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F. S. Cater, Department of Mathematics, Portland State University, Portland, Oregon 97207, USA

ON MONOTONIC AND ANALYTIC FUNCTIONS IN C^∞

Abstract

We generalize the theorem of Bernstein that any infinitely many times differentiable function on an interval, I, that is regularly monotonic on I must be a real analytic function on I.

Let f be a function in C^∞ (that is, f has derivatives of all orders) on the interval $(-d,d)$. S. Bernstein in [1] proved a classic result.

Theorem B. *For each n let $f^{(n)}$ not change sign on $(-d,d)$. Then f is a real analytic function on $(-d,d)$.*

For an easier proof of Theorem B consult [3]. Unfortunately many analytic functions on $(-d,d)$ do not satisfy the hypothesis of Theorem B. Consider, for example, the elementary functions $\sin x$ and $\cos x$ on $(-4,4)$. We will provide a variation on Theorem B whose hypothesis is satisfied by a wider class of functions including most of the elementary functions on all the interiors of compact intervals on which they are analytic. We offer:

Theorem I. *Let (c_n) be a sequence of real numbers such that the sequence $\left(\frac{c_n d^n}{n!}\right)$ is bounded and the functions $f^{(n)}-c_n$ do not change sign on the interval $(-d,d)$. Then for any $x\in(-d,d)$ we have*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n\,.$$

It seems to be difficult to construct a convergent power series on $(-1,1)$ whose sum has bounded derivatives of all orders and fails to satisfy the hypothesis of Theorem I for all appropriate sequences (c_n). Consult the problems at the end of this paper.

We also offer:

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Theorem II. *Let* (c_n) *be a sequence of real numbers such that the sequence* $\left(\frac{c_n d^n}{n!}\right)$ *is bounded and the functions* $f^{(n)} - c_n$ *do not change sign on* $(-d, 0)$ *or on* $(0, d)$*. Then for any* $u \in \left(-\frac{1}{2}d, \frac{1}{2}d\right)$ *and* $x \in \left(u - \frac{1}{2}d, u + \frac{1}{2}d\right)$*, we have*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(u)}{n!}(x-u)^n .$$

Functions in $C^\infty(a,b)$ satisfying the hypotheses of Theorem B on an interval (a,b) are called *regularly monotonic* on (a,b). We modify this definition as follows:

Definition. We say that $f \in C^\infty(a,b)$ is a *generalized regularly monotonic* function on (a,b) if at each $x \in (a,b)$, there exist a positive number d and a sequence of numbers (c_n), depending on x, such that the sequence $\left(\frac{c_n d^n}{n!}\right)$ is bounded and for any n the function $f^{(n)} - c_n$ does not change sign on $(x-d, x)$ or on $(x, x+d)$.

Theorem III. *If* f *is a generalized regularly monotonic function on* (a,b)*, then* f *is a real analytic function on* (a,b)*.*

This follows from Theorem II.

Theorem IV. *Let* $f \in C^\infty(\mathbb{R})$ *and let* f *satisfy the hypothesis of* Theorem II*. Let* $f(x+2d) = f(x)$ *for all* x*. Then* f *is a real analytic function on* $\mathbb{R}$*, and the interval of convergence of the Taylor series of* f *at any point in* $\mathbb{R}$ *has length* $\geq d$*.*

This also follows from Theorem II.

Until further notice, let the hypothesis of Theorem I be satisfied. We classify the indices $n \geq 0$ as follows. We say that n is a *glide* index if $f^{(n)} - c_n$ and $f^{(n+1)} - c_{n+1}$ have the same sign. We say that n is a *jump* index if $f^{(n)} - c_n$ and $f^{(n+1)} - c_{n+1}$ have opposite sign. (Here we discard the possibility that $f^{(n)} - c_n$ is identically zero for some index n; for then $f(x)$ would equal a polynomial in x on $(-d, d)$.)

