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ON THE DECOMPOSITION THEOREMS OF LEBESGUE AND JORDAN

Abstract

In the first part of this paper we show a powerful special case of Lebesgue's decomposition theorem, namely: if F is a VB function, satisfying Lusin's condition (N) on $[a,b]$, then $F(x)-F(a) = s_F(x) + (\mathcal{L})\int_a^x F'(t)\,dt$, where s_F is the saltus function of F. In the second part we show that if F satisfies Lusin's condition (N) on $[a,b]$ then the functions (from the decomposition theorem of Jordan) $V_F(x) := V(F;[a,x])$ and $G(x) := F(x) - V_F(x)$ also satisfy (N).

The following decomposition theorem of Lebesgue is well known:

Theorem A (Lebesgue's decomposition theorem)**.** ([7], p. 119).
If F is an additive function of bounded variation of an interval, the derivative F' is summable, and the function F is the sum of a singular additive function of an interval and of the indefinite integral of the derivative F'.

Moreover, if the function F is non-negative, we have for every interval I_o

$$F(I_o) \geq \int_{I_o} F'(t)\,dt\,,$$

equality holding only in the case in which the function F is absolutely continuous on I_o.

In the first part of this paper, for the special case of a function defined on an interval $[a,b]$, with bounded variation and satisfying Lusin's condition (N), Theorem A becomes

$$F(x)-F(a) = s_F(x) + (\mathcal{L})\int_a^x F'(t)\,dt\,,$$

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where s_F is the saltus function of F (clearly s_F is a singular function). Moreover we obtain

$$V(F;[a,x]) = S_F(x) + (\mathcal{L})\int_a^x |F^{'}(t)|\,dt\,,$$

(for the definition of S_F see Lemma 8).

The following decomposition theorem of Jordan is well known:

Theorem B (Jordan's decomposition theorem)**.** ([6], p. 218). *A function $F:[a,b]\to\mathbb{R}$ is VB if and only if it is representable as the difference of two increasing functions.*

In fact from the proof of this theorem it follows that if F is VB on $[a,b]$ then the functions $V_F(x) := V(F;[a,x])$ and $G(x) := F(x) - V_F(x)$ are increasing. The question is which properties of F will be preserved for V_F and G? It is known that if F is left, right or bilaterally continuous at a point $x\in[a,b]$ then so are V_F and G ([6], p. 223).

In the second part of this paper we show that if F satisfies Lusin's condition (N) then V_F and G also satisfy (N).

1 On Lebesgue's decomposition theorem

We assume that the reader is familiar with the notions of VB, AC and Lusin's condition (N) (see [7], [6]). We denote by $\mathcal{C}$ the set of all continuous functions. If $F:[a,b]\to\mathbb{R}$ and $x_o\in[a,b)$ (resp. $x_o\in(a,b]$) then we denote by

$$F(x_o+) = \lim_{\substack{x\to x_o\\ x>x_o}} F(x) \quad \left(\text{resp. } F(x_o-) = \lim_{\substack{x\to x_o\\ x<x_o}} F(x)\right)$$

Lemma 1. (Theorem 1 of [6], p. 205). *The sets of the discontinuity points of an increasing function $F:[a,b]\to\mathbb{R}$ is at most countable. If $x_1,x_2,x_3,\ldots$ are all of the interior discontinuity points then*

$$F(a+) - F(a) + \sum_k \big(F(x_k+) - F(x_k-)\big) + F(b) - F(b-) \le F(b) - F(a)\,.$$

Definition 1 (The saltus of an increasing function)**.** ([6], p. 206). Let $F:[a,b]\to\mathbb{R}$ be an increasing function. Let $A=\{a_1,a_2,\ldots\}$ be a countable subset of $[a,b]$ containing all the interior discontinuity points of F (that this is possible follows by Lemma 1). We define $s_F:[a,b]\to\mathbb{R}$ by $s_F(a)=0$ and for every $x\in(a,b]$,

$$s_F(x) = F(a+) - F(a) + \sum_{t\in A\cap(a,x)} \big(F(t+) - F(t-)\big) + F(x) - F(x-)\,.$$

