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ON THE MEASURABILITY OF FUNCTIONS DEFINED ON THE PRODUCT OF TWO TOPOLOGICAL SPACES

Abstract

Some conditions implying the measurability of functions defined on the product of two topological spaces are investigated.

Let $\mathbb{R}$ denote the set of all reals and let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Moreover, let μ_1 and μ_2 respectively, be σ-finite measures defined on some σ-fields $\mathcal{M}_1 \supset \mathcal{T}_X$ and $\mathcal{M}_2 \supset \mathcal{T}_Y$. Assume that

(1) for every set $A \in \mathcal{M}_1$ with $\mu_1(A) > 0$ there is a set $B \in \mathcal{T}_X$ such that $B \subset A$ and $\mu_1(B) > 0$;

(2) $\mu_1(A) > 0$ for all nonempty sets $A \in \mathcal{T}_X$.

A function $f : X \to \mathbb{R}$ is called $\mathcal{T}_X$-quasicontinuous ($\mathcal{T}_X$-cliquish) at a point $x \in X$ ([5] if for every positive real η and for every set $U \in \mathcal{T}_X$ containing x there is a nonempty set $V \in \mathcal{T}_X$ such that $V \subset U$ and $|f(v) - f(x)| < \eta$ for all points $v \in V$ ($\operatorname{osc}_V f < \eta$, where $\operatorname{osc}_V f$ denotes the diameter of the set $f(V)$).

In the proofs we will use the following Davies lemma ([2, 3]):

Lemma 1. *Suppose that the measure μ_1 is complete and a function $f : X \to \mathbb{R}$ is such that for every positive real η and for every set $A \in \mathcal{M}_1$ with $\mu_1(A) > 0$ there is a set $B \in \mathcal{M}_1$ such that $B \subset A$, $\mu_1(B) > 0$ and $\operatorname{osc}_B f < \eta$. Then the function f is μ_1-measurable.*

Remark 2. *If a function $f : X \to \mathbb{R}$ is measurable with respect to μ_1, then it is $\mathcal{T}_X$-cliquish at every point $x \in X$;*

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If the measure μ_1 is complete, then every function $f : X \to \mathbb{R}$ is $\mathcal{T}_X$-cliquish at each point is μ_1-measurable.

PROOF. Assume that the function f is μ_1-measurable and fix a positive real η, a point $x \in X$ and a set $U \in \mathcal{T}_X$ containing x. Since the function f is μ_1-measurable and $\mu_1(U) > 0$, there is an open interval I of the length $d(I) < \eta$ such that $\mu_1(f^{-1}(I) \cap U) > 0$. By (1) there is a nonempty set $V \in \mathcal{T}_X$ such that $V \subset U \cap f^{-1}(I)$. Since $d(I) < \eta$, we obtain $\mathrm{osc}_V f < \eta$.

Now, we suppose that the function f is $\mathcal{T}_X$-cliquish at every point. Fix a positive real η and a set $A \in \mathcal{M}_1$ with $\mu_1(A) > 0$. By (1) there is a nonempty set $U \in \mathcal{T}_X$ such that $U \subset A$. Fix a point $x \in U$. Since the function f is $\mathcal{T}_X$-cliquish at x, there is a nonempty set $V \in \mathcal{T}_X$ such that $V \subset U$ and $\mathrm{osc}_V f < \eta$. Since $V \in \mathcal{M}_1$ and $\mu_1(V) > 0$, we obtain by Davies's Lemma that the function f is measurable with respect to μ_1. □

Remark 3. *If a function $f : X \to \mathbb{R}$ is μ_1-measurable and if $C(f)$ denotes the set of all $\mathcal{T}_X$-continuity points of f, then $\mu_1(X \setminus C(f)) = 0$.*

PROOF. Suppose, to the contrary, that $\mu_1(X \setminus C(f)) > 0$. For a point $x \in X$ let

$$\mathrm{osc}\, f(x) = \inf\{d(f(A)); x \in A \in \mathcal{T}_X\},$$

where $d(f(A))$ denotes the diameter of the set $f(A)$. Evidently,

$$X \setminus C(f) = \bigcup_{n=1}^{\infty} \{x; \mathrm{osc}\, f(x) \geq 1/n\}.$$