The plan is to prove Theorem I under various restrictions until all cases are covered. We begin with:

Lemma 1. *Let all but finitely many indices* n *be glide indices. Then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \quad \textit{for all } x \in (-d, d).$$

PROOF. Let $d_o < 0$. It suffices to prove the conclusion on the interval $(-d_o, d_o)$ because d_o is arbitrary. Note that the series $\sum \frac{|c_n| d_o^n}{n!}$ converges because $\left(\frac{|c_n| d^n}{n!}\right)$ is bounded. Let N be an index such that $n > N$ implies that n is a glide index and $\frac{|c_n| d_o^n}{n!} < 1$. Without loss of generality, we assume that $f^{(n)} - c_n \geq 0$ for $n > N$. (The proof for the opposite inequality is analogous.)

For each N and $x \in (0, d_o)$, put

$$p_n(x) = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j - \sum_{j=0}^{n} \frac{c_j}{j!} x^j .$$

It follows that $p_{N+1}(x) \leq p_{N+2}(x) \leq p_{N+3}(x) \leq \ldots$. By Taylor's Theorem,

$$R_n(x) = f(x) - \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j = \frac{f^{(n+1)}(v)}{(n+1)!} x^{n+1}$$

for some $v \in (0, x)$. It follows that

$$R_n(x) \geq \frac{c_{n+1}}{(n+1)!} x^{n+1} \quad \text{and} \quad R_n(x) \geq -\frac{|c_{n+1} d_o^{n+1}|}{(n+1)!} > -1 .$$

Hence

$$p_n(x) + \sum_{j=0}^{n} \frac{c_j}{j!} x^j = \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j = f(x) - R_n(x) \leq f(x) + 1 .$$

From the fact that $\sum_{j=0}^{\infty} \frac{c_j}{j!} x^j$ converges, we deduce that $\big(p_n(x)\big)_n$ is a non-decreasing sequence bounded above. Hence $\big(p_n(x)\big)_n$ converges and likewise $\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$ converges for $x \in (0, d_o)$. Clearly $\sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$ converges for $x \in (-d_o, d_o)$. Put $g(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$. Then g is a real analytic function on $(-d_o, d_o)$.

It remains to prove that $f(x) = g(x)$ for $x \in (-d_o, d_o)$. Select $u > 0$ such that $2u < d_o$. For each $n > N$, let $v_n \in (u - d_o, d_o - u)$. Put $h_n(x) = f^{(n)}(x) - c_{n+1} x$ for each $n > N$. Then h_n' does not change sign on $(-d, d)$. It follows that $h_n(v_n)$ lies between $h_n(u - d_o)$ and $h_n(d_o - u)$, and therefore

$$\big|h_n(v_n)\big| \leq \big|h_n(u - d_o)\big| + \big|h_n(d_o - u)\big| .$$

Hence

$$\big|f^{(n)}(v_n) - c_{n+1} v_n\big| \leq \big|f^{(n)}(u - d_o) - c_{n+1}(u - d_o)\big| + \big|f^{(n)}(d_o - u) - c_{n+1}(d_o - u)\big|$$

and

$$\left|f^{(n)}(v_n)\right| \le \left|f^{(n)}(u-d_o)\right| + \left|f^{(n)}(d_o-u)\right| + 2\left|c_{n+1}(u-d_o)\right| + \left|c_{n+1}v_n\right|.$$

We multiply by $\frac{u^n}{n!}$ to obtain

$$\left|\frac{f^{(n)}(v_n)u^n}{n!}\right| \le \frac{|f^{(n)}(u-d_o)|u^n}{n!} + \frac{|f^{(n)}(d_o-u)|u^n}{n!} + \frac{3|c_{n+1}|du^n}{n!}.$$

But $\sum \frac{f^{(n)}(u-d_o)|u^n}{n!}$ and $\sum \frac{|f^{(n)}(d_o-u)|u^n}{n!}$ converge by the same argument as in the preceding paragraph. From the hypothesis and from $u < d$ we deduce that $\sum \frac{c_{n+1}u^n}{n!}$ converges. Finally,

$$\frac{c_{n+1}u^n}{n!} \to 0, \quad \frac{f^{(n)}(d_o-u)u^n}{n!} \to 0 \quad \text{and} \quad \frac{f^{(n)}(u-d_o)u^n}{n!} \to 0.$$