The function s_F is called the saltus of F.
Let $A_n = \{a_1, a_2, \dots, a_n\}$. We define $s_{F,n}(a) = 0$ and for every $x \in (a, b]$,

$$s_{F,n}(x) = F(a+) - F(a) + \sum_{t \in A_n \cap (a,x)} \big(F(t+) - F(t-)\big) + F(x) - F(x-)\,.$$

Remark 1. The functions s_F and $s_{F,n}$ have the following properties:

(i) s_F and $s_{F,n}$ are increasing on $[a, b]$, so VB on $[a, b]$;

(ii) $s_{F,n}(x) \to s_F(x)$ for every $x \in [a, b]$;

(iii) $s_{F,n}$ is a constant on each component of the open set $(a, b) \setminus A_n$ (therefore $s_{F,n}$ is a step-function);

(iv) The function $F - s_F$ is increasing and continuous on $[a, b]$ (see Theorem 2 of [6], p. 206).

Lemma 2 (Sarkhel and Kar)**.** ([8] or [2], [4]).
$VB \cap (N)$ is a linear space on $[a, b]$.

Lemma 3. *Let $F : [a, b] \to \mathbb{R}$ be an increasing function. Then*

$$s_F(x_o+) - s_F(x_o) = F(x_o+) - F(x_o)\,, \quad x_o \in [a, b)$$

$$s_F(x_o) - s_F(x_o-) = F(x_o) - F(x_o-)\,, \quad x_o \in (a, b]\,.$$

Proof. Let $x_o \in [a, b)$ and $x > x_o$. Then

$$s_F(x) = s_F(x_o) + F(x_o+) - F(x_o) + \sum_{t \in A \cap (x_o,x)} \big(F(t+) - F(t-)\big) + F(x) - F(x-)\,.$$

It follows that $s_F(x_o+) = \lim_{x \searrow x_o} s_F(x) = s_F(x_o) + F(x_o+) - F(x_o)$. Let $x_o \in (a, b]$ and $x < x_o$. Then

$$s_F(x_o) = s_F(x) + F(x+) - F(x) + \sum_{t \in A \cap (x,x_o)} \big(F(t+) - F(t-)\big) + F(x_o) - F(x_o-)\,.$$

It follows that $s_F(x_o) = s_F(x_o-) + F(x_o) - F(x_o-)$. □

Lemma 4. *For s_F defined above we have:*

(i) $s_F \in (N)$ on $[a, b]$ and $s_F' = 0$ a.e. on $[a, b]$;

(ii) If $F \in (N)$ on $[a, b]$ then $F - s_F \in AC$ on $[a, b]$.

Proof. (i) We have

$$s_F([a,b]) \subseteq [0, s_F(b)] \setminus \Bigg((0, s_F(a+)) \cup$$

$$\cup \Big(\cup_{k=1}^{\infty} \big((s_F(a_k-), s_F(a_k)) \cup (s_F(a_k), s_F(a_k+)) \big) \Big) \cup \big(s_F(b-), s_F(b) \big) \Bigg).$$

By Lemma 3 we have

$s_F(a+) = F(a+) - F(a)$,

$s_F(a_k+) - s_F(a_k-) = F(a_k+) - F(a_k-)$ for each k, and

$s_F(b) - s_F(b-) = F(b) - F(b-)$.

Since

$$s_F(b) = F(a+) - F(a) + \sum_{k=1}^{\infty} \big(F(a_k+) - F(a_k-) \big) + F(b) - F(b-)\,,$$

it follows that $m(s_F([a,b])) = 0$, hence $s_F \in (N)$ on $[a,b]$. Clearly s_F is derivable *a.e.* on $[a,b]$. By Krzyzewski's Lemma (see [5] or [1], p. 70), we obtain that $s_F^{'} = 0$ *a.e.* on $[a,b]$.

(ii) By Lemma 2 and Remark 1, (iv), it follows that $F - s_F \in VB \cap \mathcal{C} \cap (N) = AC$ on $[a,b]$ (see the Banach-Zarecki Theorem [7], p. 227). □

Remark 2. That $s_F^{'} = 0$ *a.e.* in the proof of Lemma 4, (i), follows also from the following theorem of Fubini: *If* $F(x) = \sum_n F_n(x)$ *is a convergent series of monotone nondecreasing functions on* $[a,b]$ *then* $F^{'}(x) = \sum_n F_n^{'}(x)$ *a.e. on* $[a,b]$ ([7], p. 117), and Remark 1, (ii), (iii).

