \{U \in \mathcal{C} : \delta_{k+1} < \operatorname{diam}(U) \leq \delta_k\}.$$

Any $U \in \mathcal{C}_k$ such that $U \cap E_{k_0} \neq \emptyset$ satisfies $\mu(U) < M\varphi(\delta_k)$. So,

$$\begin{aligned}
\mu(E_{k_0}) &\leq \sum_{\substack{U \in \mathcal{C} \\ U \cap E_{k_0} \neq \emptyset}} \mu(U) = \sum_{k=k_0}^{\infty} \sum_{\substack{U \in \mathcal{C}_k \\ U \cap E_{k_0} \neq \emptyset}} \mu(U) \\
&\leq M \sum_{k=k_0}^{\infty} \sum_{U \in \mathcal{C}_k} \varphi(\delta_k) \leq M \sum_{k=k_0}^{\infty} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \frac{\varphi(\delta_k)}{\psi(\delta_{k+1})} \\
&= M \sum_{k=k_0}^{\infty} \left(\frac{\varphi(\delta_k)}{\psi(\delta_{k+1})} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \right) \\
&\leq MA \left(\sum_{k=k_0}^{\infty} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \right) \\
&= MA \left(\sum_{U \in \mathcal{C}} \psi(\operatorname{diam}(U)) \right).
\end{aligned}$$

Now $\mu(E_{k_0}) \to \mu(E)$ as $k_0 \to \infty$. Thus, $\sum_{U \in \mathcal{C}} \psi(\operatorname{diam}(U)) \geq \mu(E)/MA$ and $\mathcal{H}^{\psi}(E) \geq \mathcal{H}^{\psi}_{\varepsilon}(E) \geq \mu(E)/MA$. $\square$

As an example, suppose that $\psi^s(t) = 2^{-(1/t)^s}$, $c > 0$, $0 < u < 1$, and let $\delta_k = cu^k$. Then for $0 < s_1 < s_2$, we have

$$\begin{aligned}
\frac{\psi^{s_2}(\delta_k)}{\psi^{s_1}(\delta_{k+1})} &= 2^{-(\frac{1}{cu^k})^{s_2} + (\frac{1}{cu^{k+1}})^{s_1}} \\
&= 2^{-(\frac{1}{cu^k})^{s_2} \left(1 - (cu^k)^{s_2} (\frac{1}{cu^{k+1}})^{s_1}\right)} \\
&= 2^{-(\frac{1}{cu^k})^{s_2} \left(1 - (\frac{1}{u})^{s_1} (cu^k)^{s_2 - s_1}\right)} \leq 2^{-\frac{1}{2}(\frac{1}{cu^k})^{s_2}}
\end{aligned}$$

for large k. This last term approaches zero as $k \to \infty$. So given an analytic set $E \subset X$, to show that $\mathcal{H}^{\psi^{s_1}}(E) = \infty$, it suffices to find a positive Borel measure μ on E, $M > 0$, and an $s_2 > s_1$ such that

$$\overline{D}_{\mu}^{\psi^{s_2}}(x, (cu^k)_k) < M$$

for every $x \in E$. This yields the following corollary.

Corollary 2.1. *Let $\psi^s(t) = 2^{-(1/t)^s}$ and let $E \subset X$ be an analytic set with a Borel measure satisfying $\mu(E) > 0$. If $\overline{D}_\mu^{\psi^s}(x, (cu^k)_k) < M < \infty$ for every $x \in E$ and every $s < s_0$, then $\mathcal{H}^{\psi^s}(E) > 0$ for every $s < s_0$.*

2.2 s-Nested Packings

Part of the importance of sequence space is that it may be used to model other spaces. In [Ed], for example, sequence spaces are used in the study of self-similar sets in $\mathbb{R}^n$. In this section, a condition on a closed subset E of a complete separable metric space X will be defined. This condition will allow the construction of a subset $E' \subset E$ which is Lipeomorphic to a certain sequence space. This result will be used later to transfer results from sequence space to more general spaces.

