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A PAIR OF ADJOINT CLASSES OF RIEMANN–STIELTJES INTEGRABLE FUNCTIONS

Abstract

The purpose of this paper is to show that the classes of Riemann integrable functions and absolutely continuous functions are adjoint with respect to the $(R\text{-}S)$ integral $\int_a^b f\,dg$.

Definition. Let A and B be two classes of functions defined on $[a, b]$. A and B are said to be adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$, if the following conditions are satisfied:

(i) If $f \in A$ and $g \in B$, then the $(R\text{-}S)\int_a^b f\,dg$ exists;

(ii) If the $(R\text{-}S)\int_a^b f\,dg$ exists for all $g \in B$, then $f \in A$; and

(iii) If the $(R\text{-}S)\int_a^b f\,dg$ exists for all $f \in A$, then $g \in B$.

If A and B are adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$, this means that on condition that the $(R\text{-}S)\int_a^b f\,dg$ exists, neither A nor B can be extended at all. For convenience, we write $(A * B)\int_a^b f\,dg$ meaning that A and B are adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$.

We introduce the following symbols for some classes of functions defined on $[a, b]$:

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$R[a,b]$	class of Riemann integrable functions on $[a,b]$
$C[a,b]$	class of continuous functions on $[a,b]$
$BV[a,b]$	class of functions of bounded variation on $[a,b]$
$AC[a,b]$	class of absolutely continuous functions on $[a,b]$.

It is known [1] that $C[a,b]$ and $BV[a,b]$ are adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$. In this paper we would like to show that $R[a,b]$ and $AC[a,b]$ are adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$. To do this, we should prove that $R[a,b]$ and $AC[a,b]$ satisfy the three conditions in the definition.

(i) If $f \in R[a,b]$ and $g \in AC[a,b]$, then $(R\text{-}S)\int_a^b f\,dg$ exists (cf. [3] for proof);

(ii) If $(R\text{-}S)\int_a^b f\,dg$ exists for all $g \in AC[a,b]$, then $f \in R[a,b]$. As a matter of fact, taking $g(x) = x \in AC[a,b]$ gives that $f \in R[a,b]$.

In order to prove statement (iii) "if the $(R\text{-}S)\int_a^b f\,dg$ exists for all $f \in R[a,b]$, then $g \in AC[a,b]$", we need the following lemmas.

Lemma 1 (Vitali)**.** [2] *Let E be set of finite outer measure and $\mathcal{J}$ a collection of intervals that cover E in the sense of Vitali. Then, given $\epsilon > 0$, there is a finite disjoint collection $\{I_1, \ldots, I_N\}$ of intervals in $\mathcal{J}$ such that*

$$m^*\left[E \sim \cup_{n=1}^{N} I_n\right] < \epsilon\,.$$

Lemma 2. *Let f be a function on $[a,b]$ such that $f' = 0$ a.e. Then, f has the following property:* (S) *Given $\epsilon > 0$, $\delta > 0$, there is a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals on $[a,b]$ such that*

$$\sum |x_k - y_k| < \delta$$

and

$$\sum |f(x_k) - f(y_k)| > |f(b) - f(a)| - \epsilon\,.$$

PROOF. Let $E \subset (a,b)$ be the set of measure $b-a$ in which $f'(x) = 0$, and ϵ and δ be arbitrary positive numbers. To each x in E there is an arbitrarily small interval $[x, x+h]$ contained in (a,b) such that $|f(x+h) - f(x)| < \epsilon \cdot h/(b-a)$. By Lemma 1, we can find a finite collection $\{[x_k, y_{k+1}]\}$ of nonoverlapping intervals of this sort which cover all of E except for a set of measure less than δ. If we label the x_k so that $x_k < x_{k+1}$, we have

