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ON THE MAXIMAL FAMILIES FOR SOME SPECIAL CLASSES OF STRONGLY QUASI–CONTINUOUS FUNCTIONS

Abstract

The maximal families (additive, multiplicative, lattice and with respect to the composition) for some special classes of strongly quasicontinuous functions are investigated.

Let $\mathcal{R}$ be the set of all reals and let μ_e (μ) denote the outer Lebesgue measure (the Lebesgue measure) in $\mathcal{R}$. Denote by

$$d_u(A,x) = \limsup_{h\to 0^+} \mu_e(A \cap (x-h, x+h))/2h$$

$$(d_l(A,x) = \liminf_{h\to 0^+} \mu_e(A \cap (x-h, x+h))/2h$$

the upper (lower) density of a set $A \subset \mathcal{R}$ at a point x. A point $x \in \mathcal{R}$ is called a density point of a set $A \subset \mathcal{R}$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family
$\mathcal{T}_d = \{A \subset \mathcal{R}; A$ is measurable and every point $x \in A$ is a density point of $A\}$ is a topology called the density topology [1]. Denote by $int(A)$ the interior (Euclidean) of the set A. The family

$$\mathcal{T}_{ae} = \{A \in \mathcal{T}_d; \mu(A - int(A)) = 0\}$$

is also a topology [5].

A function f (from $\mathcal{R}$ into $\mathcal{R}$) is called $\mathcal{T}_{ae}-$ continuous ($\mathcal{T}_d-$ continuous or approximately continuous) at a point x if it is continuous at x as the application from $(\mathcal{R}, \mathcal{T}_{ae})$ (from $(\mathcal{R}, \mathcal{T}_d)$) into $(\mathcal{R}, \mathcal{T}_e)$, where $\mathcal{T}_e$ denotes the

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Euclidean topology in $\mathcal{R}$. A function f is $\mathcal{T}_{ae}-$ continuous (everywhere on $\mathcal{R}$) if and only if it is $\mathcal{T}_d-$ continuous (everywhere) and almost everywhere (relative to μ) continuous [5]. A function f is said to be strongly quasi-continuous (in short s.q.c.) at a point x if for every set $A \in \mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in A \cap I$ [2]. If a function f is s.q.c-continuous at every point then we say that f is s.q.c-continuous.

In this paper the main results are some modifications of the results of Z.Grande in [4].

Let $\mathcal{P}(x)$ be a property of a function f at a point x (we will write $f \in \mathcal{P}(x)$) such that:

if f is continuous at x then $f \in \mathcal{P}(x)$;

if $f \in \mathcal{P}(x)$ then $-f \in \mathcal{P}(x)$;

if $f \in \mathcal{P}(x)$ and $g/I = f/I$ for some open interval I containing x then $g \in \mathcal{P}(x)$.

Denote by P the family of all functions f such that for every positive real η and for every point x and for every set $A \in \mathcal{T}_d$ containing x there is an open interval I such that $I \cap A \neq \emptyset$, $|f(t) - f(x)| < \eta$ and $f \in \mathcal{P}(t)$ for all $t \in I \cap A$.

Now, let:

- $C = \{f; f \text{ is continuous }\}$;
- $C_{ae} = \{f; f \text{ is } \mathcal{T}_{ae}- \text{ continuous }\}$;
- $Q_s = \{f; f \text{ is s.q.c.}\}$;
- $Max_{add}(P) = \{f; f + g \in P \text{ for every } g \in P\}$;
- $Max_{mult}(P) = \{f; fg \in P; \text{ for every } g \in P\}$;
- $Max_{max}(P) = \{f; \max(f,g) \in P \text{ for every } g \in P\}$;
- $Max_{min}(P) = \{f; \min(f,g) \in P \text{ for every } g \in P\}$;
- $Max_{comp}(P) = \{f; f \circ g \in P \text{ for every } g \in P\}$.

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Remark 1. *Evidently*

$$C \subset P \cup C_{ae} \subset Q_s.$$

So, every function $f \in P$ is almost everywhere continuous [2, 3].

Remark 2. *The inclusion*

$$Max_{add}(P) \cup Max_{mult}(P) \cup Max_{max}(P) \cup Max_{min}(P) \cup Max_{comp}(P) \subset P$$

is true.