It follows that $\frac{f^{(n)}(v_n)u^n}{n!} \to 0$. But $\frac{f^{(n)}(v_n)u^n}{n!}$ has the form of the remainder R_n in Taylor's Theorem,

$$f(t) = \sum_{j=1}^{n-1} \frac{f^{(j)}(t-u)u^n}{n!} + R_n$$

for any $t \in (2u-d_o, d_o-u)$. Thus f is analytic at each point in $(2u-d_o, d_o-u)$. But $u > 0$ is arbitrary, so f is analytic on $(-d_o, d_o)$. Moreover, f equals the analytic function g on some neighborhood of 0, so $f(x) = g(x)$ for $x \in (-d_o, d_o)$.

The conclusion follows from the fact that $d_o < d$ was arbitrary. □

Next we see that jump index can replace glide index in Lemma 1.

Lemma 2. *Let all but finitely many indices n be jump indices. Then*

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!}x^j \quad \textit{for all } x \in (-d,d).$$

Proof. Put $g(x) = f(-x)$ for $x \in (-d,d)$. Then $g^{(n)}(x) = (-1)^n f^{(n)}(-x)$ for all n and all $x \in (-d,d)$. It follows that each jump index for g is a glide index for f, and each glide index for g is a jump index for f. Thus all but finitely many indices are glide indices for g. By Lemma 1,

$$g(x) = \sum_{j=1}^{\infty} \frac{g^{(j)}}{j!}x^j \quad \text{for } x \in (-d,d).$$

Finally, for $x \in (-d, d)$,

$$f(x) = g(-x) = \sum_{j=0}^{\infty} \frac{g^{(j)}(0)}{j!}(-x)^j = \sum_{j=0}^{\infty} \frac{(-1)^j g^{(j)}(0)}{j!} x^j = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j \,.$$

□

PROOF. [Proof of Theorem I] In view of Lemmas 1 and 2 we can assume, without loss of generality, that there are infinitely many jump indices and infinitely many glide indices. Fix an index N that exceeds at least one jump index and exceeds at least one glide index. Fix $u \in (0, d)$. We write

$$f(u) = \sum_{j=0}^{n} \frac{f^{(j)}(0)u^j}{j!} + \frac{f^{(n+1)}(t_n)u^{n+1}}{(n+1)!} \,,$$

$$f(-u) = \sum_{j=0}^{n} \frac{f^{(j)}(0)(-u)^j}{j!} + \frac{f^{(n+1)}(s_n)(-u)^{n+1}}{(n+1)!} \,,$$

for each index n, where t_n is some point in $(0, u)$ and s_n is some point in $(-u, 0)$. Put

$$E_n(u) = \sum_{j=0}^{n} \frac{f^{(j)}(0)u^j}{j!} \,, \quad E_n(-u) = \sum_{j=0}^{n} \frac{f^{(j)}(0)(-u)^j}{j!} \,,$$

$$R_n(u) = \frac{f^{(n+1)}(t_n)u^{n+1}}{(n+1)!} \,, \quad R_n(-u) = \frac{f^{(n+1)}(s_n)(-u)^{n+1}}{(n+1)!} \,.$$

Thus $f(u) = E_n(u) + R_n(u)$ and $f(-u) = E_n(-u) + R_n(-u)$ for each index n.

Suppose that m is a jump index and $m+1, m+2, \ldots, m+v$ are glide indices. Then

$$E_{m-1}(u) - f(u) + \frac{c_m u^m}{m!} = -\frac{f^{(m)}(t_m)u^m}{m!} + \frac{c_m u^m}{m!}$$

has the same sign as

$$E_{m+1}(u) - E_m(u) - \frac{c_{m+1}u^{m+1}}{(m+1)!} = \frac{f^{(m+1)}(0)u^{m+1}}{(m+1)!} - \frac{c_{m+1}u^{m+1}}{(m+1)!} \,,$$

and likewise the same sign as

$$E_{m+2}(u) - E_{m+1}(u) - \frac{c_{m+2}u^{m+2}}{(m+2)!} \,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_{m+v}(u) - E_{m+v-1}(u) - \frac{c_{m+v}u^{m+v}}{(m+v)!}$$

and

$$f(u) - E_{m+v}(u) - \frac{c_{m+v+1}u^{m+v+1}}{(m+v+1)!} =$$

$$= \frac{f^{(m+v+1)}(t_{m+v+1})u^{m+v+1}}{(m+v+1)!} - \frac{c_{m+v+1}u^{m+v+1}}{(m+v+1)!} .$$