$$a = y_o < x_o < y_1 < x_1 < y_2 < \cdots < y_{n-1} < x_{n-1} < y_n < x_n = b$$

and

$$\sum_{k=0}^{n} |x_k - y_k| < \delta\,.$$

Now

$$|f(b) - f(a)| = \left|\sum_{k=0}^{n}[f(x_k) - f(y_k)] + \sum_{k=0}^{n-1}[f(y_{k+1}) - f(x_k)]\right| <$$

$$< \sum_{k=0}^{n}|f(x_k) - f(y_k)| + \epsilon\,.$$

Thus

$$\sum_{k=0}^{n}|f(x_k) - f(y_k)| > |f(b) - f(a)| - \epsilon\,.$$

□

Note. If the function f in Lemma 2 is continuous, then we can find a finite collection $\{[y_k, x_k]\}$ of nonoverlapping intervals in (a, b) instead of $[a, b]$ such that the above two inequalities hold, too.

We are now in a position to prove the statement (iii):

Theorem. *If the* $(R\text{-}S)\int_a^b f\,dg$ *exists for all* $f \in R[a, b]$*, then* $g \in AC[a, b]$.

Proof. First of all, the fact that

$C[a, b] \subset R[a, b]$, $BV[a, b] \subset R[a, b]$,

$(C * BV)\int_a^b f\,dg$ and the $(R\text{-}S)\int_a^b f\,dg$ exists for all $f \in R[a, b]$,

implies that $g \in C[a, b] \cap BV[a, b]$. So, it follows that $g = G + F$ with $G \in AC[a, b]$ and $F \in C[a, b]$, $F^{'} = 0$ *a.e.* on $[a, b]$. To show $g \in AC[a, b]$, it suffices to show $F = const.$ on $[a, b]$. By hypothesis, the $(R\text{-}S)\int_a^b f\,dg$ exists for all $f \in R[a, b]$ and so does the $(R\text{-}S)\int_a^b f\,dF$. Suppose that $F(x) \neq const.$ on $[a, b]$. Then, there is a point $c \in (a, b]$ such that $F(c) - F(a) \neq 0$. For convenience, let $c = b$. We shall now construct a function $f \in R[a, b]$ such that the $(R\text{-}S)\int_a^b f\,dF$ does not exist.

Let ϵ be a number with $0 < \epsilon < |F(b) - F(a)|$, and $\{\delta_n\}$ be a sequence satisfying $\delta_n \downarrow 0$ $(n \to \infty)$. Since $F \in C[a, b]$ and $F^{'} = 0$ *a.e.* on $[a, b]$, by the note of Lemma 2, the function F has property (S). That is, for $\epsilon > 0$ and $\delta_1 > 0$, there is a finite collection $\{[y_k^{(1)}, x_k^{(1)}]\}$ of nonoverlapping intervals in (a, b) such that

$$\sum_{k=0}^{n_1}\left|x_k^{(1)} - y_k^{(1)}\right| < \delta_1$$

and

$$\sum_{k=0}^{n_1} \left| F(x_k^{(1)}) - F(y_k^{(1)}) \right| > |F(b) - F(a)| - \frac{\epsilon}{2} \,.$$

We call $\{[y_k^{(1)}, x_k^{(1)}]\}$ the first level of the collection of nonoverlapping intervals. Then, for every integer $h > 1$, we can inductively find the h-th level of it, denoted by

$$\mathcal{J}_h = \left\{ \left[y_k^{(h)}, x_k^{(h)} \right] \right\}_{k=0,\ldots,n_h} ,$$

such that

(1) $\sum_{k=0}^{n_h} \left| x_k^{(h)} - y_k^{(h)} \right| < \delta_h$;

(2) $\sum_{k=0}^{n_h} \left| F(x_k^{(h)}) - F(y_k^{(h)}) \right| > |F(b) - F(a)| - \epsilon \cdot (2^h - 1)/2^h$;

(3) $I_h \subset I_{h-1}^{\sim}$, where define $I_h = \cup_k \left[y_k^{(h)}, x_k^{(h)} \right]$ and $I_h^{\sim} = \cup_k \left(y_k^{(h)}, x_k^{(h)} \right)$.