Now let E be as above, fix $c, s > 0$, $\varepsilon \in (0, 1/4)$, and $m > (1/\varepsilon)^s + 1$. Let $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$ be a sequence space with the metric d given by $r(\alpha) = c\varepsilon^n$ for every $\alpha \in \Omega^n$. An *s-nested packing* of E will be a collection of closed balls $\{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha \in \Omega^*}$ satisfying:

1. $x_\alpha \in E$ for every $\alpha \in \Omega^*$
2. $B_{c\varepsilon^n}(x_\alpha) \cap B_{c\varepsilon^n}(x_\beta) = \emptyset$ for distinct $\alpha,\ \beta \in \Omega^n$.
3. $B_{c\varepsilon^{|\alpha|}}(x_\alpha) \subset B_{c\varepsilon^{|\alpha^-|}/4}(x_{\alpha^-})$ for every $\alpha \in \Omega^*$.

This definition also depends on c, ε, and m, however the important parameter is s because dimensional bounds given later will be in terms of s. If E has such an s-nested packing, then let $E' = \cap_{n=1}^\infty \cup_{\alpha \in \Omega^n} B_{c\varepsilon^{|\alpha|}}(x_\alpha)$. The existence of s-nested packings will be established later for self-similar sets. Define a map $g : \Omega \to E'$ by $g(\omega) = \cap_{n=1}^\infty B_{c\varepsilon^n}(x_{\omega|_n})$.

Lemma 2.2. *The map g is bi-Lipschitz.*

PROOF. Let $\omega_1,\ \omega_2 \in \Omega$. Choose $n \in \mathbb{N}$ so that $d(\omega_1, \omega_2) = c\varepsilon^n$. Then $g(\omega_1),\ g(\omega_2) \in B_{c\varepsilon^n}(x_{\omega_1|_n})$, so $\rho(g(\omega_1), g(\omega_2)) \leq 2c\varepsilon^n$. For the lower bound, note that $g(\omega_i) \in B_{c\varepsilon^{n+1}/4}(x_{\omega_i|_{n+1}})$ for $i = 1, 2$ and, by the choice of n, $B_{c\varepsilon^{n+1}}(x_{\omega_1|_{n+1}}) \cap B_{c\varepsilon^{n+1}}(x_{\omega_2|_{n+1}}) = \emptyset$. So

$$\begin{aligned}
c\varepsilon^{n+1} &\leq \rho(x_{\omega_1|_{n+1}}, x_{\omega_2|_{n+1}}) \\
&\leq \rho(x_{\omega_1|_{n+1}}, g(\omega_1)) + \rho(g(\omega_1), g(\omega_2)) + \rho(g(\omega_2), x_{\omega_2|_{n+2}}) \\
&\leq \frac{c\varepsilon^{n+1}}{4} + \rho(g(\omega_1), g(\omega_2)) + \frac{c\varepsilon^{n+1}}{4}.
\end{aligned}$$

So $\rho(g(\omega_1), g(\omega_2)) \geq \frac{c\varepsilon^{n+1}}{2} = \frac{\varepsilon}{2}c\varepsilon^n$. $\square$

Next, it will be shown that this condition is non-vacuous by the construction of an s-nested packing for a self similar set $E \subset X$. Two sequence spaces Ω_1 and Ω_2 will be used to analyze the set E. The first one is the self-similar sequence space $\Omega_1 = \{1, \ldots, m_1\}^{\mathbb{N}}$ with metric d_1 given by the ratio list $(r_i)_{i=1}^{m_1}$ corresponding to the contraction ratios of $(f_i)_{i=1}^{m_1}$. Given $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Omega_1^n$, abbreviate $f_{\alpha_1} \circ \cdots \circ f_{\alpha_n}(E)$ by $\alpha(E)$. An s-nested packing will be constructed in the metric space (E, ρ) rather than (X, ρ). The reason for this is because for $x \in E$, $\alpha \in \Omega_1^*$, and $\varepsilon > 0$ we have $\alpha(B_\varepsilon(x)) = B_{r(\alpha)\varepsilon}(\alpha(x))$ as long as only balls in (E, ρ) are considered. This is due to the invariance of E under the transformations $(f_i)_{i=1}^{m_1}$ and not generally true in the larger metric space (X, ρ).

