Real Analysis Exchange
Vol. 23(2), 1997–1998, pp. 799–803

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ON ABSOLUTELY HENSTOCK INTEGRABLE FUNCTIONS

Abstract

In this note, we shall prove that every absolutely Henstock integrable function is McShane integrable, without using the measurability of Henstock integrable functions and gauge functions.

The result that every absolutely Henstock integrable function is McShane integrable is well-known. In [1, 2], the proof has been simplified without using Egoroff's theorem and truncated functions. In this note, we shall further simplify the proof in [1] without using the measurability of Henstock integrable functions and gauge functions. It has the advantage that, in a more general setting, the measurability of integrable functions and gauge functions may not make sense or its verification may be very involved. Lemma 2 in this note is crucial, though the proof is elementary.

By an *interval* E in $\mathbf{R}^m$, we mean the 'box' $[a_1, b_1] \times [a_2, b_2] \times \cdots [a_m, b_m]$, where $a_i, b_i \in \mathbf{R}$ for $i = 1, 2, \cdots, m$. We denote the volume of E by $|E|$. Given a fixed interval E an *elementary set* is a subinterval of E or a union of finite number of non-overlapping subintervals of E. We denote by $\mathcal{B}$ the collection of open sets whose complement with respect to E is an elementary set or an empty set. Given a $\xi \in E$ and a positive δ, we let $B(\xi, \delta) = \{x \in E : |x - \xi| < \delta\}$, that is the ball centered at ξ and radius δ.

A real-valued function f defined on E is said to be *Henstock integrable* to $F(E)$ if for every $\epsilon > 0$, there is a $\delta(\xi) > 0$ such that for any δ-fine full division $D = \{(I, \xi)\}$ of E, i.e. $\xi \in I \subseteq B(\xi, \delta(\xi))$, we have

$$\left|(D) \sum f(\xi)|I| - F(E)\right| < \epsilon.$$

It can be showed that whenever f is Henstock integrable on E, it is Henstock integrable on any subinterval I of E. A function f is said to be *absolutely Henstock integrable* on E if both f and $|f|$ are Henstock integrable on E.

Key Words: **Henstock, McShane, Absolutely Henstock**
Mathematical Reviews subject classification: 26A39
Received by the editors December 15, 1997

A real-valued function f defined on E is said to be *McShane integrable* to M if for every $\epsilon > 0$, there is a $\delta(\xi) > 0$ such that for any δ-fine full division $D = \{(I, \xi)\}$ of E, i.e. $I \subseteq B(\xi, \delta(\xi))$, but ξ does not necessarily belong to I, we have

$$\left|(D) \sum f(\xi)|I| - M\right| < \epsilon.$$

The following lemma can be proved readily.

Lemma 1. Let f be an Henstock integrable function on E. Then $G \in \mathcal{B}$ implies f is Henstock integrable on G.

Lemma 2. Let f be non-negative, Henstock integrable on E with $F(I) = \int_I f$, I is any subinterval of E. Let X be any subset of E. Then for any $\epsilon > 0$, there exists $G(\epsilon) \in \mathcal{B}$ with $X \subset G(\epsilon)$ such that whenever there is a finite collection $\{I_1, I_2, \cdots, I_n\}$ of non-overlapping subintervals of E with $I_i \subseteq G(\epsilon) - X$ for each i, we have

$$\sum_{i=1}^{n} F(I_i) < \epsilon.$$

PROOF. Let $X \subset E$. Define the class $B^* = \{G \in \mathcal{B} : X \subset G\}$, which is non-empty as $E \in B^*$. Since f is Henstock integrable on E, it is Henstock integrable on G, for each $G \in B^*$. Let $A = \inf\{\int_G f : G \in B^*\}$, which exists and $0 \leq A < \infty$. For a given $\epsilon > 0$, there exists $G(\epsilon) \in B^*$ such that

$$0 \leq \int_{G(\epsilon)} f - A < \epsilon/2.$$

Suppose $\{I_1, I_2, \cdots, I_n\}$ is a collection of non-overlapping subintervals of E with $I_i \subseteq G(\epsilon) - X$ for each i. Let $G' = G(\epsilon) - \cup_{i=1}^{n} I_i$. Then $G' \in B^*$. Moreover,

$$0 \leq \int_{G'} f - A \leq \int_{G(\epsilon)} f - A \leq \epsilon/2.$$

Thus, $0 \leq \sum_{i=1}^{n} F(I_i) = \int_{G(\epsilon)} f - \int_{G'} f < \epsilon$. □