Assume $\mathcal{J}_h$ is found. We wish to find $\mathcal{J}_{h+1}$. To do this, we shall make use of Lemma 2 repeatedly. Applying Lemma 2 on each interval $\left[y_i^{(h)}, x_i^{(h)} \right]$ in $\mathcal{J}_h$ $(i = 0, 1, \ldots, n_h)$, we can find a finite collection $\left\{ \left[y_{k(i)}^{(h+1)}, x_{k(i)}^{(h+1)} \right] \right\}$ of nonoverlapping intervals in $\left(y_i^{(h)}, x_i^{(h)} \right)$ such that

$$\sum_{k(i)} \left| x_{k(i)}^{(h+1)} - y_{k(i)}^{(h+1)} \right| < \frac{\delta_{h+1}}{2^{i+1}}$$

and

$$\sum_{k(i)} \left| F(x_{k(i)}^{(h+1)}) - F(y_{k(i)}^{(h+1)}) \right| > \left| F(x_i^{(h)}) - F(y_i^{(h)}) \right| - \frac{\epsilon}{2^{h+i+2}} \,.$$

Collecting all of

$$\left\{ \left[y_{k(i)}^{(h+1)}, x_{k(i)}^{(h+1)} \right] \right\} \qquad (i = 0, 1, \ldots, n_h)$$

gives us a finite collection of nonoverlapping intervals in $I_h^{\sim}$, denoted by

$$\mathcal{J}_{h+1} = \left\{ \left[y_k^{(h+1)}, x_k^{(h+1)} \right] \right\}_{k=0,\ldots,n_{h+1}} .$$

Then, we have that

$$\sum_{k=0}^{n_{h+1}} \left| x_k^{(h+1)} - y_k^{(h+1)} \right| < \delta_{h+1} \left(\frac{1}{2} + \cdots + \frac{1}{2^{n_h+1}} \right) < \delta_{h+1}$$

and
$$\sum_{k=0}^{n_{h+1}} \left|F(x_k^{(h+1)}) - F(y_k^{(h+1)})\right| >$$
$$> \sum_{k=0}^{n_h} \left|F(x_k^{(h)}) - F(y_k^{(h)})\right| - \epsilon \left(\frac{1}{2^{h+2}} + \cdots + \frac{1}{2^{n_h+h+2}} \right)$$
$$> |F(b) - F(a)| - \frac{\epsilon(2^h - 1)}{2^h} - \frac{\epsilon}{2^{h+1}}$$
$$= |F(b) - F(a)| - \frac{\epsilon(2^{h+1} - 1)}{2^{h+1}} \, .$$
It is clearly true that $I_{h+1} \subset I_h^{\sim}$.

Therefore, the above $\mathcal{J}_{h+1}$ is indeed the $(h+1)$-th level of the finite collection of nonoverlapping intervals with properties (1), (2) and (3).