Definition 2. Let $\Delta : a = x_o < x_1 < \ldots < x_n = b$ be a division of $[a,b]$, and let $g : [a,b] \to \mathbb{R}$. We denote

$$V_\Delta(g; [a,b]) = \sum_{k=0}^{n-1} |g(x_{k+1}) - g(x_k)|$$

and

$$V(g; [a,b]) = \sup_{\Delta} V_\Delta(g; [a,b]) \quad \text{(see [6], p. 215)}\,.$$

A division $\Delta_1 : a = y_o < y_1 < \ldots < y_m = b$ is said to be finer than Δ if $\{x_o, x_1, \ldots, x_n\} \subset \{y_o, y_1, \ldots, y_m\}$. Clearly $V_\Delta(g; [a,b]) \le V_{\Delta_1}(g; [a,b])$.

Lemma 5. *Let $f : [a,b] \to \mathbb{R}$, $f \in \mathcal{C} \cap VB$, and let $h : [a,b] \to \mathbb{R}$ such that h is constant on (a,b). Then*

$$V(h;[a,b]) = |h(a+) - h(a)| + |h(b) - h(b-)|$$

and

$$V(f+h;[a,b]) = V(f;[a,b]) + V(h;[a,b])\,.$$

Proof. The first part is obvious. We show the second one. Clearly

$$V(f+h;[a,b]) \le V(f;[a,b]) + V(h;[a,b])\,.$$

Let $\epsilon > 0$. Since f is continuous on $[a,b]$, there exists $\delta > 0$ such that $|f(a) - f(x)| < \epsilon/8$ if $x \in [a, a+\delta)$, and $|f(b) - f(y)| < \epsilon/8$ if $y \in (b-\delta, b]$. Let

$$\Delta : a = t_o < t_1 < t_2 < \ldots < t_n < t_{n+1} = b$$

be a division of $[a,b]$ such that $V_\Delta(f;[a,b]) > V(f;[a,b]) - \epsilon/2$. We may suppose without loss of generality that $t_1 \in (a, a+\delta)$ and $t_n \in (b-\delta, b)$ (because if Δ_1 is a finer division than Δ then $V_{\Delta_1}(f;[a,b]) \ge V_\Delta(f;[a,b])$). Then

$$V_\Delta(f+h;[a,b]) =$$

$$= |f(t_1) - f(a) + h(t_1) - h(a)| + |f(t_2) - f(t_1)| + \ldots + |f(t_n) - f(t_{n-1})| +$$

$$+ |f(b) - f(t_n) + h(b) - h(t_n)| \ge |h(t_1) - h(a)| - 2|f(t_1) - f(a)| + V_\Delta(f;[a,b]) -$$

$$-2|f(b) - f(t_n)| + |h(b) - h(t_n)| > |h(a+) - h(a)| - 2\frac{\epsilon}{8} +$$

$$+ V(f;[a,b]) - \frac{\epsilon}{2} - 2\frac{\epsilon}{8} + |h(b) - h(b-)| = V(f;[a,b]) + V(h;[a,b]) - \epsilon\,.$$

Therefore $V(f+h;[a,b]) \ge V(f;[a,b]) + V(h;[a,b])$, hence $V(f+h;[a,b]) = V(f;[a,b]) + V(h;[a,b])$. □

Corollary 1. *Let $f : [a,b] \to \mathbb{R}$, $f \in \mathcal{C} \cap VB$, and let $g : [a,b] \to \mathbb{R}$ such that g is constant on each of the intervals (a, c_1), $(c_1, c_2), \ldots, (c_m, b)$, where $a < c_1 < c_2 < \ldots < c_m < b$. Then*

$$V(g;[a,b]) = |g(a+) - g(a)| +$$

$$+ \sum_{i=1}^{m} (|g(c_i) - g(c_i-)| + |g(c_i+) - g(c_i)|) + |g(b) - g(b-)|$$

and

$$V(f+g;[a,b]) = V(f;[a,b]) + V(g;[a,b])\,.$$

Proof. We have

$$V(g;[a,b]) = V(g;[a,c_1]) + \sum_{i=1}^{m-1} V(g;[c_i,c_{i+1}]) + V(g;[c_m,b]) =$$