Now, let $s > 0$, let $r = \min\{r_i\}_{i=1}^{m_1}$, and let $c = \frac{8}{r}\max\{\operatorname{diam}(E), 1\}$. Fix $\delta \in (0, \min\{\frac{1}{4}, \frac{1}{4}\operatorname{diam}(E)\})$ such that $N_{2\delta}(E) > (c/\delta)^s + 1$. Such a δ certainly exists if $\Delta(E) > s$. Let $\varepsilon = \delta/c$ and let $m_2 = N_{2\delta}(E)$. The other sequence space of interest is $\Omega_2 = \{1, \ldots, m_2\}^{\mathbb{N}}$ with metric d_2 given by $r(\beta) = c\varepsilon^n$ for $\beta \in \Omega_2^n$.

An s-nested packing of E, using the above choices for c, ε, and m, may be constructed as follows: Choose $x_\Lambda \in E$ arbitrarily. This gives $B_c(x_\Lambda)$. The existence of $\{B_{c\varepsilon}(x_\beta)\}_{\beta \in \Omega_2^1}$ is guaranteed by the fact that $N_{2\delta}(E) > (c/\delta)^s + 1$ since $\delta = c\varepsilon$. The construction will proceed by induction on the length of β. Suppose that $B_{c\varepsilon^{|\beta|}}(x_\beta)$ have been defined for $|\beta| \leq n$. For $\beta \in \Omega_2^n$, choose $\alpha_\beta \in \Omega_1^*$ such that $x_\beta \in \alpha_\beta(E)$, and

$$\operatorname{diam}(\alpha_\beta(E)) \leq \frac{1}{8}\frac{\delta^n}{c^{n-1}} < \operatorname{diam}(\alpha_\beta^-(E)).$$

In particular, $\alpha_\beta(E) \subset B_{c\varepsilon^{|\beta|}/4}(x_\beta)$. Since $r = \min\{r_i\}$ we have:

$$\operatorname{diam}(\alpha_\beta(E)) \geq \frac{r}{8}\frac{\delta^n}{c^{n-1}} = \frac{r}{8\operatorname{diam}(E)}\frac{\delta^n}{c^{n-1}}\operatorname{diam}(E) \geq \frac{\delta^n}{c^n}\operatorname{diam}(E).$$

So $N_{2\delta(\delta^n/c^n)}(\alpha_\beta(E)) \geq N_{2\delta}(E) = m_2$. Thus $\alpha_\beta(E)$ may be packed with m_2 balls of radius $\delta^{n+1}/c^n = c\varepsilon^{n+1}$ to continue the induction.

For a useful generalization, note that if $f : X \to X$ is a bi-Lipschitz map satisfying $\rho(f(x), f(y)) \geq r\rho(x, y)$ for every $x, y \in X$, then $f(B_\varepsilon(x)) \supset B_{r\varepsilon}(f(x))$. So if $f : E \to F$ is a bi-Lipschitz bijection and E has an s-nested packing $\{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha \in \Omega^*}$, then f induces an s-nested packing of F, namely $\{B_{cr\varepsilon^{|\alpha|}}(f(x_\alpha))\}_{\alpha \in \Omega^*}$. Putting all this together we obtain.

Theorem 2.1. *If $E \subset X$ has a subset which is Lipeomorphic to a self-similar set F satisfying $\Delta(F) > s$, then E has an s-nested packing.*

3 Hausdorff Dimension of $\mathcal{K}(X)$

In this section, the relationship between $\dim(E)$ and $\dim(\mathcal{K}(E))$ is investigated. The plan is the following. First, it is shown that if Ω is a sequence space with finite Hausdorff dimension s_0 and $\psi^s(t) = 2^{-(1/t)^s}$, then

$$\mathcal{H}^{\psi^s}(\mathcal{K}(\Omega)) = \begin{cases} \infty & \text{if } s < s_0 \\ 0 & \text{if } s > s_0. \end{cases}$$

This result is then used along with the notion of an s-nested packing to obtain results in more general spaces. In this section, $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$ is a fixed self-similar sequence space with contraction ratio list $(r_1, \ldots, r_m)$ so that $\sum_1^m r_i^{s_0} = 1$, where $s_0 > 0$ is fixed.

