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κ-to-1 DARBOUX-LIKE FUNCTIONS

Abstract

We examine the existence of κ-to-1 functions $f : \mathbb{R} \to \mathbb{R}$ in the class of continuous functions, Darboux functions, functions with perfect roads, and functions with the Cantor intermediate value property. In this setting κ denotes a cardinal number (finite or infinite). We also consider different variations on this theme.

1 Continuous and Darboux Functions

We will use the standard terminology and notation as in [4]. In particular, ordinal numbers, will be identified with the set of their predecessors and cardinal numbers with the initial ordinals. Thus the first infinite cardinal ω is identified with the set of natural numbers. We will reserve the letters k and n for the natural numbers. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathfrak{c}$. The symbol $|X|$ denotes the cardinality of the set X. For a cardinal $\kappa > 0$ we say that a function $f : X \to Y$ is *κ-to-1* if $\left|f^{-1}(y)\right| = \kappa$ for every $y \in Y$. Similarly we define *$\le\kappa$-to-1* and *$<\kappa$-to-1* functions. We will use the terms *countable-to-1* and *finite-to-1* for functions that are $\le\omega$-to-1 and

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$<\omega$-to-1, respectively. A function $f\colon \mathbb{R}\to\mathbb{R}$ is *Darboux* if it has the intermediate value property; that is, if the image $f[J]$ of every connected subset J of the domain (i.e., an interval) is connected in the range. The last property serves also as a general definition of a Darboux function from a topological space X into a topological space Y.

The notion of an n-to-1 function was introduced by O. G. Harrold, Jr. in 1939 in the paper [11] where he showed that there does not exist a continuous 2-to-1 function carrying an arc into an arc or a circle. Following this paper a sequence of papers appeared in the early 1940's which studied the existence of n-to-1 continuous functions defined on various classes of continua, [6], [9], and [17]. More recent relevant papers were published in the 1980's and among those are [12], [13], and [16].

In 1922 D. C. Gillespie stated in the Bulletin of the American Math. Soc. [10] that a function having the intermediate value property will be continuous unless the set of values it assumes an infinite number of times fills at least one interval. This fact is well-known and follows from the following proposition.

Proposition 1.1. [3, thm 5.2] *If $f\colon \mathbb{R}\to\mathbb{R}$ is Darboux and all level sets $f^{-1}(y)$ of f are closed, then f is continuous.*

As a consequence of those results we see that the question

$$\textit{For which } k<\omega \textit{ does there exist a } k\textit{-to-1 Darboux function?} \tag{1}$$

is equivalent to the following

$$\textit{For which } k<\omega \textit{ does there exist a } k\textit{-to-1 continuous function?} \tag{2}$$

Our first result is the following proposition, that is probably known.

Proposition 1.2. *The following conditions are equivalent for $n<\omega$.*

(i) *There exists a continuous function $f\colon \mathbb{R}\to\mathbb{R}$ that is n-to-1.*

(ii) *There exist a set $Y\subset\mathbb{R}$ and a continuous function $f\colon \mathbb{R}\to Y$ that is n-to-1.*

(iii) *n is odd.*

Proof. The implication (i)⇒(ii) is obvious.

(ii)⇒(iii) Suppose that $f\colon \mathbb{R}\to Y$ is a continuous n-to-1 function and, by way of contradiction, assume that n is even, say $n=2k$. Clearly $n>0$. Fix a $y_0\in Y$ and the points $x_1<x_2<\cdots<x_n$ such that $f(x_i)=y_0$ for $i=1,2,\ldots,n$.

For each $m = 1, \ldots, n-1$ let $I_m = [x_m, x_{m+1}]$. So, we have a partition of $[x_1, x_n]$ onto $2k-1$ intervals I_m such that for each m either $f|I_m \geq y_0$ or $f|I_m \leq y_0$. We will suppose that the set $M = \{m\colon f|I_m \geq y_0\}$ has at least k elements, since the case when $|\{m\colon f|I_m \leq y_0\}| \geq k$ is essentially the same. Put $h_m = \max f|I_m$ and $h = \min\{h_m\colon m \in M\}$. Then $h > y_0$ and for each $y \in (y_0, h)$ and $m \in M$ the set $f^{-1}(y) \cap I_m$ has at least 2 points. So

$$(x_1, x_n) \cap f^{-1}(y) \text{ has at least } 2|M| \geq 2k \text{ points for every } y \in (y_0, h). \qquad (3)$$

Since $|f^{-1}(y)| = n = 2k$ for every y, we conclude that M has exactly k elements. Moreover, (3) implies that

$$\{x\colon f(x) > y_0\} \subset \bigcup_{m \in M} I_m \subset [x_1, x_n].$$

Thus, if $y_m = \max f|[x_1, x_n]$, then all n elements of $f^{-1}(y_m)$ belong to (x_1, x_n) and are local maxima. Therefore, for every $y < y_m$ which is close enough to y_m the set $f^{-1}(y)$ has at least $2n$ elements, a contradiction.

(iii)⇒(i) Assume that n is odd. If $n = 1$ we put $f(x) = x$. For $n > 1$ let f be the function defined by the formula $f(x) = x + n\ \operatorname{dist}(x, \mathbb{Z})$ where $\operatorname{dist}(x, \mathbb{Z})$ denotes the distance between x and the set $\mathbb{Z}$ of integers. It is easy to observe that $f^{-1}(y)$ has n elements for each $y \in \mathbb{R}$. □

Corollary 1.3. *The following conditions are equivalent.*

(i) *There exists a continuous κ-to-1 function $f\colon \mathbb{R} \to \mathbb{R}$.*

(ii) *There exist a set $Y \subset \mathbb{R}$ and a continuous function $f\colon \mathbb{R} \to Y$ that is κ-to-1.*

(iii) $\kappa \in \{\mathfrak{c}, \omega\} \cup \{2k+1\colon k < \omega\}$.

PROOF. (i)⇒(ii) is obvious.

(ii)⇒(iii) Since $f^{-1}(y)$ is a closed subset of $\mathbb{R}$ for any continuous function f, we see that $\kappa \in \{\mathfrak{c}, \omega\} \cup \omega$. But if $\kappa \in \omega$, then κ cannot be an even number by Proposition 1.2.