Proof. Since the functions $g_1(t) = 0$, $g_2(t) = 1$ and $g_3(t) = t$ for $t \in \mathcal{R}$ belong to P, for all functions $f_1 \in Max_{add}(P)$, $f_2 \in Max_{mult}(P)$ and $f_3 \in Max_{comp}(P)$ we obtain have $f_1 = f_1 + g_1 \in P$, $f_2 = f_2 g_2 \in P$ and $f_3 = f_3 \circ g_3 \in P$. So,

$$Max_{add}(P) \cup Max_{mult}(P) \cup Max_{comp}(P) \subset P.$$

If a function f is not in P then there are a positive real η, a point x and a set $A \in \mathcal{T}_d$ containing x such for every open interval I with $I \cap A \neq \emptyset$ there is a point $t \in I \cap A$ such that $|f(t) - f(x)| \geq \eta$ or f is not in $\mathcal{P}(t)$. Then the functions $\max(f, f(x) - \eta)$ and $\min(f, f(x) + \eta)$ are not in P. So, f is not in $Max_{max}(P) \cup Max_{min}(P)$ and the proof is completed. □

I. The family $Max_{add}(P)$.

In this part we suppose that the property $\mathcal{P}(x)$ is such that if $f, g \in \mathcal{P}(x)$ then $f + g \in \mathcal{P}(x)$ (then we say that $\mathcal{P}()$ has the additive property).

Theorem 1. *Assume $\mathcal{P}(x)$ has the additive property. Then*

$$C_{ae} \cap P = Max_{add}(P)$$

holds.

Proof. Let $f \in C_{ae} \cap P$ and $g \in P$ be functions. Fix a positive real η, a point x and a set $A \in \mathcal{T}_d$ containing x. Since $f \in C_{ae}$, the point x is a density point of the set $B = int(\{t; |f(t) - f(x)| < \eta/2\})$. Consequently, x is a density point of the set $B \cap A$. Since $g \in P$, there is an open interval $J \subset B$ such that $J \cap A \neq \emptyset$, $|g(t) - g(x)| < \eta/2$ and $g \in \mathcal{P}(t)$ for every $t \in J \cap A$. From the relation $f \in P$ follows that there is an open interval $I \subset J$ such that $I \cap A \neq \emptyset$ and $f \in \mathcal{P}(t)$ for all points $t \in I \cap A$. Consequently, $I \cap A \neq \emptyset$, $f + g \in \mathcal{P}(t)$ and $|(f(t) + g(t)) - (f(x) + g(x))| < \eta/2 + \eta/2 = \eta$ for all points $t \in I \cap A$. So, the function $f \in Max_{add}(P)$ and the inclusion $C_{ae} \cap P \subset Max_{add}(P)$ is proved.

For the proof of the inclusion $Max_{add}(P) \subset C_{ae} \cap P$ fix a function $f \in Max_{add}(P)$. By Remark 1 the function $f \in P$. If f is not in C_{ae} then there are a point $x \in \mathcal{R}$ and a positive number η such that the closure $cl(\{t; |f(t) - f(x)| > \eta\})$ of the set $\{t; |f(t) - f(x)| > \eta\}$ has positive upper density at a point x. We can assume that the closure

$$cl(\{t; f(t) > f(x) + \eta\})$$

has positive upper density at a point x. Since f belonging to $P \subset Q_s$ is almost everywhere continuous [2, 3], we obtain

$$\mu(cl(\{t; f(t) > f(x) + \eta\}) \setminus \{t; f(t) \geq f(x) + \eta\}) = 0$$

and consequently,

$$d_u(int(\{t; f(t) > f(x) + \eta/2\}), x) > 0.$$

Thus there is a sequence of disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; f(t) > f(x) + \eta/2\}, n = 1, 2, \ldots$, such that:

(1) x is not in I_n for $n = 1, 2, \ldots$;

(2) f is continuous at all points $a_n, b_n, n = 1, 2, \ldots$;

(3) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;

(4) $d_u(\bigcup_n I_n, x) > 0$.