Theorem 3.1. *For $M > 0$, let $\varphi_M(t) = 2^{-M(1/t)^{s_0}}$. Then, there exists an M large enough so that $\mathcal{H}^{\varphi_M}(\mathcal{K}(\Omega)) < \infty$.*

Proof. Choose $0 < u < \min\{r_i\}$ so that $1/u^{s_0} = n \in \mathbb{N}$. For every $k \in \mathbb{N}^+$, let

$$L_k = \{\alpha \in \Omega^* : r(\alpha) \le u^k < r(\alpha^-)\}$$

and let $L_0 = \{\Lambda\}$. Each $\alpha \in L_k$ satisfies

$$u^{k+1} < r(\alpha) \le u^k$$

and

$$n^{-(k+1)} < \mathcal{H}^{s_0}([\alpha]) \le n^{-k} \tag{2}$$

by equation (1). Suppose that $\#(L_1) = L$. Then, since each $\beta \in L_1$ satisfies $n^{-2} < \mathcal{H}^{s_0}([\beta]) \le n^{-1}$ for each $k \in \mathbb{N}$, the number of descendants of β in L_k cannot exceed n^k. Otherwise, their total measure would exceed $n^k n^{-(k+1)} = \frac{1}{n} \ge \mathcal{H}^{s_0}([\beta])$. Similarly, the number of descendants of β in L_k must be at least n^{k-2}. So,

$$Ln^{k-2} \le \#(L_k) \le Ln^k. \tag{3}$$

Let $A \subset L_k$ be nonempty. Associate with A a set $\widetilde{A} \subset \mathcal{K}(\Omega)$ defined by:

$$\widetilde{A} = \{C \in \mathcal{K}(\Omega) : \{\alpha \in L_k : [\alpha] \cap C \neq \emptyset\} = A\}.$$

Such a set $\widetilde{A}$ is called a *k-set* and satisfies $u^{k+1} < \operatorname{diam}(\widetilde{A}) \le u^k$. For a fixed k, the set of all k-sets covers $\mathcal{K}(\Omega)$. Since $\#(L_k) \le Ln^k$, there are no more than $2^{Ln^k} - 1$ such k-sets. This leads to the following estimate:

$$\begin{aligned} \mathcal{H}^{\varphi_M}_{u^k}(\mathcal{K}(\Omega)) &\le \left(2^{Ln^k} - 1\right) \varphi_M(u^k) \\ &\le 2^{Ln^k} 2^{-M(1/u^k)^{s_0}} = 2^{Ln^k} 2^{-Mn^k} \le 1 \end{aligned}$$

as long as $M \geq L$. Thus for $M \geq L$, it follows that $\mathcal{H}^{\varphi_M}(\mathcal{K}(\Omega)) \leq 1$. $\square$

Now for the lower bound, let $\psi^s(t) = 2^{-(1/t)^s}$ for $s > 0$.

Theorem 3.2. *$\mathcal{H}^{\psi^s}(\mathcal{K}(\Omega)) > 0$, whenever $s < s_0$.*

PROOF. L_k, L, u, n, and k-sets are defined as in the preceding proof. Given $A \subset L_k$, define $\pi(A) = \#(A)/Ln^{k-2}$. Let $n' = 1/u^s < n$, choose $p \in (0,1)$ so that $n' < pn$, then choose $j \in \mathbb{N}$ large enough so that $\left(\frac{n'}{pn}\right)^j < \frac{1}{n}$. A measure μ, concentrated on those kj-sets $\widetilde{A}$ with $\pi(A) \geq p^{kj}/n^k$, will be constructed. μ will satisfy $\overline{D}_\mu^{\psi^s}(E, (u^{kj+1})_k) \leq 1$ for every $E \in \mathcal{K}(\Omega)$ implying the result by corollary 2.1. Recall that the definition of $\overline{D}_\mu^{\psi^s}(E, (u^{kj+1})_k)$ is given in lemma 2.1.