So, there is a positive integer n such that $A_n = \{x; \mathrm{osc}\, f(x) \geq 1/n\}$ is not of measure μ_1 zero. Since the set A_n is $\mathcal{T}_X$-closed, we have $A_n \in \mathcal{M}_1$ and $\mu_1(A_n) > 0$. By (1) there is a nonempty set $U \in \mathcal{T}_X$ such that $U \subset A_n$. Fix a point $x \in U$. Since the function f is $\mathcal{T}_X$-cliquish at x, there is a nonempty set $V \in \mathcal{T}_X$ such that

$$(V \subset U \subset A_n) \ \wedge \ (\mathrm{osc}_V f < 1/n).$$

So, we obtain a contrary with the inequality $\mathrm{osc}\, f(v) \geq 1/n$ for $v \in V$. □

Now, we will consider some functions of two variables.
For this, let $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ and let μ be the completion of the product measure $\mu_1 \times \mu_2$. Assume also that:

(3) For every set $A \in \mathcal{M}$ with $\mu(A) > 0$ there is a set $B \in \mathcal{M}$ such that $(B \subset A) \ \wedge \ (\mu(B) > 0)$, all sections

$$B_x = \{y \in Y; (x, y) \in B\} \in \mathcal{T}_Y; \ x \in X,$$

and all sections

$$B^y = \{x \in X; (x,y) \in B\} \in \mathcal{T}_X;\ y \in Y.$$

Let $\mathcal{A} \subset \mathcal{M}_2$ be a family of subsets of Y of positive measure μ_2, let $y \in Y$ be a point and let $f : Y \to \mathbb{R}$ be a function. We will write:

$f \in \mathcal{B}(\mathcal{A})$ if and only if for every positive real η and for every set $U \in \mathcal{T}_Y$ there is a set $A \in \mathcal{A}$ such that

$$(\mu_2(A \cap U) > 0)\ \wedge\ (\mathrm{osc}_{A\cap U} f < \eta);$$

$f \in \mathcal{Q}_s(y, \mathcal{A})$ if and only if for every positive real η and for every set $U \in \mathcal{T}_Y$ containing y there is a nonempty set $V \in \mathcal{A}$ such that

$$(\mu_2(V \cap U) > 0)\ \wedge\ (|f(t) - f(y)| < \eta)$$

for all points $t \in V \cap U$.

Theorem 4. *Let $f : X \times Y \to \mathbb{R}$ be a function such that all sections $f^y(x) = f(x,y)$, $x \in X$ and $y \in Y$, are $\mathcal{T}_X$-quasicontinuous at every point $x \in X$. If there is a countable family $\mathcal{A} \subset \mathcal{M}_2$ of subsets of Y of positive measure μ_2 such that for every point $x \in X$ the section $f_x(y) = f(x,y)$, $y \in Y$, belongs to the family $\mathcal{B}(\mathcal{A})$, then the function f is measurable with respect to the measure μ.*

PROOF. We will prove that the function f satisfies the hypothesis of Davies Lemma. Fix a positive real η and a set $A \in \mathcal{M}$ such that $\mu(A) > 0$. By (3) there is a set $B \in \mathcal{M}$ such that $(B \subset A)\ \wedge\ (\mu(B) > 0)$ and all sections

$$B_x \in \mathcal{T}_Y,\ \ x \in X \text{ and } B^y \in \mathcal{T}_X,\ \ y \in Y.$$

Enumerate all sets of the family $\mathcal{A}$ in a sequence (finite or not)

$$A_1, \dots, A_n, \dots.$$

By our hypothesis for every point $(x,y) \in B$ there is a set $A(x,y) \in \mathcal{A}$ such that

$$\mathrm{osc}_{A(x,y)\cap B_x} f_x < \eta/8 \text{ and } \mu_2(A(x,y) \cap B_x) > 0.$$