The sum of these terms is

$$E_{m-1}(u) - E_m(u) + \frac{c_m u^m}{m!} - \sum_{j=m+1}^{m+v+1} \frac{c_j u^j}{j!} .$$

The absolute value of the sum of terms of the same sign is at least as large as the absolute value of any one of the summands, so

$$\begin{aligned} &\Big| E_{m-1}(u) - E_m(u) + \frac{c_m u^m}{m!} - \sum_{j=m+1}^{m+v+1} \frac{c_j u^j}{j!} \Big| \geq \\ &\geq \Big| E_{m+v}(u) - E_{m+v-1}(u) - \frac{c_{m+v}u^{m+v}}{(m+v)!} \Big| . \end{aligned} \tag{1}$$

Now

$$E_{m+1}(u) - E_m(u) = -\frac{f^{(m)}(0)u^m}{m!}$$

and

$$E_{m+v}(u) - E_{m+v-1}(u) = \frac{f^{(m+v)}(0)u^{m+v}}{(m+v)!} .$$

From (1) we obtain

$$\Big| \frac{f^{(m)}(0)u^m}{m!} \Big| + 2 \sum_{j=m}^{m+v+1} \frac{|c_j| u^j}{j!} \geq \Big| \frac{f^{(m+v)}(0)u^{m+v}}{(m+v)!} \Big| . \tag{2}$$

It follows that for any glide index $k > N$, there is a jump index $m < k$ such that

$$\Big| \frac{f^{(m)}(0)u^m}{m!} \Big| + 2 \sum_{j=m}^{k+1} \frac{|c_j| u^j}{j!} \geq \Big| \frac{f^{(k)}(0)u^k}{k!} \Big| . \tag{3}$$

Note that $(-u)^j u^j$ is positive for j even and negative for j odd. It follows that the roles of glide and jump index reverse in the preceding paragraph when $-u$

replaces u, $(-1)^n c_n$ replaces c_n and s_n replaces t_n. Thus for any jump index $p > N$, there is a glide index $n < p$ with

$$\Big|\frac{f^{(n)}(0)(-u)^n}{n!}\Big| + 2\sum_{j=n}^{p+1} \frac{|(-1)^j c_j|(-u)^j}{j!} \geq \Big|\frac{f^{(p)}(0)(-u)^p}{p!}\Big|. \tag{4}$$

We obtain from (3) and (4) that for any index $k > N$ there is an index $m < k$ such that inequality (3) holds.

We deduce from $|u| < d$ and from the hypothesis that $\sum_{j=0}^{\infty} \frac{|c_j|u^j}{j!} < \infty$. We conclude from (3) and (4) that for any index $k > N$ there is an index $q \leq N$ such that

$$\Big|\frac{f^{(q)}(0)u^q}{q!}\Big| + 4\sum_{j=0}^{\infty} \frac{|c_j|u^j}{j!} \geq \Big|\frac{f^{(k)}(0)u^k}{k!}\Big|. \tag{5}$$

Consequently the sequence $\left(\left|\frac{f^{(k)}(0)u^k}{k!}\right|\right)_k$ is bounded. Now $\left(\left|\frac{f^{(k)}(0)u_o^k}{k!}\right|\right)_k$ is also bounded for $u < u_o < d$, so indeed $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)u^k}{k!}$ converges. It suffices to prove that it converges to $f(u)$ for $u \in (-d, d)$.