The measure μ will be constructed recursively. The empty word Λ is the only string of length 0 leading to the one 0-set $\widetilde{\Lambda} = \mathcal{K}(\Omega)$. Define $\mu(\mathcal{K}(\Omega)) = 1$. Fix $k \in \mathbb{N}$ and suppose that μ has been defined for all kj-sets $\widetilde{A}$ such that $\mu(\widetilde{A}) > 0$ only if $\pi(A) \geq p^{kj}/n^k$. This condition is seen to be satisfied by $\widetilde{\Lambda}$ by substituting $k = 0$ into inequality 3. If $\widetilde{A}$ is a kj-set of positive measure, then distribute $\mu(\widetilde{A})$ evenly among all those $(k+1)j$-sets $\widetilde{B} \subset \widetilde{A}$ such that $\pi(B) \geq \frac{p^{(k+1)j}}{n^{k+1}}$. Such a set $\widetilde{B}$ will be called an eligible descendent of $\widetilde{A}$. A lower bound on the number of eligible descendants of $\widetilde{A}$ is needed in order to estimate $\mu(\widetilde{A})$ from above. Now $\#(A) \geq \frac{p^{kj}}{n^k} Ln^{kj-2}$, since $\pi(A) \geq \frac{p^{kj}}{n^k}$. If $\alpha \in A \subset L_{kj}$, then

$$n^{-(kj+1)} < \mathcal{H}^{s_0}([\alpha]) \leq n^{-kj},$$

by equation 2. Similarly, if $\beta \in L_{(k+1)j}$, then

$$n^{-(k+1)j-1} < \mathcal{H}^{s_0}([\beta]) \leq n^{-(k+1)j}.$$

Thus if $L_{(k+1)j,\alpha}$ is the set of descendants of α in $L_{(k+1)j}$, then

$$n^{j-1} < \#(L_{(k+1)j,\alpha}) < n^{j+1}.$$

To form an eligible descendent $\widetilde{B} \subset \widetilde{A}$ proceed as follows: Take $\left[p^j \frac{p^{kj}}{n^k} Ln^{kj-2}\right]$ of the α's $\in A$ and choose all possible descendants β to form part of the set B. This guarantees that

$$\#(B) \geq \left[\frac{p^{(k+1)j}}{n^k} Ln^{kj-2} n^{j-1}\right] = \left[\frac{p^{(k+1)j}}{n^k} Ln^{(k+1)j-3}\right],$$

so that $\pi(B) \geq \frac{p^{(k+1)j}}{n^{k+1}}$. Thus, $\widetilde{B}$ is an eligible descendant. We are now free to choose descendants of the remaining

$$\left[\frac{p^{kj}}{n^k}Ln^{kj-2}\right] - \left[\frac{p^{(k+1)j}}{n^k}Ln^{kj-2}\right] \geq (1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2}) - 1$$

α's $\in A$ in any combination. Since each $\alpha \in A$ has at least n^{j-1} descendants $\beta \in L_{(k+1)j}$ and any possible non-empty subset of these may be chosen as possible descendants, we get at least

$$(2^{n^{j-1}} - 1)^{(1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2})-1}$$

eligible descendants $\widetilde{B} \subset \widetilde{A}$. This means that any such $\widetilde{B}$ satisfies

$$\mu(\widetilde{B}) \leq (2^{n^{j-1}} - 1)^{-(1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2})-1}\mu(\widetilde{A}).$$

Applying this recursively, we see that a kj-set satisfies

$$\begin{aligned}\mu(\widetilde{A}) &\leq (2^{n^{j-1}} - 1)^{-(1-p^j)Ln^{-2}(1+p^jn^{j-1}+\cdots+(p^jn^{j-1})^{k-1})-k} \\ &= (2^{n^{j-1}} - 1)^{-(1-p^j)Ln^{-2}\frac{(p^jn^{j-1})^k-1}{p^jn^{j-1}-1}-k} \\ &\leq 2^{-L'(p^jn^{j-1})^k}\end{aligned}$$

where $L' > 0$ is a sufficiently small constant.