Since $\mu(B) > 0$, there is a positive integer n such that the set

$$D = \{(x,y) \in B; A(x,y) = A_n\}$$

is not of measure μ zero. Let

$$\mathrm{Pr}_X(D) = \{x \in X; \exists_y (x,y) \in D\}$$

and let $E \subset X$ be a μ_1-measurable covering of the set $\mathrm{Pr}_X(D)$, i.e. the set $E \in \mathcal{M}_1$, $\mathrm{Pr}_X(D) \subset E$ and every μ_1-measurable set $S \subset E \setminus \mathrm{Pr}_X(D)$ is such that $\mu_1(S) = 0$. Evidently, $\mu_1(E) > 0$. By (1) there is a nonempty set U such that $(U \in \mathcal{T}_X) \ \wedge \ (U \subset E)$. Fix a point $(x,y) \in (U \times A_n) \cap B$. Since $x \in U \cap B^y$ and the section f^y is $\mathcal{T}_X$-quasicontinuous at x, there is a nonempty set $V \subset U \cap B^y$ such that

$$(V \in \mathcal{T}_X) \ \wedge \ (\mathrm{osc}_V f^y < \eta/8).$$

If points

$$(u_1, v_1), (u_2, v_2) \in ((V \cap \mathrm{Pr}_X(D)) \times A_n) \cap B,$$

then

$$|f(u_1,v_1) - f(u_2,v_2)| \le |f(u_1,v_1) - f(u_1,y)| + |f(u_1,y) - f(u_2,y)|$$

$$+|f(u_2,y) - f(u_2,v_2)| < \eta/8 + \eta/8 + \eta/8 = 3\eta/8$$

So, there is a closed interval I of the length $d(I) \le 3\eta/8$ such that

$$f(((V \cap \mathrm{Pr}_X(D)) \times A_n) \cap B) \subset I.$$

Let J be the closed interval of the length $3\eta/4$ having the same center as I. Assume, to the contrary, that there is a point $(u,v) \in (V \times A_n) \cap B$ such that $f(u,v) \in \mathbb{R} \setminus J$. Since $u \in V \cap B^v \in \mathcal{T}_X$, by the $\mathcal{T}_X$-quasicontinuity of the section f^v at the point u we obtain the existence of a nonempty set $W \in \mathcal{T}_X$ such that

$$W \subset V \cap B^v \ \wedge \ \forall_{w \in W} f(w,v) \in \mathbb{R} \setminus J.$$

But

$$(\mu_1(W) > 0) \ \wedge \ (W \subset U \subset E),$$

so there is a point

$$s \in W \cap \mathrm{Pr}_X(D) \subset V \cap \mathrm{Pr}_X(D).$$

Observe that

$$(v \in A_n) \wedge (s \in W \subset B^v) \wedge ((s,v) \in B).$$

Since $s \in \mathrm{Pr}_X(D)$, we have $f(s,v) \in I$. So, we obtain a contradiction to $f(s,v) \in \mathbb{R} \setminus J$.

By Fubini's theorem the μ-measurable set $P = (V \times A_n) \cap B$ is of positive measure μ. So,

$$P \subset A \ \wedge \ \mu(P) > 0 \text{ and } \operatorname{osc}_P f \le d(J) < \eta$$

and, by Davies's Lemma, the function f is μ-measurable.

Remark 5. *Observe that in Theorem 1, it is suffices to assume only that the sections f_x belong to the family $\mathcal{B}(\mathcal{A})$ with the exception of a set of measure μ_1 zero.*

Remark 6. *If $X = Y = \mathbb{R}$, $\mu_1 = \mu_2$ is Lebesgue measure in $X = Y$ and $\mathcal{T}$ is the density topology (see [1]), then the conditions (1), (2), (3) are satisfied. There is a nonmeasurable (in the Lebesgue sense) set $A \subset \mathbb{R}^2$ such that all sections A_x, A^y, $x, y \in \mathbb{R}$, are empty or contain only one point (see [6]). Let $\mathcal{A}$ be the family of all open intervals with rational endpoints. If $f(x, y) = 1$ for $(x, y) \in A$ and 0 otherwise on $\mathbb{R}^2$, then the function f is nonmeasurable and the sections f_x and f^y, $x, y \in \mathbb{R}$, are measurable and belong to $\mathcal{B}(\mathcal{A})$.*