If $u \in (0, d)$ and n is a jump index, then

$$R_n(u) - \frac{c_n u^n}{n!} = \frac{f^{(n)}(t_n)u^n}{n!} - \frac{c_n u^n}{n!}$$

and

$$R_{n+1}(u) - \frac{c_{n+1}u^{n+1}}{(n+1)!} = \frac{f^{(n+1)}(t_{n+1})u^{n+1}}{(n+1)!} - \frac{c_{n+1}u^{n+1}}{(n+1)!}$$

have opposite sign. Because there are infinitely many jump indices, it follows that $R_n(u) \to 0$ and $f(u) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)u^j}{j!}$. On the other hand, if n is a glide index, then

$$R_n(-u) - \frac{c_n(-u)^n}{n!} \quad \text{and} \quad R_{n+1}(-u) - \frac{c_{n+1}(-u)^{n+1}}{(n+1)!}$$

have opposite sign. Because there are infinitely many glide indices, it follows that

$$R_n(-u) \to 0 \quad \text{and} \quad f(-u) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)(-u)^j}{j!}.$$

This gives the desired result for $u \in (-d, d)$. □

Before tackling Theorem II we need a nuts and bolts type lemma.

Lemma 3. *Let g be a twice differentiable function on $[-r, r]$ such that g' and g'' do not change sign on interval $(-r, 0)$ or on interval $(0, r)$. Let s be a number such that $0 < s < 1$. Then*

$$(1-s)\big|g(0)\big| \leq \big|g(r)\big| + \big|g(-r)\big| + \big|g(sr)\big| .$$

PROOF. The argument is divided into several cases.
CASE 1. $g' \geq 0$ on $(-r, 0)$ and $g' \geq 0$ on $(0, r)$. Here g is nondecreasing on $(-r, r)$ and hence $(1-s)|g(0)| \leq |g(0)| \leq |g(-r)| + |g(r)|$.
CASE 2. $g' \leq 0$ on $(-r, 0)$, $g' \leq 0$ on $(0, r)$. Apply Case 1 to $-g$.
CASE 3. $g' \geq 0$ on $(-r, 0)$, $g' \leq 0$ on $(0, r)$, $g'' \leq 0$ on $(-r, 0)$ and on $(0, r)$, and $g(0) < 0$. Here $g < 0$ on $(-r, r)$ and $g(0)$ is the maximum value of g. Hence $(1-s)|g(0)| \leq |g(0)| \leq |g(r)|$.
CASE 4. $g' \leq 0$ on $(-r, 0)$, $g' \geq 0$ on $(0, r)$, $g'' \geq 0$ on $(-r, 0)$ and on $(0, r)$, and $g(0) > 0$. Apply Case 3 to $-g$.
CASE 5. $g' \geq 0$ on $(-r, 0)$, $g' \leq 0$ on $(0, r)$, $g'' \leq 0$ on $(-r, 0)$ and on $(r, 0)$, and $g(0) > 0$. Here $g(0)$ is the maximum value of g on $(-r, r)$. Moreover, g is concave down, so $(1-s)g(0) + sg(r) \leq g(sr)$ and $(1-s)|g(0)| \leq |g(sr)| + s|g(r)| \leq |g(sr)| + |g(r)|$.
CASE 6. $g' \leq 0$ on $(-r, 0)$, $g' \geq 0$ on $(0, r)$, $g'' \geq 0$ on $(-r, 0)$ and on $(0, r)$, and $g(0) < 0$. Apply Case 5 to $-g$.

If $g(0) = 0$, there is nothing to prove. We have covered all possibilities. □

PROOF. [Proof of Theorem II] Let s be a number such that $0 < s < 1$. We divide our argument into cases and find an inequality in each case. Fix an index n.
CASE 1. $f^{(n+1)} - c_{n+1}$ does not change sign on $(-d, d)$. Here the function $f^{(n)}(x) - c_{n+1}x$ is monotonic on $(-d, d)$, and hence

$$\big|f^{(n)}(0)\big| \leq \big|f^{(n)}\big(-\tfrac{1}{2}d\big) + \tfrac{1}{2}c_{n+1}d\big| + \big|f^{(n)}\big(\tfrac{1}{2}d\big) - \tfrac{1}{2}c_{n+1}d\big| ,$$

and

$$\big|f^{(n)}(0)\big| \leq \big|f^{(n)}\big(-\tfrac{1}{2}d\big)\big| + \big|f^{(n)}\big(\tfrac{1}{2}d\big)\big| + \big|c_{n+1}\big|d .$$