Put

$$g(t) = \{ \begin{array}{ccc} -f(x) + \eta/2 & if & (t = x) \vee (t \in I_n, n = 1, 2, \ldots) \\ -f(t) & otherwise. & \end{array}$$

Fix a positive real η, a point t and a set $A \in \mathcal{T}_d$ containing t. For a positive integer n and a point $t = a_n$ or $t = b_n$ the function g is unilaterally continuous at t and g/I_n is constant. So, there is an open interval $I \subset I_n$ with $I \cap A \neq \emptyset$. Evidently, $g \in \mathcal{P}(u)$ and $|g(u) - g(x)| = 0 < \eta$ for each point $u \in I \cap A$. If $t \in int(I_n)$ for some positive integer n we proceed the same as above. If $t \neq x$ and t is not in I_n for $n = 1, 2, \ldots$ then there is an open interval I with $I \cap I_n \neq \emptyset$ for $n = 1, 2, \ldots$, $I \cap A \neq \emptyset$ and such that $|f(u) - f(t| < \eta$ and $f \in \mathcal{P}(u)$ for $u \in I \cap A$. Since $g/I = -f/I$, we obtain $|g(u) - g(t)| = |f(u) - f(t)| < \eta$ and $g \in \mathcal{P}(u)$ for all points $u \in I \cap A$. If $t = x$ then, by (4), there is a positive integer n with $A \cap int(I_n) \neq \emptyset$. Since $g(u) = -f(x) + \eta/2$ for $u = x$ and for $u \in int(I_n)$, we have $g \in \mathcal{P}(u)$ and $|g(u) - g(t)| = 0 < \eta$ for $u \in A \cap \int(I_n)$. So, $g \in P$. Moreover, $f(x) + g(x) = \eta/2$, $f(t) + g(t) \geq \eta$ for $t \in I_n$, $n = 1, 2, \ldots$ and $f(t) + g(t) = 0$ otherwise on $\mathcal{R}$. So, $f + g$ is not in P and consequently f is not in $Max_{add}(P)$. This contradiction finishes the proof. □

II.The families $Max_{max}(P)$ and $Max_{min}(P)$.

In this part we suppose about the property $\mathcal{P}(x)$ that if $f, g \in \mathcal{P}(x)$ then also $max(f, g), min(f, g) \in \mathcal{P}(x)$ (then we say that $\mathcal{P}()$ has the lattice property).

Theorem 2. *Let $\mathcal{P}(x)$ has the lattice property. Then*

$$Max_{max}(P) = Max_{min}(P) = C_{ae} \cap P$$

holds.

PROOF. For the proof of the inclusion

$$C_{ae} \cap P \subset Max_{max}(P) \cap Max_{min}(P).$$

we take a function $f \in C_{ae} \cap P$ and a function $g \in P$. Fix a positive real η, a point x and a set $A \in \mathcal{T}_d$ containing x. Let $h = \max(f, g)$. Consider the following cases:

(1)$f(x) > g(x)$**.** Then let $r = f(x) - g(x)$ and let $s = \min(r/2, \eta)$. Since $f \in C_{ae}$, x is a density point of the set $B = int(\{t; |f(t) - f(x)| < s\})$. From the relation $g \in P$ follows that there is an open interval $J \subset B$ such that $J \cap A \neq \emptyset$, $g \in \mathcal{P}(t)$ and $|g(t) - g(x)| < s$ for all points $t \in J \cap A$. Since $f \in P$, there is an open interval $I \subset J$ with $I \cap A \neq \emptyset$ and $f \in \mathcal{P}(t)$ for all points $t \in I \cap A$. Observe that for $u \in I \cap A$ we have

$$f(u) > f(x) - s \geq g(x) + 2s - s = g(x) + s > g(u),$$

whence $h(u) = f(u)$. Moreover, $h(x) = f(x)$, $h \in \mathcal{P}(u)$ and

$$|h(u) - h(x)| = |f(u) - f(x)| < s \leq \eta$$

for all point $u \in I \cap A$.

(2)$f(x) < g(x)$**.** In this case the proof is analogous as above.

(3)$f(x) = g(x)$**.** In this case we put $s = \eta$ and we find an open interval as above. Then $I \cap A \neq \emptyset$ and for $u \in I \cap A$ we obtain $h \in \mathcal{P}(u)$ and

$$|h(u) - h(x)| \leq \max(|f(u) - f(x)|, |g(u) - g(x)|) < s = \eta.$$

So, $h = \max(f, g) \in P$. The prof that $\min(f, g) \in P$ is analogous.