Theorem 7. *Let $f : X \times Y \to \mathbb{R}$ be a function such that the sections f^y, $y \in \mathbb{R}$, are μ_1-measurable. Suppose that there is a countable family $\mathcal{A} \subset \mathcal{M}_2$ of sets of positive measure μ_2 such that for every positive real η and for each nonempty set $U \in \mathcal{M}_2$ with $\mu_2(U) > 0$ there is a set $A \in \mathcal{A}$ such that $A \subset U$ and for each point $x \in \mathbb{R}$ the relation $\operatorname{osc}_A f_x < \eta$ holds. Then the function f is μ-measurable.*

PROOF. Let $B \in \mathcal{M}$ be a set of positive measure μ and let η be a positive real. There is a μ-measurable set $D \subset B$ such that $\mu(D) > 0$ and all sections $D_x \in \mathcal{T}_Y$, $x \in X$, and $D^y \in \mathcal{T}_X$, $y \in Y$. By Fubini's theorem and our hypothesis there are an interval I and a set $A \in \mathcal{A}$ such that the set

$$E = \{u \in D^y; A \subset D_u \wedge f_u(A) \subset I\}$$

is not of measure μ_1 zero and $d(I) < \eta/8$.

Let $F \subset A$ be a nonempty set belonging to $\mathcal{A}$ such that for each point $x \in X$ the inequality $\operatorname{osc}_F f_x < \eta/8$ is true. If $G \supset E$ is μ_1-measurable covering of the set E, then there is a nonempty set $H \subset G$ belonging to $\mathcal{T}_X$. Let $K = (H \times F) \cap D$ and let J be a closed interval having the same center as I and such that $d(J) = 3\eta/4$. The set K is μ-measurable and $\mu(K) > 0$. Observe that if for a point $x \in \mathbb{R}$ there is a point $t \in F$ such that $f(x, t) \in \mathbb{R} \setminus J$, then $f_x(F) \subset \mathbb{R} \setminus I$. Since the sections f^v are μ_1-measurable, the set

$$S = \{u \in H : \exists_{t \in F} f(u, t) \in \mathbb{R} \setminus J\}$$

is of measure μ_1 zero. Consequently, $f(K \setminus (S \times F)) \subset J$. Since

$$(K \setminus (S \times F) \subset B) \wedge (\mu(K \setminus (S \times F)) > 0),$$

by Davies's Lemma, the function f is μ-measurable.

Theorem 8. *Let $f : X \times Y \to \mathbb{R}$ be a function such that all sections f^y, $y \in Y$, are μ_1-measurable. If there is a countable family $\mathcal{A} \subset \mathcal{M}_2$ of sets of positive measure μ_2 such that for every point $x \in X$ and for every point $y \in Y$ the relation $f_x \in \mathcal{Q}_s(y, \mathcal{A})$ is true, then the function f is μ-measurable.*

PROOF. Fix a positive real η and a set $A \in \mathcal{M}$ with $\mu(A) > 0$. By (3) there is a μ-measurable set $B \subset A$ such that $\mu(B) > 0$ and

$$\forall_{(x,y)\in B}[(B_x \in \mathcal{T}_Y) \wedge (B^y \in \mathcal{T}_X)].$$

Fix a point $(x, y) \in B$. By Remark 1 the section f^y is $\mathcal{T}_X$-cliquish at the point x. From the $\mathcal{T}_X$-cliquishness of the section f^y at x it follows that there is a nonempty set D such that

$$D \subset B^y \ \wedge D \in \mathcal{T}_X \ \wedge \operatorname{osc}_D f^y < \eta/20.$$

By our hypothesis for every point $u \in D$ there is a set $A(u) \in \mathcal{A}$ such that

$\mu_2(A(u) \cap B_u) > 0$;

$|f(u, t) - f(u, y)| < \eta/20$ for every point $t \in A(u) \cap B_u$.