For an absolutely Henstock integrable function f, we have $\sum\left|F(I)\right| \leq \sum G(I)$ where $G(I) = \int_I \left|f\right|$. Furthermore, the constant function $c(x) \equiv 1$ is Henstock integrable and $C(I) = \int_I c = |I|$. Thus, we have

Corollary 1. Suppose f is an absolutely Henstock integrable function on E with $F(I) = \int_I f$, where I is any subinterval of E. Let X be any subset of E. Then for any $\epsilon > 0$, there exists $G(\epsilon) \in \mathcal{B}$ with $X \subset G(\epsilon)$ such that whenever there is a finite collection $\{I_1, I_2, \cdots, I_n\}$ of non-overlapping subintervals of E with $I_i \subseteq G(\epsilon) - X$ for each i, we have

$$\sum_{i=1}^{n} \left|F(I_i)\right| < \epsilon \text{ and } \sum_{i=1}^{n} |I_i| < \epsilon.$$

Theorem 1. If f is absolutely Henstock integrable on E, then f is McShane integrable on E.

Proof. The idea is similar to that of Theorem 1 in [1]. Let $\epsilon > 0$ be given. Since f is Henstock integrable, there is $\delta(\xi)$ such that $0 < \delta(\xi) \leq 1$ and $(D)\sum\left|f(\xi)|I| - F(I)\right| < \epsilon$ whenever $D = \{(I, \xi)\}$ is a partial δ-fine division of E, by Henstock's Lemma.

For each integer k, we define the set $X(\epsilon, k) = \{x \in E : (k-1)\epsilon \leq f(x) < k\epsilon\}$ so that for any x, x' in $X(\epsilon, k)$, we have $\left|f(x) - f(x')\right| < \epsilon$.

Now, for each positive integer n greater than or equal to 2, we define the set $X(\epsilon, k, n) = \{x \in X(\epsilon, k) : 1/n < \delta(x) \leq 1/(n-1)\}$.

Next, we divide E into $p(n)$ subintervals $I(n, q)$, where $q = 1, 2, \cdots, p(n)$ such that $\text{diag}I(n, q) \leq 1/n$ where $\text{diag}I$ is the diagonal of I. We denote by $X(\epsilon, k, n, q)$ the set $X(\epsilon, k, n) \cap I^o(n, q)$, where $I^o(n, q)$ is the interior of $I(n, q)$. We note that whenever $\xi \in X(\epsilon, k, n, q)$, we have $I(n, q) \subset B(\xi, \delta(\xi))$.

By Corollary 1, with E replaced by $I(n, q)$, there exists $G(\epsilon, k, n, q)$ such that $X(\epsilon, k, n, q) \subset G(\epsilon, k, n, q) \subset I(n, q)$ and satisfying the property in Corollary 1, with ϵ replaced by $\epsilon/(|k|+1)2^{|k|+n+q}$.

We shall now define $\delta'(\xi)$ for each $\xi \in E$. Note that $\xi \in X(\epsilon, k, n, q)$ or ξ is on the boundary of $I(n, q)$ for some (n, q). Let S be the union of the boundary of all $I(n, q)$. We define δ' on S in such a way that for any δ'-fine partial division $D = \{(I, \xi)\}$, with $\xi \in S$, we have

$$(D)\sum\left|f(\xi)|I|\right| < \epsilon/2 \text{ and } (D)\sum\left|F(I)\right| < \epsilon/2.$$

For $\xi \in X(\epsilon, k, n, q)$, we shall define $\delta'(\xi)$ such that $0 < \delta'(\xi) < \delta(\xi)$ and $B(\xi, \delta'(\xi)) \cap I(n, q) \subseteq G(\epsilon, k, n, q) \subset I(n, q)$.