(iii)⇒(i) For an odd number $\kappa \in \omega$ the existence of f follows from Proposition 1.2. For $\kappa = \omega$ it is enough to take $f(x) = x \sin x$. So assume that $\kappa = \mathfrak{c}$ and let $f_0\colon [0,1] \to [0,1]$ be such that $f_0(0) = 0$, $f_0(1) = 1$, and $|f_0^{-1}(y)| = \mathfrak{c}$ for each $y \in [0,1]$. An example of such a function is given in Bruckner's book [2, pp. 148–150]. Then $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = E(x) + f_0(x - E(x))$ is continuous and $\mathfrak{c}$-to-1, where $E(x)$ denotes the integer part of x. □

Corollary 1.3 gives the full answer for questions (1) and (2). However, the following more general problem might be also of interest.

Problem 1.1. *For which maps* $j\colon \mathbb{R} \to \{\mathfrak{c}, \omega\} \cup \omega$ *does there exist a continuous function* $f_j\colon \mathbb{R} \to \mathbb{R}$ *such that* $\left|f_j^{-1}(y)\right| = j(y)$ *for every* $y \in \mathbb{R}$*?*

To investigate this problem we will use the following terminology. For a map $j\colon \mathbb{R} \to \mathfrak{c} \cup \{\mathfrak{c}\}$ we say that a function $f\colon X \to \mathbb{R}$ is *j-to-1* provided $|f^{-1}(y)| = |j(y)|$ for every $y \in \mathbb{R}$. Corollary 1.3 answers the above question for constant maps j. Some light on the general version of Problem 1.1 is shed by the following fact.

Proposition 1.4. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be a Darboux function,* $y \in \mathbb{R}$*, and* $\kappa = \left|f^{-1}(y)\right|$*. If* $\kappa < \omega$ *and* $B_\kappa = \{z \in \mathbb{R}\colon \left|f^{-1}(z)\right| \geq \kappa\}$*, then there exists an* $\varepsilon > 0$ *such that either* $(y-\varepsilon, y] \subset B_\kappa$ *or* $[y, y+\varepsilon) \subset B_\kappa$*. In particular,* B_κ *is an* F_σ*-set for each* $\kappa < \omega$*.*

Proof. Let $X = f^{-1}(y)$ and choose a positive δ such that the intervals $\{[x-\delta, x+\delta]\}_{x \in X}$ are pairwise disjoint. Let $X^* = \bigcup_{x \in X}\{x-\delta, x+\delta\}$ and put $X^+ = \{x \in X^*\colon f(x) > y\}$ and $X^- = \{x \in X^*\colon f(x) < y\}$. Then at least one of the sets X^+ and X^- has at least κ elements. Assume that $|X^+| \geq \kappa$ and let $y_1 = \min\{f(x)\colon x \in X^+\}$. Then $y_1 > y$ and $[y, y_1] \subset B_\kappa$. The case for $|X^-| \geq \kappa$ is similar.

Now, the set B_κ is F_σ since it is a countable union of nontrivial intervals; the components of B_κ. □

For continuous finite-to-1 functions we have a full answer to Problem 1.1. It is a consequence of the following improvement of Proposition 1.4.

Proposition 1.5. *Let* f *be a finite-to-1 continuous function from* $\mathbb{R}$ *onto* $\mathbb{R}$ *and for* $k < \omega$ *let* $B_k = \{z \in \mathbb{R}\colon \left|f^{-1}(z)\right| \geq k\}$*. Then for all* $k, l \in \omega$ *with* $l \leq 2k+1$ *and* $y \in \mathbb{R} \setminus \operatorname{int} B_l$ *with* $k = \left|f^{-1}(y)\right|$ *there exists an* $\varepsilon > 0$ *such that either* $(y-\varepsilon, y) \subset B_t$ *or* $(y, y+\varepsilon) \subset B_t$*, where* $t = 2k - l + 1$ *if* l *is even and* $t = 2k - l + 2$ *if* l *is odd.*

Proof. Suppose that $j(y) = \left|f^{-1}(y)\right| = k$ and

$$\text{there exists a sequence } y_n \searrow y \text{ with } j(y_n) \leq l - 1. \tag{4}$$

Note that since f is finite-to-1 and onto $\mathbb{R}$, we have either

$$\lim_{x \to -\infty} f(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = \infty$$

or

$$\lim_{x \to -\infty} f(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} f(x) = -\infty.$$

Now $f^{-1}(y)$ partitions $\mathbb{R}$ into $k+1$ open intervals $J_0, \ldots, J_k$ of which $k-1$, say $J_1, \ldots, J_{k-1}$, are bounded. Also, for every $j \in \{0, \ldots, k\}$ we have either $f|J_j > y$ or $f|J_j < y$. Moreover, by the above limit consideration, either $f|J_0 > y$ or $f|J_k > y$. Consequently, by condition (4), the set $M \subset \{1, \ldots, k-1\}$ of all i for which $f|J_i > y$ has at most $\mathrm{E}((l-2)/2)$ elements. Thus $N = \{1, \ldots, k-1\} \setminus M$ has at least $k - 1 - \mathrm{E}((l-2)/2)$ elements. Let $\varepsilon > 0$ be such that $\min f|J_i < y - \varepsilon$ for every $i \in N$. Then every value z from $(y - \varepsilon, y)$ is taken at least twice on each interval J_i with $i \in N$. Moreover, such a value is in at least one of the unbounded intervals by the above limit consideration. Thus, $|f^{-1}(z)| \geq 2k - 2 - 2\mathrm{E}((l-2)/2) + 1$. Finally, note that $2k - 2 - 2\mathrm{E}((l-2)/2) + 1$ is equal to $2k - l + 1$ if l is even and it is equal to $2k - l + 2$ if l is odd. □

Corollary 1.6. *Let f be a finite-to-1 continuous function from $\mathbb{R}$ onto $\mathbb{R}$. Then for every even $k \in \omega$ and $y \in \mathbb{R}$ with $k = \left|f^{-1}(y)\right|$ (that is $y \in B_k \setminus B_{k+1}$) there exists an $\varepsilon > 0$ such that either $(y - \varepsilon, y) \subset B_{k+1}$ or $(y, y + \varepsilon) \subset B_{k+1}$.*

In particular, for every $n \in \omega$ the set $B_{2n} \setminus B_{2n+1}$ has an empty interior.