$$\big|g(a+)-g(a)\big| + \big|g(c_1)-g(c_1-)\big| + \big|g(c_1+)-g(c_1)\big| + \big|g(c_2)-g(c_2-)\big| + \ldots +$$

$$+\big|g(c_{m-1}+)-g(c_{m-1})\big| + \big|g(c_m)-g(c_m-)\big| + \big|g(c_m+)-g(c_m)\big| + \big|g(b)-g(b-)\big| =$$

$$= \big|g(a+)-g(a)\big| + \sum_{i=1}^{m}\big(|g(c_i)-g(c_i-)| + |g(c_i+)-g(c_i)|\big) + \big|g(b)-g(b-)\big|$$

(this equality was used before in [6], p. 231, but without proof). Now by Lemma 5

$$V(f+g;[a,b]) = V(f+g;[a,c_1]) + \sum_{i=1}^{m-1} V(f+g;[c_i,c_{i+1}]) + V(f+g;[c_m,b]) =$$

$$= V(f;[a,c_1]) + V(g;[a,c_1]) + \sum_{i=1}^{m-1}(V(f;[c_i,c_{i+1}]) + V(g;[c_i,c_{i+1}]))+$$

$$+V(f;[c_m,b]) + V(g;[c_m,b]) = V(f;[a,b]) + V(g;[a,b])$$

□

Definition 3 (The saltus of a VB function)**.** ([6], p. 219).
Let $F:[a,b]\to\mathbb{R}$, $F\in VB$. Then we define $s_F:[a,b]\to\mathbb{R}$ by

$$s_F(x) = s_{V_F}(x) - s_G(x),$$

where $V_F(x) = V(F;[a,x])$ and $G(x) = V_F(x) - F(x)$. Clearly V_F and G are increasing (see Theorem B).

Lemma 6. ([6], p. 233). *Let $g_n, g:[a,b]\to\mathbb{R}$. If $\{g_n\}_n$ converges pointwise to g on $[a,b]$ and there exists a positive number α such that $V(g_n;[a,b])<\alpha$, $(\forall)$ $n=1,2,\ldots,$ then $V(g;[a,b])\le\alpha$.*

Lemma 7. *Let $g, g_n:[a,b]\to\mathbb{R}$ such that $\{g_n\}_n$ converges pointwise to g on $[a,b]$ and $V(g_n;[a,b])\nearrow\alpha$ for some positive number α. If*

$$\liminf_n V(g_n-g;[a,b]) = 0$$

then $V(g;[a,b])=\alpha$.

Proof. By Lemma 6 it follows that $V(g;[a,b]) \leq \alpha$. We show the converse inequality. For $\epsilon > 0$ let n_ϵ be a positive integer such that

$$V(g_n;[a,b]) > \alpha - \frac{\epsilon}{2}\,,\ (\forall)\ n \geq n_\epsilon\,.$$

For each $n \geq n_\epsilon$ we have

$$\alpha - \frac{\epsilon}{2} < V(g_n;[a,b]) = V(g-(g-g_n);[a,b]) \leq$$
$$\leq V(g;[a,b]) + V(g-g_n;[a,b])\,.$$

Since $\liminf_n V(g-g_n;[a,b]) = 0$, it follows that there exists a positive number $m \geq n_\epsilon$ such that $V(g-g_m;[a,b]) < \epsilon/2$. Therefore

$$\alpha - \frac{\epsilon}{2} \leq V(g;[a,b]) + \frac{\epsilon}{2}\,,$$

so $\alpha \leq V(g;[a,b])$. It follows that $\alpha = V(g;[a,b])$. □

Lemma 8. *Let F, G, and V_F be as in Definition 3.*

(i) The set A of the interior discontinuity points of F is at most countable and contains the sets of interior discontinuity points of V_F and G. Moreover, if $x \in A \cup \{a\}$ then there exists $F(x+)$ and for each $x \in A \cup \{b\}$ there exists $F(x-)$.