Now, if $E \in \mathcal{K}(\Omega)$ and k is fixed, let

$$A_E = \{\alpha \in L_{kj} : [\alpha] \cap E \neq \emptyset\}.$$

Then $\widetilde{A}_E$ is a kj-set and, so, satisfies $\operatorname{diam}(\widetilde{A}_E) > u^{kj+1}$. So any set $\widetilde{F} \subset \mathcal{K}(\Omega)$ such that $\operatorname{diam}(\widetilde{F}) < u^{kj+1}$ and $E \in \widetilde{F}$, must also satisfy $\widetilde{F} \subset \widetilde{A}_E$. Thus, $\mu_{u^{kj+1}}(E) < \mu(\widetilde{A}_E)$. Let $n' = 1/u^s < n$. Then,

$$\begin{aligned}\frac{\mu_{u^{kj+1}}(E)}{\psi^s(u^{kj+1})} \leq \frac{\mu(\widetilde{A}_E)}{\psi^s(u^{kj+1})} &\leq \frac{2^{-L'(p^jn^{j-1})^k}}{2^{-(n')^{kj+1}}} \\ &= 2^{(n')^{kj+1}-L'(p^jn^{j-1})^k} \to 0\end{aligned}$$

as $k \to \infty$, since $p^jn^{j-1} > (n')^j$ by assumption. $\square$

The next order of business is to extend these theorems to more general sets E. For the upper bound, let us suppose that $E \subset F$ where F is the self-similar set given by the maps $(f_1, \ldots, f_m)$ with ratio list $(r_1, \ldots, r_m)$. Let (Ω, ρ) be the corresponding self-similar sequence space. In this situation, it is shown in [Ed] that there is a surjective Lipschitz map $h : \Omega \to F$. Since a Lipschitz map is continuous and the continuous image of a compact set is compact, h extends naturally to a Lipschitz map $\widetilde{h} : \mathcal{K}(\Omega) \to \mathcal{K}(F)$. Thus the upper bound for $\mathcal{K}(\Omega)$ holds for $\mathcal{K}(E)$. By composing the map h with another if necessary, it is also clear that F need not be strictly self-similar, but only the Lipschitz image of a self similar set. This is summarized in the following theorem.

Theorem 3.3. *Let $E \subset F \subset X$, where X is a separable metric space and F is the Lipschitz image of a self-similar set with ratio list $(r_1, \ldots, r_m)$ such that $\sum_{i=1}^m r_i^{s_0} = 1$. Let $\varphi_M(t) = 2^{-M(1/t)^{s_0}}$. Then, there is an $M > 0$ large enough so that $\mathcal{H}^{\varphi_M}(\mathcal{K}(E)) < \infty$.*

For the lower bound, suppose that E has an s_0-nested packing. Then we may extract a subset $E' \subset E$, which is bi-Lipschitz equivalent to a self-similar sequence space (Ω, ρ) of finite Hausdorff dimension s_0 by lemma 2.2. Again, the bi-Lipschitz map $g : \Omega \to E'$ extends to a bi-Lipschitz map $\widetilde{g} : \mathcal{K}(\Omega) \to \mathcal{K}(E')$. Thus, we have the following theorem.

Theorem 3.4. *Let $\psi^s(t) = 2^{-(1/t)^s}$. Suppose that E has an s_0-nested packing. Then for $s < s_0$ we have $\mathcal{H}^{\psi^s}(\mathcal{K}(E)) > 0$.*

As noted in corollary 1.1, these theorems apply to self-similar sets. It is natural to ask whether these theorem hold (or fail) for other types of sets. The next theorem is an example showing that no general estimate can hold. For every $\varphi \in \Phi$, there is countable metric space X so that $\mathcal{H}^\varphi(\mathcal{K}(X)) > 0$. It is interesting to note that the following metric space yields this same unexpected behavior for the entropy dimensions (see [Mcc]). Suppose that $X = \{x_0, x_1, \ldots, x_\infty\}$ is a countable metric space with metric ρ satisfying $\rho(x_n, x_\infty) = a_n \searrow 0$ and $\rho(x_n, x_m) \geq a_n$ for $m < n < \infty$. Clearly $\mathcal{H}^\varphi(X) = 0$ for every $\varphi \in \Phi$. But the following is also true:

Theorem 3.5. *Let (X, ρ) be as above and suppose $\varphi \in \Phi$ satisfies $\varphi(a_n) = 2^{-n}$. Then $\frac{1}{2} \leq \mathcal{H}^\varphi(\mathcal{K}(X)) \leq 1$.*

PROOF. A set $T \in \mathcal{K}(X)$ is isolated if and only if $x_\infty \notin T$. Let

$$\mathcal{K}'(X) = \{T \in \mathcal{K}(X) : x_\infty \in T\}.$$

Then $\mathcal{K}(X) \setminus \mathcal{K}'(X)$ is countable so that $\mathcal{H}^\varphi(\mathcal{K}(X) \setminus \mathcal{K}'(X)) = 0$.