Since the family $\mathcal{A}$ is countable and $\mu_1(D) > 0$, there is a set $E \in \mathcal{A}$ such that the set $F = \{u \in D; A(u) = E\}$ is not of measure μ_1 zero. Let G be μ_1-measurable covering of the set F and let H be a nonempty set such that $H \subset G \cap D \wedge H \in \mathcal{T}_X$. If points (u_1, v_1), (u_2, v_2) belong to the set $(F \times E) \cap B$, then

$$|f(u_1, v_1) - f(u_2, v_2)| \le |f(u_1, v_1) - f(u_1, y)| + |f(u_1, y) - f(u_2, y)|$$
$$+|f(u_2, y) - f(u_2, v_2)| < \eta/20 + \eta/20 + \eta/20 = 3\eta/20.$$

So, there is a closed interval I such that

$$d(I) \le 3\eta/20 \ \wedge \ f((F \times E) \cap B) \subset I.$$

Let J be the closed interval having the same center as I and such that $d(J) = 3\eta/4$. Put $K = (H \times E) \cap B$. The set $K \in \mathcal{M}$ and by Fubini's theorem $\mu(K) > 0$. Let $P \subset K$ be a μ-measurable set such that $\mu(P) > 0$ and

$$\forall_{(u,v)\in P}[(P_u \in \mathcal{T}_Y) \wedge (P^v \in \mathcal{T}_X)].$$

We will prove that $\mu(P \setminus f^{-1}(J)) = 0$. Assume, to the contrary, that the set $L = P \setminus f^{-1}(J)$ is not of measure μ zero. Then for every point $(u, v) \in L$ there is a set $B(u, v) \in \mathcal{A}$ such that

$\mu_2(B(u, v) \cap P_u) > 0$;

$f(u, w) \in \mathbb{R} \setminus J$ for every point $w \in B(u, v) \cap P_u$.

Since the family $\mathcal{A}$ is countable, there is a set $N \in \mathcal{A}$ such that the set

$$M = \{(u, v) \in L; B(u, v) = N\}$$

is not of measure μ zero. Let M_1 be a μ_1-measurable covering of the projection $\mathrm{Pr}_X(M)$ and let $Q \in \mathcal{T}_X$ be a nonempty set contained in the set $M_1 \cap H$. Evidently,

$S = (Q \times N) \cap P \in \mathcal{M}$;

$\mu(S) > 0$;

$f((M \times N) \cap P) \subset \mathbb{R} \setminus J$.

Fix a point $(u, v) \in S$. Since the section f^v is $\mathcal{T}_X$-cliquish at the point u, there is a nonempty set $U \in \mathcal{T}_X$ such that

$$(U \subset Q \cap P^v) \ \wedge \ (\mathrm{osc}_U f^v < \eta/20).$$

There are points (s, v), (t, v) belonging to S with $s \in U \cap D$ and $t \in U \cap \mathrm{Pr}_X(M)$. Then $(f(s, v) \in I) \ \wedge \ (f(t, v) \in \mathbb{R} \setminus J)$. Consequently, we obtain

$$|f(s, v) - f(t, v)| \geq 3\eta/8 - 3\eta/40 > \eta/5,$$

and $\mathrm{osc}_U f^v > \eta/20$. This contradiction shows that $\mu(P \setminus f^{-1}(J)) = 0$. The set $P \cap f^{-1}(J) \subset A$ is μ-measurable, $\mu(P \cap f^{-1}(J)) > 0$ and

$$\mathrm{osc}_{(P \cap f^{-1}(J))} f \leq d(J) < \eta.$$

Hence, by Davies's Lemma, the function f is μ-measurable. □

Particular cases of properties $\mathcal{B}(\mathcal{A})$ and $\mathcal{Q}_s(\mathcal{A})$ are investigated in [3, 4].

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