Suppose we have a δ'-fine McShane division $D' = \{(I, \xi)\}$ of E. If $\xi \notin I$ and $\xi \in X(\epsilon, k, n, q)$, then either $I \cap X(\epsilon, k, n, q) = \phi$ or $I \cap X(\epsilon, k, n, q) \neq \phi$. For the former, $I \subseteq G(\epsilon, k, n, q) - X(\epsilon, l, n, q)$ so that, summing over all such I, $\sum_1\left|F(I)\right| < \epsilon$, $\left|\sum_1 f(\xi)|I|\right| < \epsilon$, by Corollary 1.

For I with $I \cap X(\epsilon, k, n, q) \neq \phi$, there exists $\xi' \in I \cap X(\epsilon, k, n, q)$. Hence we have

$$\xi' \in I \subset B(\xi, \delta'(\xi)) \subset I(n, q) \subset B(\xi', \delta(\xi'))$$

so that (I,ξ') is $\delta(\xi')$-fine and $\xi' \in I$. Summing over all I with $I \cap X(\epsilon,k,n,q) \neq \phi$, we get

$$\begin{aligned}\Big|\sum_2 f(\xi)|I| - F(I)\Big| &\le \Big|\sum_2 (f(\xi)-f(\xi'))|I| + f(\xi')|I| - F(I)\Big| \\ &\le \sum_2 \Big|\,(f(\xi)-f(\xi'))|I|\Big| + \Big|\sum_2 f(\xi')|I| - F(I)\Big|.\end{aligned}$$

The full summation can now be done in the following way:

$$\Big|(D')\sum_{(I,\xi)} f(\xi)|I| - F(I)\Big| \le \Big|\sum_{\xi\in S} f(\xi)|I| - F(I)\Big| + \Big|\sum_{\xi\notin S} f(\xi)|I| - F(I)\Big|.$$

By our choice of δ', the first sum, $\big|\sum\limits_{\xi\in S} f(\xi)|I| - F(I)\big| \le \epsilon$, whereas for the second sum, we get

$$\begin{aligned}\Big|\sum_{\xi\notin S} f(\xi)|I| - F(I)\Big| &\le \Big|\sum_{\xi\in I} f(\xi)|I| - F(I)\Big| + \Big|\sum_{\xi\notin I} f(\xi)|I| - F(I)\Big| \\ \le \Big|\sum_{\xi\in I} f(\xi)|I| - F(I)\Big| &+ \Big|\sum_1 f(\xi)|I| - F(I)\Big| + \Big|\sum_2 f(\xi)|I| - F(I)\Big| \\ &\le \Big|\sum_{\xi\in I} f(\xi)|I| - F(I)\Big| + \Big|\sum_1 f(\xi)|I| - F(I)\Big| \\ &\quad + \Big|\sum_2 (f(\xi)-f(\xi'))|I| + f(\xi')|I| - F(I)\Big| \\ \le \Big|\sum_{\xi\in I} f(\xi)|I| - F(I)\Big| &+ \Big|\sum_2 f(\xi')|I| - F(I)\Big| + \Big|\sum_1 f(\xi)|I| - F(I)\Big| \\ &\quad + \Big|\sum_2 (f(\xi)-f(\xi'))|I|\Big|.\end{aligned}$$

The total of the first two summations is less than ϵ, since $(I,\xi),(I,\xi')$ form a partial δ-fine division, with $\xi \in I, \xi' \in I$. The subsummation $\sum_1$ is less than 2ϵ, by Corollary 1. For the last summation, $\big|\sum_2 (f(\xi)-f(\xi'))|I|\big| \le \sum_2 \epsilon|I| \le \epsilon|E|$ since $\xi,\xi' \in X(\epsilon,k,n,q)$ so that $\big|f(\xi)-f(\xi')\big| < \epsilon$. Hence f is McShane integrable on E. $\square$

References

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