Real Analysis Exchange
Vol. 23(1), 1998-99, pp. 295–301

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AN ELEMENTARY PROOF OF THE BANACH–ZARECKI THEOREM

Abstract

In this paper we shall give a new, elementary proof of the Banach–Zarecki theorem, based on the following classical result [3] (p. 183): If $\{A_i\}_i$ is a sequence of decreasing sets in a measurable space $(X, \mathcal{M}, \mu)$ and $\mu(A_1) < +\infty$ then $\mu(\cap_i A_i) = \lim_{i\to\infty} \mu(A_i)$.

There is a very rich literature concerning the Banach–Zarecki Theorem, such as the books of Saks [5] (p. 227), Natanson [4] (p. 250), Foran [3] (p. 357), Ene [1] (pp. 81, 104) and a paper of Varberg [6] (p. 835). This theorem asserts that *if a continuous and VB function satisfies Lusin's condition (N) on an interval then it is also AC on that interval.*

The proofs in [5], [3], [6] and [1] (p. 81) are based on the following result (see Theorem 6.5 of [5], p. 227; Theorem 1 of [6], p. 834; Theorem 8.1 of [3]): *If a function F is derivable at every point of a measurable set D, then* $m^*(F(D)) \leq (\mathcal{L}) \int_D |F'(x)| dx$.

In [4], the Banach–Zarecki Theorem is proved in a totally different way, namely using Lebesgue's Convergence Theorem as well as the fact that *the Banach indicatrix for a continuous and VB function on $[a, b]$ is summable* (see Theorem 3 of [4], p. 225).

In [1] (p. 104), the Banach–Zarecki Theorem is a consequence of some general notion (AC_∞, VB_∞ etc.). Here the Banach indicatrix has also an important role, but the proof is different from that in [4].

In this paper we shall give a new, elementary proof of the Banach–Zarecki theorem, based on the following classical result [3] (p. 183): *If $\{A_i\}_i$ is a sequence of decreasing sets in a measurable space $(X, \mathcal{M}, \mu)$ and $\mu(A_1) < +\infty$ then $\mu(\cap_i A_i) = \lim_{i\to\infty} \mu(A_i)$.*

Key Words: (N), VB, AC, Banach–Zarecki Theorem
Mathematical Reviews subject classification: 26A45, 26A46, 26A48
Received by the editors February 27, 1997

Let $m(A)$ denote the Lebesgue measure of the set A, whenever $A \subset \mathbb{R}$ is Lebesgue measurable, and $m^*(X)$ the outer measure of the set X. We denote by $\mathcal{O}(F;[a,b])$ the oscillation of the function F on $[a,b]$, and by $V(F;[a,b])$ the variation of F on $[a,b]$. For the definitions of VB and AC see [5].

Lemma 1. *Let $f : X \mapsto Y$ and $Y_1 = \{y \in Y : f^{-1}(y)$ contains more than one point$\}$. If $(X_i)_{i\in I}$ is a family of subsets of X then*

$$(\cap_{i\in I} f(X_i)) \setminus Y_1 \ \subseteq\ f(\cap_{i\in I} X_i) \ \subseteq\ \cap_{i\in I}\, f(X_i)\,.$$

PROOF. Let $y \in (\cap_{i\in I} f(X_i)) \setminus Y_1$. Then there exists a unique point $x \in X$ such that $y = f(x)$. Since $y \in f(X_i)$ for each $i \in I$, it follows that there exists $x_i \in X_i$ such that $f(x_i) = y$, so $x_i = x$. We obtain that $x \in X_i$ for each $i \in I$, hence $x \in \cap_{i\in I} X_i$. It follows that $y = f(x) \in f(\cap_{i\in I} X_i)$. The last (well-known) inclusion is easy to verify. □

Lemma 2. *Let $F : [a,b] \mapsto \mathbb{R}$ be an increasing function and let $B = \{y \in [F(a),F(b)] : F^{-1}(y)$ contains more than one point$\}$. Then:*

(i) B is at most countable, hence B is a Borel set;

(ii) If $Z \subset (a,b)$ is a G_δ-set then $F(Z)$ is a Borel set.