Turn now to $\mathcal{K}'(X)$. For fixed $n \in \mathbb{N}$, each set $A \subset \{x_0, \ldots, x_{n-1}\}$ determines a set

$$\widetilde{A}_n = \{T \in \mathcal{K}'(X) : A = T \cap \{x_0, \ldots, x_{n-1}\}\}.$$

Note that if $S, T \in \widetilde{A}_n$, then any point $x_k \in S$ with $k \geq n$ satisfies $\rho(x_k, x_\infty) \leq a_n$. So $\operatorname{dist}(x_k, T) \leq a_n$, since $x_\infty \in T$. Since S and T agree on A, it follows that $\operatorname{dist}(x_k, T) \leq a_n$ for every $x_k \in S$ and vice versa. So $\operatorname{diam}(\widetilde{A}_n) \leq a_n$. In fact, $A \cup \{x_\infty\}$ and $A \cup \{x_n, x_\infty\} \in \widetilde{A}_n$, so that $\operatorname{diam}(\widetilde{A}_n) = a_n$. Now there are 2^n such A's contained in $\{x_0, \ldots, x_{n-1}\}$. So

$$\mathcal{H}^{\varphi}_{a_n}(\mathcal{K}'(X)) \leq 2^n \varphi(a_n) = 2^n 2^{-n} = 1.$$

So $\mathcal{H}^{\varphi}(\mathcal{K}'(X)) \leq 1$.

For the lower bound, a measure μ on $\mathcal{K}'(X)$ will be constructed recursively. Let $\mu(\mathcal{K}'(X)) = 1$. Fix $m \in \mathbb{N}$ and suppose that μ has been constructed so that $A \subset \{x_0, \ldots, x_{n-1}\}$ implies $\mu(\widetilde{A}_n) = 2^{-n}$ for every $n \leq m$. Note that if $A \subset \{x_0, \ldots, x_{m-1}\}$, then

$$\widetilde{A}_m = \{T \in \widetilde{A}_m : x_m \in T\} \cup \{S \in \widetilde{A}_m : x_m \notin S\}.$$

Divide $\mu(\widetilde{A}_m)$ evenly between these two sets. In this way μ is constructed so that $\mu(\widetilde{A}_n) = 2^{-n}$ for any $n \in \mathbb{N}$ and $A \subset \{x_0, \ldots, x_{n-1}\}$.

Now suppose that $\widetilde{B} \subset \mathcal{K}'(X)$ satisfies $a_{n+1} < \operatorname{diam}(\widetilde{B}) \leq a_n$. Let $T \in \widetilde{B}$ and let $A = T \cap \{x_0, \ldots, x_{n-1}\}$. Then $\widetilde{B} \subset \widetilde{A}_n$, so

$$\mu(\widetilde{B}) \leq 2^{-n} = 2 \cdot 2^{-(n+1)} = 2\varphi(a_{n+1}) < 2\varphi(\operatorname{diam}(\widetilde{B})).$$

Thus if $\{\widetilde{B}_k\}_{k=1}^{\infty}$ is an ε-cover of $\mathcal{K}'(X)$, we have

$$\sum_{k=1}^{\infty} \varphi(\operatorname{diam}(\widetilde{B}_k)) \geq \frac{1}{2} \sum_{k=1}^{\infty} \mu(\widetilde{B}_k) \geq \frac{1}{2}$$

and so

$$\mathcal{H}^{\varphi}(\mathcal{K}'(X)) \geq \mathcal{H}^{\varphi}_{\varepsilon}(\mathcal{K}'(X)) \geq \frac{1}{2}. \qquad \square$$

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