PROOF. This is a consequence of Proposition 1.5 with $l = k + 1$. Indeed, suppose that k is even. For every $y \in B_k \setminus B_{k+1}$, either $y \in \mathrm{int}(B_{k+1})$ or y is an end-point of some interval contained in B_{k+1}. □

Corollary 1.7. *Let f be a finite-to-1 continuous function from $\mathbb{R}$ onto $\mathbb{R}$. If $|f^{-1}(y)| = 2k + 1$ and $y \notin \mathrm{int}\, B_{2k+1}$, then there exists an $\varepsilon > 0$ such that either $(y - \varepsilon, y) \subset B_{2k+3}$ or $(y, y + \varepsilon) \subset B_{2k+3}$.*

PROOF. Consider Proposition 1.5 with $l = 2k + 1$. □

Theorem 1.8. *Let $j \colon \mathbb{R} \to \{1, 2, 3, \ldots\}$. The following conditions are equivalent.*

(a) *There exists a continuous j-to-1 function $f \colon \mathbb{R} \to \mathbb{R}$.*

(b) *For every $k \in \omega$*

  (i) *$C_k = j^{-1}(\{k, k+1, k+2, \ldots\})$ is a (possibly empty) union of pairwise disjoint non-trivial intervals,*

  (ii) *$j^{-1}(2k)$ has an empty interior, and*

  (iii) *if $y \in j^{-1}(2k+1) \setminus \mathrm{int}\, C_{2k+1}$ then y is an end-point of a component of $\mathrm{int}\, C_{2k+3}$.*

PROOF. (a)⇒(b) Clearly $j^{-1}(\{k, k+1, \ldots\}) = \{z \in \mathbb{R} \colon \left|f^{-1}(z)\right| \geq k\} = B_k$ and, by Proposition 1.4, the component intervals of B_k are the non-trivial intervals, proving (i). Conditions (ii) and (iii) follow immediately from Corollaries 1.6 and 1.7, respectively.

(b)⇒(a) Let

- $\mathcal{J}_k$ be the family of all components of C_{2k+1} $\mathcal{J}_k = \{J_{k,1}, J_{k,2}, \ldots\}$,
- $D_k = \operatorname{int} C_{2k+1} = \bigcup\{\operatorname{int} J \colon J \in \mathcal{J}_k\}$,
- E be the set of all endpoints of intervals belonging to some $\mathcal{J}_k$ and
- $E_k = \bigcup\{\operatorname{bd}(J) \colon J \in \mathcal{J}_k\}$.

Note that E is countable, $E = \bigcup_k E_k$, and $C_{2k} \subset D_k \cup E_k$ for each positive integer k. The desired function f will be defined as a limit of functions f_k from $\mathbb{R}$ onto $\mathbb{R}$. We start with f_0 being the identity function. Assume that f_k is defined. To construct f_{k+1} we take an arbitrary interval $J_{k+1,i}$ from $\mathcal{J}_{k+1}$ and represent $\hat{J} = J_{k+1,i} \cup (\operatorname{bd} J_{k+1,i} \cap C_{2k-1})$ as a union of closed intervals $J^1_{k+1,i}, J^2_{k+1,i}, J^3_{k+1,i}, \ldots$ with disjoint interiors. We will assume also that

(α) the length of $J^m_{k+1,i}$ is less than 2^{-k-1},

(β) the endpoints of the intervals $J^m_{k+1,i}$ are disjoint from E, with the exception of the endpoints of $\hat{J}$, if they belong to $J^m_{k+1,i}$,

(γ) if $J^n_{k+1,i} \cap J^m_{k,j} \neq \emptyset$ then $J^n_{k+1,i} \subset J^m_{k,j}$ and

(δ) for every m there is an interval $I^m_{k+1,i} \subset f_k^{-1}(J^m_{k+1,i})$ such that $f_k|I^m_{k+1,i}$ is linear, $f_k[I^m_{k+1,i}] = J^m_{k+1,i}$, and $I^m_{k+1,i} \subset I^n_{k,j}$ whenever $I^m_{k+1,i} \cap I^n_{k,j} \neq \emptyset$.

Also, we can order the family of all $J^m_{k+1,i}$ in the type of $\mathbb{Z}$, if $\hat{J}$ is open, in the type of ω (or ω^*) if $\hat{J}$ contains only left (right) endpoints, and in a finite type, when $\hat{J}$ contains both endpoints.

The function f_{k+1} is obtained by modifying f_k on every interval $I^m_{k+1,i}$ The modification is obtained by replacing a $f_k|I^m_{k+1,i}$ by a function with graph of shape of letter N. (Or its mirror image.)

By (α), the sequence $(f_k)_k$ is uniformly convergent to a continuous function $f \colon \mathbb{R} \to \mathbb{R}$.

Observe that for each $k \in \omega$ and $y \in \mathbb{R}$ we have

$$|f_k^{-1}(y)| = \begin{cases} |f_{k-1}^{-1}(y)| + 2 & \text{if } y \in D_k \\ |f_{k-1}^{-1}(y)| + 1 & \text{if } y \in E_k \cap C_{2k-1} \\ |f_{k-1}^{-1}(y)| & \text{otherwise.} \end{cases} \tag{5}$$

Thus, we easily obtain (by induction) the equations

$$|f_k^{-1}(y)| = \begin{cases} 2k+1 & \text{if } y \in D_k \\ 2k & \text{if } y \in E_k \cap C_{2k} \\ 2k-1 & \text{if } y \in E_k \cap (C_{2k-1} \setminus C_{2k}) \\ |f_{k-1}^{-1}(y)| & \text{otherwise.} \end{cases} \tag{6}$$

Note also the following properties of the sequence $(f_k)_k$.

$$\text{If } k < n \text{ and } y \in \mathbb{R} \text{ then } |f_k^{-1}(y)| \le |f_n^{-1}(y)|. \tag{7}$$

$$\text{If } x \notin \bigcup_m \bigcup_i I_{k,i}^m \text{ then } f_k(x) = f_{k-1}(x). \tag{8}$$

Statement (7) and condition (δ) imply that for each $x \in \mathbb{R}$ there is a k_0 with $x \in \bigcup_m \bigcup_i I_{k_0,i}^m \setminus \bigcup_{k>k_0} \bigcup_m \bigcup_i I_{k,i}^m$. Thus, by (8),

$$\text{for each } x \in \mathbb{R} \text{ there is } k_0 \in \omega \text{ such that } f_k(x) = f_{k_0}(x) \text{ for } k > k_0. \tag{9}$$

Moreover,

$$\text{if } y \in \bigcup_m \bigcup_i J_{k_0,i}^m \setminus \bigcup_{k>k_0} \bigcup_m \bigcup_i J_{k,i}^m \text{ then } f^{-1}(y) = f_{k_0}^{-1}(y); \tag{10}$$

so f is finite-to-1. We will verify that f is j-to-1. Assume that $y \in C_k$ and consider two cases. If k is odd, say $k = 2k_0+1$ then, by (6), $|f_{k_0+1}^{-1}(y)| \ge 2k_0+1$, and, by (7), $|f_k^{-1}(y)| \ge 2k_0+1$ for all $k > k_0$ so, by (10), $|f^{-1}(y)| \ge 2k_0+1$. Similarly, if k is even, say $k = 2k_0$, then $|f_{k_0}^{-1}(y)| \ge 2k_0$; so $|f^{-1}(y)| \ge 2k_0$. Therefore,

$$(\forall k \in \omega) \quad C_k \subset \{y \in \mathbb{R} \colon |f^{-1}(y)| \ge k\}. \tag{11}$$