(ii) Let $A = \{a_1, a_2, a_3, \ldots\}$. Then $s_F(a) = 0$ and for $x \in (a,b]$,

$$s_F(x) = F(a+) - F(a) + \sum_{t\in(a,x)\cap A} \big(F(t+) - F(t-)\big) + F(x) - F(x-)\,.$$

(iii) If A is infinite, let $s_{F,n} : [a,b] \to \mathbb{R}$ such that $s_{F,n}(a) = 0$ and for $x \in (a,b]$,

$$s_{F,n}(x) = F(a+) - F(a) + \sum_{t\in(a,x)\cap A_n} \big(F(t+) - F(t-)\big) + F(x) - F(x-)\,,$$

where $A_n = \{a_1, a_2, \ldots, a_n\}$. Let

$$r_{F,n}(x) = s_F(x) - s_{F,n}(x) = \sum_{t\in(a,x)\cap(A\setminus A_n)} \big(F(t+) - F(t-)\big)\,.$$

We define $S_F : [a,b] \to \mathbb{R}$ by $S_F(a) = 0$ and for every $x \in (a,b]$,

$$S_F(x) = |F(a+) - F(a)| +$$

$$+\sum_{t\in A\cap(a,x)}(|F(t+)-F(t)|+|F(t)-F(t-)|)+|F(x)-F(x-)|$$

(this series is convergent, see [6], p. 235*). Then we have*

1) s_F *is* VB *on* $[a,b]$*;*

2) $s_F\in(N)$ *on* $[a,b]$ *and* $s_F^{'}=0$ *a.e. on* $[a,b]$*;*

3) $F-s_F$ *is continuous on* $[a,b]$*;*

4) $s_{F,n}(x)\to s_F(x)$ *for every* $x\in[a,b]$*;*

5) $s_{F,n}$ *is a constant on each component of the open set* $(a,b)\setminus A_n$ *(therefore* $s_{F,n}$ *is a step-function) and*

$$V(s_{F,n};[a,b])=|F(a+)-F(a)|+$$

$$+\sum_{i=1}^{n}\big(|F(a_i)-F(a_i-)|+|F(a_i+)-F(a_i)|\big)+|F(b)-F(b-)|\,.$$

6) If $n\to\infty$ *then*

$$V(r_{F,n};[a,b])\le\sum_{t\in(a,b)\cap(A\setminus A_n)}\big(|F(t+)-F(t)|+|F(t)-F(t-)|\big)\to 0\,;$$

7) $V(s_{F_n};[a,b])\nearrow S_F(b)$*;*

8) $V(s_F;[a,b])=S_F(b)$*.*

Proof. (i) See [6] (Corollary 2, p. 219 and Theorem 1, p. 223).

(ii) See [6] (p. 219).

(iii) 1) See the definition of s_F;

2) See Lemma 4, (i) and Lemma 2.

3) See [6] (p. 220).

4) This is obvious.

5) The first part is evident. We show the second part. For $i=1,2,\dots,n$,

$$s_{F,n}(a+)-s_{F,n}(a)=F(a+)-F(a)\,;$$

$$s_{F,n}(a_i)-s_{F,n}(a_i-)=F(a_i)-F(a_i-)\,;$$

$$s_{F,n}(a_i+)-s_{F,n}(a_i)=F(a_i+)-F(a_i)\,;$$

$$s_{F,n}(b)-s_{F,n}(b-)=F(b)-F(b-)\,.$$

By Corollary 1 we obtain

$$V(s_{F,n};[a,b])=|F(a+)-F(a)|+$$

$$\sum_{i=1}^{n}\big(|F(a_i)-F(a_i-)|+|F(a_i+)-F(a_i)|\big)+|F(b)-F(b-)|\,.$$

6) Let $a \le x_1 < x_2 \le b$. Then

$$r_{F,n}(x_2)-r_{F,n}(x_1)=\sum_{t\in[x_1,x_2)\cap(A\setminus A_n)}\big(F(t+)-F(t-)\big)\le$$

$$\le \sum_{t\in[x_1,x_2)\cap(A\setminus A_n)}\big(|F(t+)-F(t)|+|F(t)-F(t-)|\big)\,,$$

hence

$$V(r_{F,n};[a,b])\le \sum_{t\in[a,b)\cap(A\setminus A_n)}\big(|F(t+)-F(t)|+|F(t)-F(t-)|\big)=$$

$$=\sum_{t\in(a,b)\cap(A\setminus A_n)}\big(|F(t+)-F(t)|+|F(t)-F(t-)|\big)\to 0 \ \text{ for } \ n\to\infty\,.$$

7) This is evident.
8) See 4), 6), 7) and Lemma 7. □

Remark 3. $S_F \in (N)$ on $[a,b]$ and $S_F^{'} = 0$ *a.e.* on $[a,b]$ (the proof is similar to that of Lemma 4, (i)).