CASE 2. $f^{(n+1)} - c_{n+1}$ has opposite sign on $(-d, 0)$ and $(0, d)$, and $f^{(n+2)} - c_{n+2}$ has opposite sign on $(-d, 0)$ and $(0, d)$. Here $f^{(n+1)}(0) - c_{n+1} = 0$. Put

$$g(x) = f^{(n)}(x) - c_{n+1}x - \tfrac{1}{2}c_{n+2}x^2 .$$

Then $g'(0) = 0$ and g'' has opposite sign on $(-d, 0)$ and $(0, d)$. It follows that g is monotonic on $(-d, d)$, and

$$\big|g(0)\big| \leq \big|g\big(-\tfrac{1}{2}d\big)\big| + \big|g\big(\tfrac{1}{2}d\big)\big| ,$$

and hence

$$\left|f^{(n)}(0)\right| = \left|f^{(n)}\left(-\tfrac{1}{2}d\right) + \frac{c_{n+1}d}{2} - \frac{c_{n+2}d^2}{8}\right| + \left|f^{(n)}\left(\tfrac{1}{2}d\right) - \frac{c_{n+1}d}{2} - \frac{c_{n+2}d^2}{8}\right| .$$

We obtain

$$\left|f^{(n)}(0)\right| \le \left|f^{(n)}\left(-\tfrac{1}{2}d\right)\right| + \left|f^{(n)}\left(\tfrac{1}{2}d\right)\right| + \left|c_{n+1}\right|d + \left|c_{n+2}\right|d^2 .$$

CASE 3. $f^{(n+1)} - c_{n+1}$ has opposite sign on $(-d, 0)$ and on $(0, d)$, and $f^{(n+2)} - c_{n+2}$ does not change sign on $(-d, d)$. It follows that $f^{(n+1)}(0) - c_{n+1} = 0$, and if

$$g(x) = f^{(n)}(x) - c_{n+1}x - \tfrac{1}{2}c_{n+2}x^2 ,$$

then g'' does not change sign on $(-d, d)$, $g'(0) = 0$, and g' has one sign on $(-d, 0)$ and the opposite sign on $(0, d)$. By Lemma 3,

$$(1-s)|g(0)| \le |g(-\tfrac{1}{2}d)| + |g(\tfrac{1}{2}d)| + |g(\tfrac{1}{2}sd)| ,$$

and hence

$$(1-s)\left|f^{(n)}(0)\right| \le \left|f^{(n)}\left(-\tfrac{1}{2}d\right)\right| + \left|f^{(n)}\left(\tfrac{1}{2}d\right)\right| + \left|f^{(n)}\left(\tfrac{1}{2}sd\right)\right| + \\ + 2\left|c_{n+1}\right|d + 2\left|c_{n+2}\right|d^2 . \tag{1}$$

In any of these cases (1) holds. For $0 < u < \frac{1}{2}sd$,

$$(1-s)\frac{|f^{(n)}(0)|u^n}{n!} \le \frac{|f^{(n)}(-\frac{1}{2}d)|u^n}{n!} + \frac{|f^{(n)}(\frac{1}{2}d)|u^n}{n!} + \\ + \frac{|f^{(n)}(\frac{1}{2}sd)|u^n}{n!} + + \frac{2c_{n+1}du^n}{n!} + \frac{2|c_{n+2}|d^2u^n}{n!} , \tag{2}$$

and we deduce from Theorem I and the hypothesis on c_n that $\lim \frac{|f^{(n)}(0)|u^n}{n!} = 0$. But s is an arbitrary number in the interval $(0, 1)$, so $\lim \frac{|f^{(n)}(0)|u^n}{n!} = 0$ for $0 < u < \frac{1}{2}d$.