Moreover, if $Z = \cap_{i=1}^{\infty} G_i$, where each G_i is an open set, then

$$m(F(Z)) = m(\cap_{i=1}^{\infty} F(G_i))\,.$$

PROOF. (i) For $y \in B$ let $x_y^{'} = \inf(F^{-1}(y))$ and $x_y^{''} = \sup(F^{-1}(y))$. By hypotheses $\emptyset \neq (x_y^{'}, x_y^{''}) \subset F^{-1}(y)$. Since

$$(x_{y_1}^{'}, x_{y_1}^{''}) \cap (x_{y_2}^{'}, x_{y_2}^{''}) = \emptyset \quad \text{whenever} \quad y_1 \neq y_2\,,$$

it follows that B is at most countable.

(ii) Let $Z = \cap_{i=1}^{\infty} G_i$, where each G_i is an open set. Then by Lemma 1 we have

$$(\cap_{i=1}^{\infty} F(G_i)) \setminus B \ \subseteq\ F(\cap_{i=1}^{\infty} G_i) \ \subseteq\ \cap_{i=1}^{\infty}\, F(G_i)\,. \tag{1}$$

Since F is increasing, each $F(G_i)$ is a countable union of intervals (some of them might be degenerate). Hence each $F(G_i)$ is a Borel set and by (1), $F(\cap_{i=1}^{\infty} G_i)$ is also a Borel set.

The last part follows by (i) and (1) □

Lemma 3. *Let $F : [a,b] \mapsto \mathbb{R}$ be a continuous and increasing function. Then $F \in (N)$ on $[a,b]$ if and only if $m(F(Z)) = 0$ whenever Z is a compact subset of $[a,b]$ and $m(Z) = 0$.*

PROOF. "⇒" This is obvious.

"⇐" Suppose on the contrary that $F \notin (N)$ on $[a, b]$. Then there exists a set $Z \subset [a, b]$, with $m(Z) = 0$ such that $m^*(F(Z)) > 0$. We may suppose without loss of generality that Z is of G_δ–type (because for Z there exists a G_δ–set Z_1 of measure zero, such that $Z \subset Z_1$, so $m^*(F(Z_1)) > 0$). Then $F(Z)$ is a Borel set of positive measure (see Lemma 2). Let K be a compact subset of $F(Z)$ of positive measure. Then $K_1 := F^{-1}(K)$ is a compact subset of Z (because F is continuous), and $F(K_1) = K$, a contradiction. □

Remark 1. In [2], Foran introduces the following condition: *A function is said to satisfy condition $(N^{'})$ provided the image of closed sets of measure* 0 *is of measure* 0.

Using this condition, Lemma 3 from above can be stated as follows: *Let $F : [a, b] \mapsto \mathbb{R}$ be a continuous and increasing function. Then $F \in (N)$ if and only if $F \in (N^{'})$.*

Foran showed that conditions (N) and $(N^{'})$ coincide for Baire functions (in fact his results are much stronger). Hence Lemma 3 is a special case of this result. On the other hand, our proof in this particular case is elementary.

Lemma 4. *Let $F : [a, b] \mapsto \mathbb{R}$ be such that $F([a, b])$ is an interval. Suppose that P is a perfect set containing the points a and b, and let $\{(a_i, b_i)\}_i$ be the intervals contiguous to P. Then*

$$|F(b) - F(a)| \le m(F([a, b])) \le m^*(F(P)) + \sum_{i=1}^{\infty} \mathcal{O}(F; [a_i, b_i]) .$$

PROOF. We have

$$F([a, b]) = F(P \cup (\cup_{i=1}^{\infty}(a_i, b_i)) = F(P) \cup (\cup_{i=1}^{\infty} F((a_i, b_i))) ,$$

hence

$$|F(b) - F(a)| \le m(F([a, b]) \le m^*(F(P)) + \sum_{i=1}^{\infty} m^*(F((a_i, b_i))) \le$$

$$\le m^*(F(P)) + \sum_{i=1}^{\infty} \mathcal{O}(F; [a_i, b_i]) .$$

□

Lemma 5. *([4], p. 224). Let $f : [a, b] \mapsto \mathbb{R}$ be a continuous function. Subdivide $[a, b]$ by means of the points*

$$x_o = a < x_1 < x_2 < \cdots < x_n = b \quad \textit{with} \quad \max(x_{k+1} - x_k) = \lambda ,$$

and form the sums

$$V = \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|.$$

If $\lambda \to 0$ then each of the sums V tends to the total variation $V(f;[a,b])$ of the function $f(x)$ (we do not suppose that the variation is finite).