Since by Remark 1 the inclusion $Max_{max}(P) \cup Max_{min}(P) \subset P$ is true, we will show the inclusion $Max_{max}(P) \cup Max_{min}(P) \subset C_{ae}$. We will show only that $Max_{max}(P) \subset C_{ae}$, because the proof of the inclusion $Max_{min}(P) \subset C_{ae}$ is similar. Let $f \in Max_{max}(P)$ be a function. By Remark 1 the function $f \in P$. If f is not in C_{ae} then there are a point x and a positive number η such that

$$d_u(cl(\{t; |f(t) - f(x)| > \eta\}), x) > 0.$$

If

$$d_u(cl(\{t; f(t) > f(x) + \eta\}), x) > 0$$

then the same as in the proof of Theorem 1 there are disjoint closed intervals

$$I_n = [a_n, b_n] \subset \{t; f(t) > f(x) + \eta/2\},$$

such that conditions (1) – (4) from the proof of Theorem 1 are satisfied. Let

$$g(t) = \begin{cases} f(x) - \eta & if \quad (t = x) \vee (t \in I_n, n = 1, 2, \dots,) \\ f(x) + \eta & otherwise. \end{cases}$$

Analogously as in the proof of Theorem 1 we can show that $g \in P$. Moreover, $\max(f(x), g(x)) = f(x)$ and $\max(f(t), g(t)) \geq f(x) + \eta/2$ for $t \neq x$. So, $\max(f, g)$ is not in P and consequently, f is not in $Max_{max}(P)$. Now consider the case where

$$d_u(cl(\{t; f(t) < f(x) - \eta\}), x) > 0$$

. Then there are disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; f(t) < f(x) - \eta/2\}$, $n = 1, 2, \dots$, which satisfy conditions (1)–(4) from the proof of Theorem 1. Let the function g be defined the same as above. Then $g \in P$, $\max(f(x), g(x)) = f(x)$, $\max(f(t), g(t)) \leq f(x) - \eta/2$ for $t \in I_n, n = 1, 2, \dots$, and $\max(f(t), g(t)) \geq f(x) + \eta$ otherwise on $\mathcal{R}$. So,in this case also $\max(f, g)$ is not in P, and consequently f is not in $Max_{max}(P)$. This contradiction finishes the proof. □

III. The family $Max_{comp}(P)$**.**

In this part we suppose that for every continuous function g and for every function $f \in \mathcal{P}(x)$ we have $g \circ f \in \mathcal{P}(x)$; $\mathcal{P}()$ is invariant with respect to composition with continuous function.

Theorem 3. *Assume $\mathcal{P}(x)$ is invariant with respect to composition with continuous function. Then*

$$Max_{comp}(P) = C$$

holds.

PROOF. Let g be a continuous function and let $f \in P$ be a function. Fix a positive real η, a point x and a set $A \in \mathcal{T}_d$ containing x. Since g is continuous at $f(x)$, there is a positive real r such that if $|u - f(x)| < r$ then $|g(u) - g(f(x))| < \eta$. From the relation $f \in P$ follows that there is an open interval I such that $I \cap A \neq \emptyset$, $f \in \mathcal{P}(t)$ and $|f(t) - f(x)| < r$ for all points $t \in I \cap A$. Observe that for every point $t \in I \cap A$ we obtain $g \circ f \in \mathcal{P}(t)$ and $|g(f(t)) - g(f(x))| < \eta$. So, $g \circ f \in P$ and consequently $C \subset Max_{comp}(P)$.

Suppose that a function f is not continuous at a point y. Then there is a sequence of points $y_n \neq y, n = 1, 2, \dots,$ such that $\lim_{n \to \infty} y_n = y$ and $\lim_{n \to \infty} f(y_n) \neq f(y)$. Let $I_n = [a_n, b_n], n = 1, 2, \dots,$ be disjoint closed intervals such that

– $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$;

– $a_n b_n > 0$ for $n = 1, 2, \ldots,$;

– $d_u(\bigcup_n I_n, 0) > 0$.

Put

$$g(x) = \begin{cases} y_n & if \quad x \in I_n, n = 1, 2, \ldots, \\ y & if \quad x = 0 \\ y_1 & otherwise. \end{cases}$$

Fix a positive real η, a point x and a set $A \in \mathcal{T}_d$ containing x. If $x \neq 0$ then g is unilaterally continuous and consequently, there is an open interval I such that $I \cap A \neq \emptyset$, g is continuous at every point $t \in I$ and $g(t) = g(x)$ for each point $t \in I$. If $x = 0$ then there is a positive integer n such that $|y_n - y| < \eta$ and $I_n \cap A \neq \emptyset$. Consequently, there is an open interval $I \subset I_n$ with $I \cap A \neq \emptyset$. Observe that the reduced function g/I is continuous and $|g(u) - g(x)| = |y_n - y| < \eta$ for $u \in I$. This shows that $g \in P$. But $f \circ g$ is not in P, since $f \circ g$ is not s.q.c. at $x = 0$. So, $Max_{comp}(P) \subset C$, and the proof is completed. □