Lemma 9. *Let $F:[a,b]\to\mathbb{R}$, $F\in VB\cap(N)$. Then $F-s_F\in AC$ on $[a,b]$.*

Proof. By Remark 1, (iv), the functions $V_F - s_{V_F}$ and $G - s_G$ are continuous on $[a,b]$. Let

$$\varphi=(V_F-s_{V_F})-(G-s_G)\,.$$

Then φ is continuous on $[a,b]$. But

$$F=V_F-G=V_F-s_{V_F}+s_{V_F}-(G-s_G+s_G)=$$

$$=\varphi+(s_{V_F}-s_G)=\varphi+s_F\,.$$

By Lemma 4, (i) the functions s_{V_F} and s_G belong to $VB\cap(N)$ on $[a,b]$, hence by Lemma 2, $s_F\in VB\cap(N)$. Again by Lemma 2, it follows that $\varphi=F-s_F\in VB\cap(N)\cap\mathcal{C}=AC$ on $[a,b]$ (see the Banach-Zarecki Theorem [7], p. 227). □

Lemma 10. *Let $F:[a,b]\to\mathbb{R}$, $F\in VB\cap(N)$ and let $F=H-h$, where H and h are some increasing functions given by Theorem B. Then $s_F=s_H-s_h$, hence s_F does not depend on the choice of the functions H and h if $F\in VB\cap(N)$.*

Proof. By Lemma 9, $F-s_F \in AC$ on $[a,b]$. But $F-(s_H-s_h)=(H-s_H)-(h-s_h)$ is continuous (see Remark 1, (iv)). By Lemma 4, (i) and Remark 1, (i), $s_H, s_h \in VB \cap (N)$ on $[a,b]$. Since $F \in VB \cap (N)$, by Lemma 2 it follows that $F-(s_H-s_h) \in VB \cap (N)$ on $[a,b]$. From the Banach–Zarecki theorem we obtain that $F-(s_H-s_h) \in AC$ on $[a,b]$. Hence $s_F-(s_H-s_h) \in AC$ on $[a,b]$. By Lemma 4, (i), it follows that

$$s_F^{'} = s_H^{'} = s_h^{'} = 0 \ a.e. \ \text{on } [a,b].$$

Then $s_F-(s_H-s_h)$ is constant on $[a,b]$ (see for example Theorem 2 of [6], p. 246). Since $s_F(a)=s_H(a)=s_h(a)=0$, it follows that the above constant is zero, hence $s_F = s_H - s_h$ on $[a,b]$. □

Theorem 1. *Let $F:[a,b]\to\mathbb{R}$, $F \in VB$. Then $F \in (N)$ on $[a,b]$ iff*

$$F(x)-F(a) = s_F(x) + (\mathcal{L})\int_a^x F^{'}(t)\,dt\,. \tag{1}$$

Proof. "⇒" Suppose that $F \in (N)$ on $[a,b]$ and let $\varphi = F - s_F$. Then $\varphi \in AC$ on $[a,b]$ (see Lemma 9), and by Lemma 4, (i) we have that $s_F^{'}=0$ *a.e.* on $[a,b]$. It follows that $\varphi^{'} = F^{'}$ *a.e.* on $[a,b]$. By Theorem 3 of [6] (p. 255), we obtain that

$$\varphi(x) = \varphi(a) + (\mathcal{L})\int_a^x \varphi^{'}(t)\,dt = F(a) + (\mathcal{L})\int_a^x F^{'}(t)\,dt\,.$$