Lemma 6. *Let $F : [a,b] \mapsto \mathbb{R}$ be a continuous and VB function, and let $H : [a,b] \mapsto \mathbb{R}$. $H(x) = V(F;[a,x])$. Then $F \in (N)$ if and only if $H \in (N)$.*

PROOF. "$\Rightarrow$" Clearly H is increasing and continuous on $[a,b]$ (see Theorem 1 of [4], p. 223). Suppose on the contrary that $H \notin (N)$. By Lemma 3 it follows that there exists a compact set $Z \subset [a,b]$ of measure zero, such that $H(Z)$ is a compact set (because H is continuous) of positive measure. Let $c = \inf(Z)$, $d = \sup(Z)$. We may suppose without loss of generality that Z is a perfect set (if necessary eliminating the isolated points of Z, that are at most countable). Let $\{(a_i,b_i)\}_i$ be the intervals contiguous to Z. Let $\alpha = m(H(Z)) > 0$. For each positive integer n let $[c_i,d_i]$, $i = 1,2,\dots,n$ be the closed subintervals of $[c,d]$ left after extracting the open intervals (a_i,b_i), $i = 1,2,\dots,n-1$. Let $\lambda_n = \max_{i=1}^{n}(d_i - c_i)$. Since $m(Z) = 0$ it follows that $\lim_{n\to\infty} \lambda_n = 0$. Let $c = x_o < x_1 < x_2 < \dots < x_p = d$ be a division of $[c,d]$ that contains all c_i and d_i, each (c_i,d_i) containing no x_j, and such that $x_j - x_{j-1} < \lambda_n$, for each $j = 1,2,\dots,p$. Let

$$S_n := \sum_{j=1}^{p} |F(x_j) - F(x_{j-1})|$$

and

$$V_n := \sum_{i=1}^{n} |F(d_i) - F(c_i)| + \sum_{i=1}^{n-1} V(F;[a_i,b_i])\,. \tag{2}$$

Then $S_n \le V_n \le V := V(F;[c,d])$. By Lemma 5 we have

$$\lim_{n\to\infty} S_n = V\,, \quad \text{so} \quad \lim_{n\to\infty} V_n = V\,.$$

It follows that there exists a positive integer n_o such that

$$V_n > V - \frac{\alpha}{2}\,, \quad \text{whenever} \quad n \ge n_o\,. \tag{3}$$

By Theorem 5 of [4], p. 217, we obtain

$$V = \sum_{i=1}^{n} V(F;[c_i,d_i]) + \sum_{i=1}^{n-1} V(F;[a_i,b_i]) =$$

$$= \sum_{i=1}^{n} (H(d_i) - H(c_i)) + \sum_{i=1}^{n-1} V(F;[a_i,b_i]\,. \tag{4}$$

By (2), (3) and (4), for $n \geq n_o$ it follows that

$$\sum_{i=1}^{n} |F(d_i) - F(c_i)| > \sum_{i=1}^{n} (H(d_i) - H(c_i)) - \frac{\alpha}{2}\,. \tag{5}$$

Clearly

$$\sum_{i=1}^{n} (H(d_i) - H(c_i)) > m(H(Z)) = \alpha\,,$$

hence, by (5)

$$\sum_{i=1}^{n} |F(d_i) - F(c_i)| > \frac{\alpha}{2}\,. \tag{6}$$

By Lemma 4 (since $m^*(F(Z)) = 0$), for each $i = 1, 2, \ldots, n$ we have

$$|F(d_i) - F(c_i)| \leq \sum_{\{j:[a_j,b_j] \subset [c_i,d_i]\}} \mathcal{O}(F;[a_j,b_j])\,,$$

hence

$$\sum_{i=1}^{n} |F(d_i) - F(c_i)| \leq \sum_{j=n}^{\infty} \mathcal{O}(F;[a_j,b_j])\,. \tag{7}$$

By (6) and (7) it follows that for each $n \geq n_o$ we have

$$\frac{\alpha}{2} \leq \sum_{j=n}^{\infty} \mathcal{O}(F;[a_j,b_j])\,. \tag{8}$$

But $\sum_{i=1}^{\infty} \mathcal{O}(F;[a_j,b_j]) \leq V$, so

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \mathcal{O}(F;[a_j,b_j]) = 0\,. \tag{9}$$

By (8) and (9) we obtain that $\frac{\alpha}{2} \leq 0$, a contradiction. Therefore $H \in (N)$ on $[a,b]$.