Now suppose that $y \notin C_k$. Then for $k_0 = \mathrm{E}(\frac{k}{2})$ we have $|f_{k_0}^{-1}(y)| < 2k_0 \le k$ and $y \notin \bigcup_m \bigcup_i J_{k_0,i}^m$. Thus (10) implies $|f^{-1}(y)| = |f_{k_0}^{-1}(y)| < k$. Hence

$$(\forall k \in \omega) \quad \{y \in \mathbb{R} \colon |f^{-1}(y)| \ge k\} \subset C_k. \tag{12}$$

Finally, by (11) and (12) we obtain the statement

$$(\forall k \in \omega) \quad C_k = \{y \in \mathbb{R} \colon |f^{-1}(y)| \ge k\}.$$

Thus f is j-to-1. □

Proposition 1.4 also yields the following result on Darboux countable-to-1 functions.

Corollary 1.9. *If a Darboux function $f \colon \mathbb{R} \to \mathbb{R}$ is countable-to-1 and $j \colon \mathbb{R} \to \omega \cup \{\omega\}$ is defined by $j(y) = |f^{-1}(y)|$ then j is Borel measurable.*

Note that in Corollary 1.9 the assumption that f is countable-to-1 is essential. This is even the case when f is continuous, since there is a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ for which the set $A_{\mathfrak{c}} = \{y\colon |f^{-1}(y)| = \mathfrak{c}\}$ is analytic non-Borel. This follows from the fact that the set

$$A = \left\{x \in 2^\omega\colon \left|\mathrm{pr}_1^{-1}(x)\right| = \mathfrak{c}\right\} = \varphi^{-1}(\{C \in K(2^\omega)\colon |C| > \omega\})$$

is analytic non-Borel, where φ is a homeomorphism between 2^ω and the space $K(2^\omega)$ of all non-empty compact subsets of 2^ω (with the Hausdorff metric) and pr_1 is the projection of the graph of φ onto the first coordinate. This is a consequence of a theorem of Hurewicz that the set $\{C \in K(2^\omega)\colon |C| > \omega\}$ is analytic non-Borel. (See [14, thm 27.5, p. 210]. This fact was pointed out to the authors by S. Solecki.) On the other hand for continuous f the sets $A_\kappa = \{z \in \mathbb{R}\colon \left|f^{-1}(z)\right| = \kappa\}$ are not too bad; they all are Borel for $\kappa \in \omega$ (This follows from Proposition 1.4.) and analytic for $\kappa = \mathfrak{c}$. Indeed, from the Mazurkiewicz-Sierpiński theorem it follows that $A_{\mathfrak{c}}$ is analytic. (See e.g. [15, thm 3, p. 496], or [14, thm 29.19, p. 231].) Consequently, the set A_ω must be co-analytic. Moreover, if $A_{\mathfrak{c}}$ is non-Borel, then A_ω is non-Borel (so non-analytic), too.

Note that the results above follow also for Borel functions with the Darboux property. Nothing good, though, can be said of the set $B_{\mathfrak{c}}$ for a general Darboux function $f\colon \mathbb{R} \to \mathbb{R}$, as follows for the next proposition.

Proposition 1.10. *For every set $Z \subset \mathbb{R}$ there exists a Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ with $Z = \{y\colon |f^{-1}(y)| = \mathfrak{c}\}$.*

PROOF. Let $\{A_\xi\colon \xi < \mathfrak{c}\}$ be a partition of $\mathbb{R}$ into countable dense sets. Take an $h\colon \mathfrak{c} \to \mathbb{R}$ such that $|h^{-1}(z)| = \mathfrak{c}$ for $z \in Z$ and $|h^{-1}(z)| = 1$ for $z \notin Z$. Define f by putting $f(x) = h(\xi)$ for every $x \in A_\xi$ and $\xi < \mathfrak{c}$. Then f satisfies the conclusion. □

Note also that the main part of Proposition 1.4 is false for an infinite κ.

Remark 1.1. For $\kappa \in \{\omega, \mathfrak{c}\}$ there exists a continuous $\leq\kappa$-to-1 function f from $\mathbb{R}$ onto $\mathbb{R}$ for which $B_\kappa = \{0\}$. Moreover, for every countable set $B \subset \mathbb{R}$ there exists a continuous function f from $\mathbb{R}$ onto $\mathbb{R}$ with the property that $B = \{z \in \mathbb{R}\colon \left|f^{-1}(z)\right| = \mathfrak{c}\}$.

PROOF. First assume that $\kappa = \omega$ and define f by putting $f(0) = 0$ and $f(x) = x^2 \sin(x^{-1})$ for $x \neq 0$. Then f has the desired properties.

For $\kappa = \mathfrak{c}$ first fix a perfect set P and $a < b$ such that $P \subset [a,b] \subset (0,1)$ and let $g\colon [a,b] \to \mathbb{R}$ be such that $g(x) = \mathrm{dist}(x,P)$ is the distance between x and P. Now it is easy to find an extension f from $\mathbb{R}$ onto $\mathbb{R}$ for which $B_{\mathfrak{c}} = \{0\}$.

To see the additional part let $B = \{b_n\colon n < \omega\}$ and define f_0 on a set $K = \bigcup_{n<\omega}(n + [a,b])$ by putting

$$f_0(n + x) = b_n + g(x) \quad \text{for every } n < \omega \text{ and } x \in [a,b].$$

Extend f_0 to f from $\mathbb{R}$ onto $\mathbb{R}$ such that f is linear on each of the intervals $[n+b,(n+1)+a]$ and $f|(-\infty,a]$ is ω-to-1 and onto $\mathbb{R}$. It is easy to see that $B=\{z\in\mathbb{R}\colon\left|f^{-1}(z)\right|=\mathfrak{c}\}$. □

In the remainder of this section we consider the analogous problems for continuous functions from $\mathbb{R}^n$ or $[0,1]^n$ into $\mathbb{R}$ and from $[0,1]$ into $\mathbb{R}$. See [6], [11], [12], [13], [16] and [17]. J. H. Roberts in [17], proved that there does not exist a continuous 2-to-1 function defined on a closed 2-cell but left open the case for arbitrary n-cells. Paul Civin in [6], proved that there does not exist a continuous 2-to-1 function defined on a closed 3-cell and stated that it can easily be demonstrated that a continuous function defined on $\mathbb{R}$ is not 2-to-1. However, Civin noted that for $\mathbb{R}^n$ with n equal to 2 or 3 this question is unknown.