IV. The family $Max_{mult}(P)$**.**

In this part we suppose about the property $\mathcal{P}(x)$ that:

– if $f, g \in \mathcal{P}(x)$ then $fg \in \mathcal{P}(x)$;

– if $f \in \mathcal{P}(x)$ and I is an open interval such that 0 is not in $f(I)$ then the function

$$g(t) = \begin{cases} 1/f(t) & for \quad t \in I \\ 0 & otherwise. \end{cases}$$

belongs to P.

Remark 3. *If a function $f \in P$ is not $\mathcal{T}_{ae}-$ continuous at a point $x \in \mathcal{R}$ at which $f(x) \neq 0$ then there is a function $g \in P$ such that the product fg is not in P.*

PROOF. The same as in the proof of Theorem 1 we prove that there exist a positive real η and disjoint closed intervals $I_n = [a_n, b_n] \subset \{t; |f(t) - f(x)| > \eta/2\}$ which satisfy conditions (1)–(4) from the proof of Theorem 1. Put

$$g(t) = \begin{cases} 1 & if \quad (t = x) \vee (t \in I_n, n = 1, 2, \ldots,) \\ 0 & otherwise. \end{cases}$$

Observe that $g \in P$. Since $f(x)g(x) = f(x) \neq 0$ and for every point $t \neq x$ we have $f(t)g(t) = 0$ or $|f(t)g(t) - f(x)g(x)| = |f(t) - f(x)| > \eta$, the function fg is not s.q.c. at x, so fg is not in P. This completes the proof. □

Remark 4. *Let $f \in P$ be a function and let $x \in \mathcal{R}$ be a point such that $f(x) = 0$. If $d_u(\{t; f(t) = 0\}, x) > 0$ then for every function $g \in P$, for every positive real η and for every set $A \in \mathcal{T}_d$ containing x there is an open interval I such that $I \cap A \neq \emptyset$, the product $fg \in \mathcal{P}(t)$ and $|f(t)g(t)| < \eta$ for each point $t \in I \cap A$.*

Proof. Fix a function $g \in P$, a positive real η and a set $A \in \mathcal{T}_d$ containing x. The functions $f, g \in P$, so they are almost everywhere continuous. Observe that the set $B = \{t; t \in A, f(t) = 0$ and f is continuous at $t\}$ is of positive measure. There are a nonempty set $D \subset B$ belonging to $\mathcal{T}_d$ and a point $u \in D$ such that $f(u) = 0$ and the function g is continuous at u. Let J be an open interval containing u such that there is a positive real K with $|g(t)| < K$ for all points $t \in J$. Evidently, $u \in J \cap A \in \mathcal{T}_d$. Since $f \in P$ and $f(u) = 0$, there is an open interval $I_1 \subset J$ such that $I_1 \cap A \neq \emptyset$, $f \in \mathcal{P}(t)$, and $|f(t)| < \eta/K$ for all points $t \in I_1 \cap A$. But $g \in P$ and $\emptyset \neq I_1 \cap A \in \mathcal{T}_d$, so there is an open interval $I \subset I_1$ such that $I \cap A \neq \emptyset$ and $g \in \mathcal{P}(t)$ for each point $t \in I \cap A$. For $t \in I \cap A$ we have $fg \in \mathcal{P}(t)$ and $|f(t)g(t) - f(x)g(x)| = |f(t)g(t)| < (\eta/K)K = \eta$. This completes the proof. □

In the proof next Remark 4 we will apply the following Lemma which is proved in [4] :

Lemma 1. *Let $A \subset \mathcal{R}$ be a closed set and let $x \in A$ be a point such that $d_u(A, x) = 0$. Then there is a sequence of disjoint closed intervals $I_n = [a_n, b_n] \subset (x - 2, x + 2)$, $n = 1, 2, \ldots$, such that:*

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;
- $d_u(\bigcup_n I_n, x) = 0$;
- $(A \setminus \{x\}) \cap [x - 1, x + 1] \subset \bigcup_n int(I_n)$.