"$\Leftarrow$" Let $Z \subset [a,b]$, $m(Z) = 0$. We may suppose without loss of generality that Z is a G_δ-set of the form $Z = \cap_{i=1}^\infty G_i$, G_i open sets and $G_1 \supset G_2 \supset \cdots$. By Lemma 2,

$$m(H(Z)) = m(\cap_{i=1}^\infty H(G_i)) = 0\,.$$

Since the sequence of sets $\{H(G_i)\}_i$ is decreasing it follows that

$$\lim_{i\to\infty} m(H(G_i)) = 0\,.$$

Let $G_i := \cup_{j=1}^\infty (a_j^i, b_j^i)$. For each i we have

$$m(F(Z)) = m(F(\cup_{j=1}^\infty (Z \cap (a_j^i, b_j^i))) \le \sum_{j=1}^\infty m^*(F; Z \cap (a_j^i, b_j^i)) <$$

$$< \sum_{j=1}^\infty \mathcal{O}(F; [a_j^i, b_j^i]) = \sum_{j=1}^\infty (H(b_j^i) - H(a_j^i)) = m(H(G_i))\,.$$

Therefore $m(F(Z)) = 0$. □

Lemma 7. *Let $F : [a,b] \mapsto \mathbb{R}$ be a continuous and increasing function, satisfying Lusin's condition (N). Then $F \in AC$ on $[a,b]$.*

PROOF. We shall follow the first part of the proof of the Banach–Zarecki Theorem ([4], p. 250). Suppose that $F \notin AC$ on $[a,b]$. Then there is a number $\epsilon_o > 0$ having the following property: for every $\delta > 0$ there exists a finite family of pairwise disjoint open intervals $\{(a_k, b_k)\}_k$, $k = 1, 2\ldots, n$ such that

$$\sum_{k=1}^n (b_k - a_k) < \delta \quad \text{and} \quad \sum_{k=1}^n (F(b_k) - F(a_k)) \ge \epsilon_o\,.$$

Let $\sum_{i=1}^\infty \delta_i$ be a convergent series of positive terms, and for every δ_i, let $\{(a_k^i, b_k^i)\}_k$, $k = 1, 2, \ldots, n_i$ be a collection of pairwise disjoint open intervals such that

$$\sum_{k=1}^{n_i} (b_k^i - a_k^i) < \delta_i \quad \text{and} \quad \sum_{k=1}^{n_i} (F(b_k^i) - F(a_k^i)) \ge \epsilon_o\,.$$

Let

$$E_i = \sum_{k=1}^{n_i} (a_k^i, b_k^i) \quad \text{and} \quad A = \cap_{i=1}^\infty (\cup_{i=n}^\infty E_i)\,.$$

It is easy to verify that $m(A) = 0$. By hypotheses it follows that $m(F(A)) = 0$. Let $G_n := \cup_{i=n}^{\infty} E_i$. Then

$$m(F(G_n)) \geq m(F(E_n)) = \sum_{k=1}^{n_i} (F(b_k^i) - F(a_k^i) \geq \epsilon_o \,.$$

By Lemma 2 we have

$$0 = F(A) = m(\cap_{n=1}^{\infty} F(G_n)) = \lim_{n \to \infty} m(F(G_n)) \geq \epsilon_o \,,$$

a contradiction. □

Lemma 8. *Let $F, H : [a,b] \mapsto \mathbb{R}$, $H(x) = V(F; [a,x])$. If $H \in AC$ on $[a,b]$ then $F \in AC$ on $[a,b]$.*

PROOF. This follows by the fact that $|F(\beta) - F(\alpha)| \leq H(\beta) - H(\alpha)$, whenever $[\alpha, \beta] \subseteq [a,b]$. □

The Banach–Zarecki Theorem. *Let $F : [a,b] \mapsto \mathbb{R}$. If F is a continuous and VB function, satisfying Lusin's condition (N) then F is AC on $[a,b]$.*

PROOF. Let $H : [a,b] \mapsto \mathbb{R}$, $H(x) := V(F; [a,x])$. By Lemma 6, H is continuous and increasing, and satisfies condition (N) on $[a,b]$. By Lemma 7, $H \in AC$ on $[a,b]$. By Lemma 8 it follows that $F \in AC$ on $[a,b]$. □

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