We will start with the following easy remark.

Proposition 1.11. *Let $n>1$ and $X=\mathbb{R}^n$ or $X=[0,1]^n$. If $f\colon X\to\mathbb{R}$ is Darboux then $f[X]$ is an interval and for every interior point y of $f[X]$ the set $f^{-1}(y)$ has cardinality $\mathfrak{c}$.*

Proof. $f[X]$ is an interval since X it is connected. To see the other part take an interior point y of $f[X]$. Then the set $X\setminus f^{-1}(y)$ disconnects X since $f[X\setminus f^{-1}(y)]\subset f[X]\setminus\{y\}$. Thus $f^{-1}(y)$ has cardinality $\mathfrak{c}$. □

Remark 1.2. Proposition 1.11 remains true for an arbitrary Darboux function $f\colon X\to\mathbb{R}$ provided X cannot be disconnected by any set of cardinality less than $\mathfrak{c}$.

Corollary 1.12. *Let $n>1$ and $j\colon\mathbb{R}\to\mathfrak{c}\cup\{\mathfrak{c}\}$. The following conditions are equivalent.*

(i) *There exists a continuous nonconstant j-to-1 function $f\colon[0,1]^n\to\mathbb{R}$.*

(ii) *There are $-\infty<a<b<\infty$ such that $j|(a,b)=\mathfrak{c}$, $j|\mathbb{R}\setminus[a,b]=0$ and $|j(a)|,|j(b)|\in\{\omega,\mathfrak{c}\}\cup\omega\setminus\{0\}$.*

Proof. (i)⇒(ii) The range of f is a closed interval $[a,b]$ by Proposition 1.11 and compactness of $[0,1]^n$. Then, again by Proposition 1.11, we have also $j|(a,b)=\mathfrak{c}$, while for $d\in\{a,b\}$ we have $|j(d)|\in\{\omega,\mathfrak{c}\}\cup\omega\setminus\{0\}$ since $f^{-1}(d)$ is a non-empty closed subset of $\mathbb{R}^n$ and $|j(d)|=|f^{-1}(d)|$.

(ii)⇒(i) Let A and B be closed subsets of $[0,1]^n$ with distance $d>1$ and such that $|j(a)|=|A|$ and $|j(b)|=|B|$. For $C\in\{A,B\}$ define $F_C=\{x\in[0,1]^n\colon\operatorname{dist}(x,C)\le .5\}$, where $\operatorname{dist}(x,C)$ is the distance of x to C. Then $\operatorname{dist}(F_A,F_B)=d-1>0$. For $x\in F_A$ define $g(x)=\operatorname{dist}(x,A)\in[0,.5]$ and for $x\in F_B$ put $g(x)=d-\operatorname{dist}(x,B)\in[d-.5,d]$. Then, by the Tietze Extension

Theorem, we can extend g continuously onto $[0,1]^n$ such that it assumes on $[0,1]^n \setminus (F_B \cup F_B)$ only the values from $[.5, d-.5]$. Now if h is a homeomorphism between $[0,d]$ and $[a,b]$ then $f = h \circ g$ has the desired properties. □

A slight modification of the above argument gives also the following characterization.

Corollary 1.13. *Let $n > 1$ and $j \colon \mathbb{R} \to \mathfrak{c} \cup \{\mathfrak{c}\}$. The following conditions are equivalent.*

(i) *There exists a continuous nonconstant j-to-1 function $f \colon \mathbb{R}^n \to \mathbb{R}$.*

(ii) *There are $-\infty \le a < b \le \infty$ such that $j|(a,b) \equiv \mathfrak{c}$, $j|\mathbb{R} \setminus [a,b] \equiv 0$, and $|j(c)| \in \{\omega, \mathfrak{c}\} \cup \omega$ for $c \in \{a,b\} \cap \mathbb{R}$.*

The corresponding characterization of Darboux functions is slightly different.

Corollary 1.14. *Let $n > 1$ and $j \colon \mathbb{R} \to \mathfrak{c} \cup \{\mathfrak{c}\}$. The following conditions are equivalent.*

(i) *There exists a Darboux nonconstant j-to-1 function $f \colon \mathbb{R}^n \to \mathbb{R}$.*

(ii) *There are $-\infty \le a < b \le \infty$ such that $j|(a,b) = \mathfrak{c}$ and $j|\mathbb{R} \setminus [a,b] = 0$.*

Proof. (i)⇒(ii) This follows immediately from Proposition 1.11.

(ii)⇒(i) Let a and b be as in (ii). Recall that every connectivity function $f \colon \mathbb{R}^n \to \mathbb{R}$, with $n > 1$, is Darboux. (See [8].) In [5] there has been constructed a connectivity function $g \colon \mathbb{R}^n \to \mathbb{R}$ such that for some dense G_δ set $G \subset \mathbb{R}^n$ any modification of g on G results still a connectivity function. Now, if h is a homeomorphism from $\mathbb{R}$ onto (a,b) then $f_0 = h \circ g$ has a property that a function $f \colon \mathbb{R}^n \to [a,b]$ is connectivity provided f which agrees with f_0 outside of G. (Compare also [18, thm. 1].) Now, take disjoint sets $A, B \subset G$ such that $|j(a)| = |A|$ and $|j(b)| = |B|$. Define $f(x) = a$ for $x \in A$, $f(x) = b$ for $x \in B$, and $f(x) = f_0(x)$ for $x \in \mathbb{R}^n \setminus (A \cup B)$. Then f is connectivity; so Darboux, and it has all other required properties. □

In the remainder of this section we will consider functions $f \colon [0,1] \to \mathbb{R}$.

Proposition 1.15. *Assume that $n > 1$. There is no continuous function $f \colon [0,1] \to \mathbb{R}$ which is n-to-1.*

Proof. For $n = 2$ it is easy and well-known. (See [11] and [16].) Suppose that $n > 2$. Let $y_1 = \max_{x \in [0,1]} f(x)$ and let $f^{-1}(y_1) = \{x_1, \ldots, x_n\}$ where $x_1 < \cdots < x_n$. Now, if $y_0 = \max\{\min f|[x_i, x_{i+1}] \colon i = 1, \ldots, n-1\}$ then for each $y \in (y_0, y_1)$, $f^{-1}(y)$ has at least $2(n-1) > n$ points, a contradiction. □

Following Theorem 5.2 in [3], Bruckner and Ceder stated that there exists a continuous function defined on $[0,1]$ such that each value between 0 and 1 is taken on infinitely often. Such a function can be constructed by suitably modifying the well-known Cantor function on its intervals of constancy. For completeness we will include such a construction in the following proposition.