We now define a function f_1 on $[a, b]$ by
$$f_1(x) = \begin{cases} \text{sign}\left[F(x_k^{(h)}) - F(y_k^{(h)})\right] & \text{if } x = y_k^{(h)} \text{ for } h \geq 1 \text{ and } k = 0, 1, \ldots, n_h \\ 0 & \text{if } x \in [a, b] \sim \cup_{k,h}\left\{y_k^{(h)}\right\} \ . \end{cases}$$
Since $I_{h+1} \subset I_h^{\sim}$ and $\mathcal{J}_h$ is the collection of nonoverlapping intervals for $h \geq 1$, and so $y_k^{(h)} \neq y_j^{(i)}$ if $(h, k) \neq (i, j)$. Whence, the function f_1 is well defined on $[a, b]$. We must show $f_1 \in R[a, b]$. If $x_o \in I_h^{\sim} \sim I_{h+1}$ for an integer $h \geq 0$ (denote $I_0 = [a, b]$), then in view of the definition of f_1, there is an open interval $O(x_0, \eta) = (x_0 - \eta, x_0 + \eta) \subset I_h^{\sim} \sim I_{h+1}$ such that $f_1(x) = 0$ if $x \in O(x_0, \eta)$. Thus f_1 is continuous at x_0. Let E be the set of this sort of points x_0. It is not hard to see that
$$E = (a, b) \sim \cap_{h=0}^{\infty} I_h \sim \cup_{k,h}\left\{y_k^{(h)}\right\} \sim \cup_{k,h}\left\{x_k^{(h)}\right\} ,$$
where the last two terms are countable sets.
Also, from
$$m(I_h) = \sum_{k=0}^{n_h} \left|x_k^{(h)} - y_k^{(h)}\right| < \delta_h \to 0 \quad (h \to \infty)$$
and
$$I_{h+1} \subset I_h \qquad (h \geq 0)$$
we have
$$m\left(\cap_{h=0}^{\infty} I_h\right) = 0$$
and so $m(E) = b - a$.

Hence, the bounded function f_1 is continuous almost everywhere on $[a,b]$, that is, $f_1 \in R[a,b]$. We shall now show that the $(R\text{-}S)\int_a^b f_1\,dF$ does not exist.

Given $\lambda > 0$. There is a positive integer h such that $0 < \delta_h < \lambda$. Let P be a subdivision, $a = x_0 < x_1 < \cdots < x_n = b$, of $[a,b]$ with $\max_i\{\Delta x_i\} < \lambda$ such that each interval $\left[y_k^{(h)}, x_k^{(h)}\right]$ of $\mathcal{J}_h$ is one of the subintervals of P. Let σ be a Stieltjes sum, corresponding to P. Then, $\sigma = \sum_{i=0}^{n-1} f_1(\xi_i)[F(x_{i+1}) - F(x_i)]$, where $\xi_i \in [x_i, x_{i+1}]$. If $[x_i, x_{i+1}] = \left[y_k^{(h)}, x_k^{(h)}\right]$, then we choose $\xi_i = y_k^{(h)}$. Otherwise, there is a point $\xi_i \in [x_i, x_{i+1}]$ such that $f_1(\xi_i) = 0$. Thus, we have that

$$\begin{aligned}
\sigma &= \sum_{k=0}^{n_h} \operatorname{sign}\left[F(x_k^{(h)}) - F(y_k^{(h)})\right] \cdot \left[F(x_k^{(h)}) - F(y_k^{(h)})\right] \\
&= \sum_{k=0}^{n_h} \left|F(x_k^{(h)}) - F(y_k^{(h)})\right| \\
&> |F(b) - F(a)| - \frac{\epsilon(2^h - 1)}{2^h} \\
&> |F(b) - F(a)| - \epsilon > 0\,.
\end{aligned}$$

On the other hand, however, if we choose $\xi_i \in [x_i, x_{i+1}]$ such that $f_1(\xi_i) = 0$ for $i = 0, 1, \ldots, n-1$, then, this leads to another Stieltjes sum $\sigma_1 = 0$. The fact that when $\lambda \to 0$, $\sigma - \sigma_1 \geq |F(b) - F(a)| - \epsilon > 0$ implies the $(R\text{-}S)\int_a^b f_1\,dF$ does not exist. This contradicts that the $(R\text{-}S)\int_a^b f\,dF$ exists for all $f \in R[a,b]$. Hence, $F = const.$ on $[a,b]$, and so $g = G + F$ is absolutely continuous on $[a,b]$. Thus the theorem is proved. □

Consequently, we see that $R[a,b]$ and $AC[a,b]$ are adjoint with respect to the $(R\text{-}S)\int_a^b f\,dg$.

References

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