Now $f^{(n)}(x) - c_{n+1}x$ is monotonic on $(0, d)$, so if $v_n \in \left(0, \frac{1}{2}d\right)$, then

$$\left|f^{(n)}(v_n) - c_{n+1}v_n\right| \le \left|f^{(n)}(0)\right| + \left|f^{(n)}\left(\tfrac{1}{2}d\right) - \tfrac{1}{2}c_{n+1}d\right| ,$$

$$\left|f^{(n)}(v_n)\right| \le \left|f^{(n)}(0)\right| + \left|f^{(n)}\left(\tfrac{1}{2}d\right)\right| + 2\left|c_{n+1}\right|d ,$$

and
$$\frac{|f^{(n)}(v_n)|u^n}{n!} \le \frac{|f^{(n)}(0)|u^n}{n!} + \frac{|f^{(n)}(\frac{1}{2}d)|u^n}{n!} + \frac{2|c_{n+1}|du^n}{n!}\,.$$
But $\lim \frac{f^{(n)}(\frac{1}{2}d)u^n}{n!} = 0$ can be deduced from Theorem I, and $\lim \frac{c_{n+1}du^n}{n!} = 0$ can be deduced from the hypothesis. Thus
$$\lim \frac{|f^{(n)}(v_n)|u^n}{n!} = 0 \quad \text{for } 0 < u < \tfrac{1}{2}d.$$
We deduce from this and Taylor's Theorem that for any $u \in \big(0, \frac{1}{2}d\big)$,
$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)u^n}{n!}\,. \tag{3}$$
Analogous arguments prove (3) for $u \in \big(-\frac{1}{2}d, 0\big)$. (Or consider $f(-x)$.) Thus f is analytic at 0. It follows from Theorem I that f is a real analytic function on $(-d, d)$.

But $f^{(n)}(x) - c_{n+1}x$ is monotonic on $(0, d)$. Fix $v \in \big(0, \frac{1}{2}d\big)$. Then
$$\big|f^{(n)}(v) - c_{n+1}v\big| \le \big|f^{(n)}(0)\big| + \big|f^{(n)}\big(\tfrac{1}{2}d\big) - \tfrac{1}{2}c_{n+1}d\big|$$
and
$$\big|f^{(n)}(v)\big| \le \big|f^{(n)}(0)\big| + \big|f^{(n)}\big(\tfrac{1}{2}d\big)\big| + \big|c_{n+1}(v+d)\big|\,.$$
Thus for $|u| < \frac{1}{2}d$,
$$\sum_{n=0}^{\infty} \frac{|f^{(n)}(v)|u^n}{n!} \le \sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|u^n}{n!} + \sum_{n=0}^{\infty} \frac{|f^{(n)}(\frac{1}{2}d)|u^n}{n!} + \sum_{n=0}^{\infty} \frac{|c_{n+1}|(v+d)u^n}{n!}\,.$$
We have that all the series on the right side converge, so $\sum_{n=0}^{\infty} \frac{f^{(n)}(v)u^n}{n!}$ must converge also. But f is analytic on $(-d, d)$, and it follows that
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(v)}{n!}(x-v)^n$$
for $v \in \big(0, \frac{1}{2}d\big)$ and $x \in \big(v - \frac{1}{2}d, v + \frac{1}{2}d\big)$. The argument for $v \in \big(-\frac{1}{2}d, 0\big)$ is analogous. □

To prove Theorem IV, apply Theorem II to f on each interval $\big((n-1)d, nd\big)$ where n is an integer, positive, negative or 0. To prove Theorem III, apply Theorem II locally to f. We leave the details.

We conclude with some problems that might be topics for further study.

1) Does there exist a power series on $(-1,1)$ whose sum F has bounded derivatives of all orders such that for any sequence of real numbers (c_n) for which $\left(\frac{c_n}{n!}\right)$ is bounded, $F^{(n)} - c_n$ must change sign on $(-1,1)$ for infinitely many n?

2) Does there exist a real analytic function on (a,b) that is not a generalized regularly monotonic function on (a,b)?

3) If the answer to 2) is yes, can monotonicity be used to give a necessary and sufficient condition for a real function in C^∞ to be analytic?

I conjecture that the answers are 1) yes, 2) yes, and 3) no.

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