Proposition 1.16. *If $\kappa \in \{\omega, \mathfrak{c}\}$ then there is a continuous function $f\colon [0,1] \to [0,1]$ such that $|f^{-1}(y)| = \kappa$ for each $y \in [0,1]$.*

PROOF. An example of a continuous function $f\colon [0,1] \to [0,1]$ such that $|f^{-1}(y)| = \mathfrak{c}$ for each $y \in [0,1]$ can be found in Bruckner's book [2, pp. 148–150].

Thus assume that $\kappa = \omega$ and define function $g\colon \mathbb{R} \to \mathbb{R}$ by a formula $g(x) = (x^2+1)^{-1} \sin x$ Notice that $\lim_{x\to-\infty} g(x) = \lim_{x\to\infty} g(x) = 0$. In particular, g takes value 0 infinitely many times and all other values only finitely many times. Let $C \subset I$ be the Cantor ternary set, i.e.,

$$C = \left\{ \sum_{i=1}^{\infty} \frac{k_i}{3^i} \colon k_i \in \{0,2\} \ \text{ for every } i = 1,2,\ldots \right\}$$

and let f_0 be the Cantor function from C *onto* $[0,1]$; that is, given by a formula $f_0(\sum_{i=1}^{\infty} \frac{k_i}{3^i}) = \sum_{i=1}^{\infty} \frac{k_i}{2^{i+1}}$. Thus f_0 is continuous, increasing and if $I = (a,b)$ is a component of $[0,1] \setminus C$ then $f_0(a) = f_0(b)$. Extend f_0 to f by putting on any such interval $f(x) = f_0(a) + (b-a)g(h_I(x))$, where h_I is an increasing homeomorphism from $I = (a,b)$ onto $\mathbb{R}$. It is easy to see that f is continuous and ω-to-1. □

Corollary 1.17. *Let $\kappa \le \mathfrak{c}$ be a cardinal number. The following conditions are equivalent.*

(i) *There exists a continuous nonconstant κ-to-1 function $f\colon [0,1] \to [0,1]$.*

(ii) $\kappa \in \{\omega, \mathfrak{c}\}$.

2 Perfect Road Functions

Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ has a *perfect road* at $x \in \mathbb{R}$ if there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral limit point for which the restriction $f|P$ of f to P is continuous at x. The function $f\colon \mathbb{R} \to \mathbb{R}$ has the perfect road property if it has a perfect road at each point $x \in \mathbb{R}$. (See, e.g., [8].)

Theorem 2.1. *For every function $j\colon \mathbb{R} \to \mathfrak{c} \cup \{\mathfrak{c}\} \setminus \{0\}$ there exists a j-to-1 function $f_j\colon \mathbb{R} \to \mathbb{R}$ with the perfect road property.*

Proof. Let $\{\langle I_n, J_n\rangle : n<\omega\}$ be a one-to-one enumeration of all sets of the form $(p,q)\times(r,s)$, where p,q,r,s are rationals, $p<q$, and $r<s$. Inductively choose the sequences $\{P_n : n<\omega\}$ and $\{Q_n : n<\omega\}$ of pairwise disjoint perfect nowhere dense sets such that $P_n\subset I_n$ and $Q_n\subset J_n$ for every $n<\omega$.

Let $g\colon \bigcup_{n<\omega} P_n \to \bigcup_{n<\omega} Q_n$ be a function such that $g|P_n$ is a homeomorphism between P_n and Q_n for every $n<\omega$. Notice that

$$\text{every extension } f\colon \mathbb{R}\to\mathbb{R} \text{ of } g \text{ has the perfect road property.} \tag{13}$$

Indeed, to show that f has a perfect road from the left at a point $x\in\mathbb{R}$ find a sequence $\{n_j\}_{j<\omega}$ such that $I_{n_j}<I_{n_k}$ and $J_{n_j}<J_{n_k}$ for every $j<k<\omega$ and that $\lim_{j\to\infty} I_{n_j}=x$ and $\lim_{j\to\infty} J_{n_j}=f(x)$. Then $f\left|\left(\{x\}\cup\bigcup_{j<\omega} P_{n_j}\right)\right.$ is continuous at x. The right hand side perfect road at x can be found similarly, proving (13).

To find an appropriate extension f_j of g note that $G=\mathbb{R}\setminus\bigcup_{n<\omega} P_n$ has cardinality $\mathfrak{c}$. Thus there exists a partition $\{X_y : y\in\mathbb{R}\}$ of G such that $|X_y|=|j(y)|$ if $y\notin\bigcup_{n<\omega} Q_n$ and $|X_y|=|j(y)|-1$ for $y\in\bigcup_{n<\omega} Q_n$. Finally put

$$f_j(x)=\begin{cases} g(x) & \text{for } x\in\bigcup_{n<\omega} P_n\\ y & \text{for } x\in X_y \text{ and } y\in\mathbb{R}.\end{cases}$$

It is easy to observe that f_j has the desired properties. □

Next we will consider a question for which functions $j\colon\mathbb{R}\to\mathfrak{c}\cup\{\mathfrak{c}\}$ a function f_j as in Theorem 2.1 can be Borel measurable. Clearly, if $f\colon\mathbb{R}\to\mathbb{R}$ is a Borel onto function then the function $j_f\colon\mathbb{R}\to\mathfrak{c}\cup\{\mathfrak{c}\}$ defined by $j_f(y)=|f^{-1}(y)|$ must be "nice." In particular, $j_f\colon\mathbb{R}\to\mathcal{K}_0=(\omega\setminus\{0\})\cup\{\omega,\mathfrak{c}\}$. In the following theorems we shall consider $\mathcal{K}_0$ as the topological space with the discrete topology.