Therefore we have (1).
"⇐" Suppose that (1) holds. The function $F(a)+(\mathcal{L})\int_a^x F^{'}(t)\,dt$ is AC on $[a,b]$ (see for example Theorem 1 of [6], p. 252), hence it is $VB \cap (N)$ on $[a,b]$ (for the (N) part see Theorem 3 of [6], p. 249). By Lemma 8, (iii), 2), 1), it follows that $s_F \in VB \cap (N)$ on $[a,b]$. From (1) and Lemma 2, we obtain that $F(x) = s_F(x) + F(a) + (\mathcal{L})\int_a^x F^{'}(t)\,dt$ is $VB \cap (N)$ on $[a,b]$. □

2 On Jordan's decomposition theorem

Lemma 11. (Theorem 8 of [6], p. 259). *Let $f:[a,b]\to\overline{\mathbb{R}}$ be a summable function, and let*

$$F(x) = (\mathcal{L})\int_a^x f(t)\,dt\,, \quad x \in [a,b]\,.$$

Then

$$V(F;[a,x]) = (\mathcal{L})\int_a^x |f(t)|\,dt\,, \quad x \in [a,b]\,.$$

Lemma 12. *Let $f, F : [a,b] \to \mathbb{R}$, $F, f \in VB$. If f is continuous on $[a,b]$ then*

$$V(f + s_F; [a,b]) = V(f; [a,b]) + V(s_F; [a,b])\,.$$

Particularly

$$V(F; [a,b]) = V(s_F; [a,b]) + V(F - s_F; [a,b])\,.$$

Proof. Clearly $(f + s_{F,n})(x) \to (f + s_F)(x)$ if $n \to \infty$ (see Lemma 8, (iii), 4)). By Corollary 1 and Lemma 8, (iii), 7), 8), we have

$$V(f + s_{F,n}; [a,b]) = V(f; [a,b]) + V(s_{F,n}; [a,b]) \nearrow V(f; [a,b]) + S_F(b) =$$
$$= V(f; [a,b]) + V(s_F; [a,b])\,.$$

By Lemma 8, (iii), 6) and Lemma 7, it follows that

$$V(f + s_F; [a,b]) = V(f; [a,b]) + V(s_F; [a,b])\,.$$

We show the second part. The function $f := F - s_F$ is continuous and VB on $[a,b]$ (see Lemma 8, (iii), 3)). Therefore

$$V(F; [a,b]) = V(F - s_F + s_F; [a,b]) = V(F - s_F; [a,b]) + V(s_F; [a,b])\,.$$

□

Theorem 2. *Let $F : [a,b] \to \mathbb{R}$, $F \in VB \cap (N)$. Then*

(i) $V_F(x) = S_F(x) + (\mathcal{L}) \int_a^x |F^{'}(t)|\, dt$;

(ii) $V_F \in (N)$ *on* $[a,b]$.

Proof. (i) From Theorem 1 we obtain

$$(F - s_F)(x) = F(a) + (\mathcal{L}) \int_a^x F^{'}(t)\, dt\,. \tag{2}$$

We have

$$V_F(x) = V(F; [a,x]) = V(s_F; [a,x]) + V(F - s_F; [a,x]) =$$
$$= S_F(x) + V(F - s_F; [a,x]) = S_F(x) + V(F - s_F - F(a); [a,x]) =$$
$$= S_F(x) + (\mathcal{L}) \int_a^x |F^{'}(t)|\, dt$$

(for the second equality see Lemma 12; for the third equality see Lemma 8, (iii), 8); the fourth equality is obvious; the last equality follows by Lemma 11 and (2).

(ii) This follows since $S_F \in (N)$ (see Remark 3), by Lemma 2 and (i). □

Corollary 2. *Let $F : [a,b] \to \mathbb{R}$, $F \in VB$. Then $F \in (N)$ on $[a,b]$ if and only if $V_F \in (N)$ on $[a,b]$.*

Proof. "⇒" See Theorem 2, (ii).
"⇐" This follows as the implication "⇐" in Lemma 6 of [3]. □

Remark 4. Corollary 2 extends Lemma 6 of [3], since here F is not supposed to be continuous on $[a,b]$.

Theorem 3 (A Jordan type theorem)**.** *Let $F : [a,b] \to \mathbb{R}$, $F \in VB \cap (N)$. Then F is the difference of two increasing functions, each satisfying Lusin's condition (N).*

Proof. By Theorem 2, (ii) we have that $V_F \in (N)$ on $[a,b]$, and by Lemma 2, the function $G = V_F - F \in (N)$. □

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