Remark 5. *Suppose that a function $f \in P$ is not $\mathcal{T}_{ae}-$ continuous at a point x at which $f(x) = 0$. If*

$$d_u(\{t; f(t) = 0\}, x) = 0$$

then there is a function $g \in P$ such that the product fg is not in P.

Proof. Since f is almost everywhere continuous, we obtain

$$\mu(cl(\{t; f(t) = 0\} \setminus \{t; f(t) = 0\}) = 0$$

and

$$d_u(cl(\{t; f(t) = 0\}), x) = 0.$$

By Lemma 1 there are disjoint closed intervals $I_n = [a_n, b_n] \subset (x - 2, x + 2) \setminus \{x\}$, $n = 1, 2, \ldots$, such that

- $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$;
- $[x-1, x+1] \cap cl(\{t; f(t) = 0\}) \setminus \{x\} \subset \bigcup_n int(I_n)$;
- $d_u(\bigcup_n I_n, x) = 0$.

Since the function f is not $\mathcal{T}_{ae}-$ continuous at x, there are a positive real η and disjoint closed intervals $J_n = [c_n, d_n] \subset (\{t; |f(t)| \geq \eta/2\} \cap (x-1, x+1)) \setminus \bigcup_k I_k$ such that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = x$ and $d_u(\bigcup_n J_n, x) > 0$. Moreover, we can assume that f is continuous at all points a_n, b_n, c_n, d_n, $n = 1, 2, \ldots$.

Put

$$g(t) = \begin{cases} \eta & if \quad (t = x) \vee (t \in J_n, n \geq 1) \\ 1 & if \quad (t \leq x-1) \vee (t \geq x+1) \vee (t \in I_n, n \geq 1) \\ 1/f(t) & otherwise. \end{cases}$$

By the methods used above we can show that the function $g \in P$. But the product fg is not s.q.c. at x, since $f(x)g(x) = 0$, $f(t)g(t) = 1$ for $t \in (x-2, x+2) \setminus \bigcup_n (I_n \cup J_n) \setminus \{x\}$, $|f(t)g(t)| \geq \eta^2/2$ for $t \in J_n$, $n \geq 1$ and $d_u(\bigcup_n I_n, x) = 0$. So, the product fg is not in P and the proof is finished. □

Remark 6. *If a function $f \in P$ is $\mathcal{T}_{ae}-$ continuous at a point x then for all functions $g \in P$, for every set $A \in \mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I \cap A \neq \emptyset$, $fg \in \mathcal{P}(t)$ and $|f(t)g(t) - f(x)g(x)| < \eta$ for all points $t \in I \cap A$.*

PROOF. Fix a positive real η, and a set $A \in T_d$ such that $x \in A$.Since f is $\mathcal{T}_{ae}$-continuous at x, so x is a density point of the set

$$B = int(\{t; |f(t) - f(x)| < (\eta/2)(1/|c| + 1)\}),$$

where $c = g(x)$. Consequently, x is a density point of the set $B \cap A$. Since $f \in P$, there is an open interval $I \subset B$ such that $I \cap A \neq \emptyset$ and $f \in \mathcal{P}(t)$ for all points $t \in I \cap A$. Let $g \in P$ be any function. Since $g \in P$, there is an open interval $J \subset I$ such that $J \cap A \neq \emptyset$, $|g(t) - g(x)| < (\eta/2)(1/|f(t)| + 1)$ and $g \in \mathcal{P}(t)$ for all $t \in J \cap A$. Consequently we obtain $fg \in \mathcal{P}(t)$ and

$$|f(t)g(t) - f(x)g(x)| \leq |f(t)||g(t) - g(x)| + |g(x)||f(t) - g(t)|$$

$$< |f(t)|(\eta/2)(1/|f(t)| + 1) + |g(x)|(\eta/2)(1/|g(x)| + 1) < \eta$$

for all $t \in J \cap A$. So, $fg \in P$ and the proof is completed. □

From Remarks 1 – 6 it follows immediately:

Theorem 4. *A function $f \in Max_{mult}(P)$ if and only if it is in P and satisfies the following condition:*

(F) if f is not $\mathcal{T}_{ae}-$ continuous at a point x then $f(x) = 0$ and $d_u(\{t; f(t) = 0\}, x) > 0$.

Remark 7. *If the property $\mathcal{P}(x)$ denotes that $f(x) \in \mathcal{R}$ then all above results are true for $P = Q_s$ (see [4]).*

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