Theorem 2.2. *If $j\colon\mathbb{R}\to\mathcal{K}_0$ is a Borel function then there exists a Borel j-to-1 function $f_j\colon\mathbb{R}\to\mathbb{R}$ with the perfect road property.*

Proof. Let $g\colon\bigcup_{n<\omega} P_n\to\bigcup_{n<\omega} Q_n$ satisfy the condition (13). Note that $\hat{\jmath}\colon\mathbb{R}\to\omega\cup\{\omega,\mathfrak{c}\}$ given by $\hat{\jmath}(y)=j(y)$ if $y\notin\bigcup_{n<\omega} Q_n$ and $\hat{\jmath}(y)=j(y)-1$ if $y\in\bigcup_{n<\omega} Q_n$ is Borel as well. Partition $G=\mathbb{R}\setminus\bigcup_{n<\omega} P_n$ into Borel sets $\{B_\kappa : \kappa\in\mathcal{K}_0\}$ such that $|B_\kappa|=\kappa\otimes|\hat{\jmath}^{-1}(\kappa)|$ for every $\kappa\in\mathcal{K}_0$. We claim that

$$\text{for every } \kappa\in\mathcal{K}_0 \text{ there is a } \kappa\text{-to-1 Borel function } f_\kappa\colon B_\kappa\to\hat{\jmath}^{-1}(\kappa). \tag{14}$$

First note that (14) immediately implies the theorem, since then $f_j\colon\mathbb{R}\to\mathbb{R}$ defined by

$$f_j(x)=\begin{cases} g(x) & \text{for } x\in\bigcup_{n<\omega} P_n\\ f_\kappa(x) & \text{for } x\in B_\kappa \text{ and } \kappa\in\mathcal{K}_0\end{cases}$$

clearly has the desired properties.

To prove (14) we will consider three cases.

If $\kappa < \mathfrak{c}$, then partition B_κ into Borel sets $\{B_\kappa^i : i < \kappa\}$ each of cardinality $|\hat{\jmath}^{-1}(\kappa)|$ and for every $i < \kappa$ define f_κ on B_κ^i as a Borel isomorphism between B_κ^i and $\hat{\jmath}^{-1}(\kappa)$. (Recall that any two Borel sets of the same size are Borel isomorphic. See, e.g., [15, p. 451] or [14, thm 15.6, p. 90].)

If $\kappa = \mathfrak{c}$ and $\lambda = |\hat{\jmath}^{-1}(\mathfrak{c})| < \mathfrak{c}$ then $\lambda \le \omega$. Partition $B_\mathfrak{c}$ onto λ Borel sets $\{B_\mathfrak{c}^y : y \in \hat{\jmath}^{-1}(\mathfrak{c})\}$ each of cardinality $\mathfrak{c}$ and define $f_\mathfrak{c}(x) = y$ for $x \in B_\mathfrak{c}^y$.

If $\kappa = \hat{\jmath}^{-1}(\mathfrak{c}) = \mathfrak{c}$ then define $f_\mathfrak{c}$ as a $\mathfrak{c}$-to-1 Borel function from $B_\mathfrak{c}$ onto $\hat{\jmath}^{-1}(\mathfrak{c})$. Such an $f_\mathfrak{c}$ can be constructed as follows. Let $\mathcal{N}$ denote the space of all irrationals. Let φ be a Borel isomorphism between $B_\mathfrak{c}$ and $\mathcal{N} \times \mathcal{N}$, $\varphi = (\varphi_1, \varphi_2)$, and let ψ be a Borel isomorphism between $\mathcal{N}$ and $\hat{\jmath}^{-1}(\mathfrak{c})$. Define $f_\mathfrak{c} = \psi \circ \varphi_1$. Then $f_\mathfrak{c}$ is $\mathfrak{c}$-to-1 and Borel measurable. □

We finish this section with the following remark.

Proposition 2.3. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable. Then*

(i) *the set $j_f^{-1}(\mathfrak{c})$ is analytic and*

(ii) *the set $j_f^{-1}(1)$ can be co-analytic and non-Borel.*

Proof. The statement (i) is a consequence of the Mazurkiewicz-Sierpiński theorem (see [14, thm 29.19, p. 231]), because the graph of a Borel measurable function from $\mathbb{R}$ into $\mathbb{R}$ is a Borel subset of $\mathbb{R}^2$. (See [14, thm. 14.12, p. 88].)

To prove (ii) fix an analytic non-Borel set $A \subset \mathbb{R}$. (Such sets exist by the Suslin theorem [14, thm. 14.2, p. 85].) There exists a continuous function $h : \mathcal{N} \to \mathbb{R}$ with $h[\mathcal{N}] = A$. (See [14, p. 85].) Let $\varphi : \mathcal{N} \to \mathcal{N} \times 2$ be a homeomorphism, $\varphi = \langle \varphi_0, \varphi_1 \rangle$, and let $\mathcal{N}_i = \varphi_1^{-1}(i)$ for $i = 0, 1$. Observe that $f_0 : \mathcal{N}_0 \to \mathbb{R}$ defined by $f_0(x) = h(\varphi_0(x))$ is continuous and $f_0[\mathcal{N}_0] = A$. Let $f_1 : \mathbb{R} \setminus \mathcal{N}_0 \to \mathbb{R}$ be a Borel isomorphism. (See the isomorphism theorem [14, thm 15.6, p. 90].) Put $f = f_0 \cup f_1$. Then f is Borel measurable and $j_f^{-1}(1) = \mathbb{R} \setminus A$. Thus j_f^{-1} is co-analytic and, by the Lusin Separation Theorem [14, thm 14.7, p. 87], it is non-Borel. □

3 CIVP Functions

Recall the following definitions that are introduced in [7] and [19], respectively. (See also [8].)

- $f : \mathbb{R} \to \mathbb{R}$ has the *Cantor intermediate value property* (CIVP), if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C] \subset K$.

- $f\colon \mathbb{R}\to\mathbb{R}$ has the *strong Cantor intermediate value property* (SCIVP), if for every $x,y\in\mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f[C]\subset K$ and $f|C$ is continuous.

The notion *Cantor set* means a perfect nowhere dense set. Note that in the definitions above the Cantor sets can be replaced by perfect sets.

Theorem 3.1. *For every function $j\colon \mathbb{R}\to\mathfrak{c}\cup\{\mathfrak{c}\}\setminus\{0\}$ there exists a j-to-1 function $f_j\colon\mathbb{R}\to\mathbb{R}$ with CIVP.*[1]

Proof. Let $\mathcal{P}$ be a family of pairwise disjoint perfect subsets of $\mathbb{R}$ such that $|\mathbb{R}\setminus\bigcup\mathcal{P}|=\mathfrak{c}$ and $|\{P\in\mathcal{P}\colon P\subset(a,b)\}|=\mathfrak{c}$ for any $a<b$. Take an enumeration $\{\langle U_\xi,Q_\xi\rangle\colon \xi<\mathfrak{c}\}$ of $\{(a,b)\colon a<b\}\times\{Q\subset\mathbb{R}\colon Q \text{ is perfect}\}$.

For every $\xi<\mathfrak{c}$ choose $P_\xi\in\mathcal{P}$ such that $P_\xi\subset U_\xi$ and $P_\xi\neq P_\eta$ for $\xi\neq\eta$. Finally, partition $\mathbb{R}$ into Bernstein sets $\{B_\xi\colon \xi\le\mathfrak{c}\}$ and define a function $f_0\colon\bigcup_{\xi<\mathfrak{c}}P_\xi\to\mathbb{R}$ such that for every $\xi<\mathfrak{c}$ the restriction $f_0|P_\xi$ is a bijection between P_ξ and $B_\xi\cap Q_\xi$. Then f_0 is one-to-one, since sets B_ξ are pairwise disjoint. Notice that

$$\text{any extension } f\colon\mathbb{R}\to\mathbb{R} \text{ of } f_0 \text{ has the CIVP.} \tag{15}$$

Indeed, fix $a<b$ such that $f(a)\neq f(b)$ and a perfect set K between $f(a)$ and $f(b)$. There exists $\xi<\mathfrak{c}$ such that $(a,b)=U_\xi$ and $K=Q_\xi$. Then P_ξ is a perfect set between a and b and $f[P_\xi]\subset K$.

To finish the proof let $G=\mathbb{R}\setminus\bigcup_{\xi<\mathfrak{c}}P_\xi$ and $Z=\mathbb{R}\setminus f_0[\bigcup_{\xi<\mathfrak{c}}P_\xi]$. Observe that $|G|=|Z|=\mathfrak{c}$ because $\mathbb{R}\setminus\bigcup\mathcal{P}\subset G$ and $B_\mathfrak{c}\subset Z$. Partition G into sets $\{X_y\colon y\in\mathbb{R}\}$ such that $|X_y|=|j(y)|$ for $y\in Z$ and $|X_y|=|j(y)|-1$ for $y\in\mathbb{R}\setminus Z$. Finally, it is easy to verify that the function $f_j\colon\mathbb{R}\to\mathbb{R}$ defined by

$$f_j(x)=\begin{cases} f_0(x) & \text{for } x\in\bigcup_{\xi<\mathfrak{c}}P_\xi\\ y & \text{for } x\in X_y \text{ and } y\in\mathbb{R}\end{cases}$$

satisfies all assertions of the theorem. □

It seems to be reasonable to ask whether f_j in Theorem 3.1 can be Borel if $j\colon\mathbb{R}\to\mathcal{K}_0$ is Borel. However it is easy to see that if a Borel function $f\colon\mathbb{R}\to\mathbb{R}$ has the CIVP, then it has also the SCIVP. (See e.g., [8, p. 500].) The case of SCIVP functions will be covered in the next section.

[1] Although Theorem 3.1 implies Theorem 2.1, their proofs are different and we used the proof of Theorem 2.1 in Theorem 2.2.

4 SCIVP Functions

The analog of Theorem 3.1 does not hold. This follows from the following analog of Proposition 1.1.

Theorem 4.1. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a countable-to-1 SCIVP function. If $f^{-1}(y)$ is closed for every $y\in\mathbb{R}$ then f is continuous.*

PROOF. Suppose that f is discontinuous at some $x\in\mathbb{R}$ from the right. We can assume that $\limsup_{h\to 0^+} f(x+h)=L>f(x)$. Choose $M\in(f(x),L)$, $m\in(f(x),M)$, and $x_0\in(x,x+1)$ such that $f(x_0)>M$. Let $Q_0\subset(m,M)\subset(f(x),f(x_0))$ be perfect. By SCIVP there exists a perfect set $P_0\subset(x,x_0)$ such that $f|P_0$ is continuous and $f[P_0]\subset Q_0$. Observe that $|f[P_0]|=\mathfrak{c}$; so we can choose a perfect subset Q_1 of $f[P_0]\subset Q_0$. Next find $x_1\in(x,x+1/2)$ such that $(x,x_1)\cap P_0=\emptyset$ and $f(x_1)>M$. Then $Q_1\subset Q_0\subset(m,M)\subset(f(x),f(x_1))$. Thus, by SCIVP we can find perfect sets $P_1\subset(x,x_1)$ and $Q_2\subset f[P_1]\subset Q_1$. In this way for every $n<\omega$, $n>0$, we define by induction

- $x_n\in(x,x+1/(n+1))$ such that $(x,x_n)\cap P_{n-1}=\emptyset$ and $f(x_n)>M$,
- perfect sets $P_n\subset(x,x_n)$ and $Q_n\subset f[P_n]\subset Q_{n-1}\subset(m,M)$.

Let $y\in\bigcap_{n<\omega}Q_n$. Then $f^{-1}(y)\cap P_n\neq\emptyset$ for every n; so x belongs to the closure of $f^{-1}(y)$. But $x\notin f^{-1}(y)$, since $f(x)<m<y$, a contradiction. $\square$

Corollary 4.2. *If $f\colon\mathbb{R}\to\mathbb{R}$ is SCIVP and finite-to-1 then it is continuous.*

Clearly there exist discontinuous SCIVP functions which are ω-to-1. For example, the function

$$f(x)=\begin{cases}\sin(1/x) & \text{for } x\neq 0\\ 0 & \text{for } x=0\end{cases}$$

has this property.

Proposition 4.3. *There exists an ω-to-1 SCIVP function $f\colon\mathbb{R}\to\mathbb{R}$ that is nowhere continuous.*

PROOF. Let $\langle P_n\rangle_n$ be a sequence of pairwise disjoint nowhere dense perfect sets such that every non-degenerate interval contains some P_n. For every n let $\hat{P}_n=P_n\setminus\{\min(P_n),\max(P_n)\}$ and let f_n be a continuous non-decreasing Cantor-like function from $\hat{P}_n$ onto $\mathbb{R}$ that is $\le$ 2-to-1. Moreover, let g be an injection from $\mathbb{R}\setminus\bigcup_n\hat{P}_n$ onto $\mathbb{R}$. Put $f=g\cup\bigcup_n f_n$. Then

- f maps intervals onto the whole real line; so it is nowhere continuous,

- f is ω-to-one, and
- f has SCIVP. Indeed, let $a < b$, $K \subset (f(a), f(b))$ be a perfect set, and $P_n \subset (a, b)$. Then there exists a perfect set $C \subset P_n$ with $f[C] \subset K$. $\square$

Also, it is well-known that there exist SCIVP functions that are $\mathfrak{c}$-to-1. (Actually there are continuous functions with this property.) Moreover there exist nowhere continuous SCIVP functions $f\colon \mathbb{R} \to \mathbb{R}$ that are $\mathfrak{c}$-to-1. An example of such a function can be found in [1]. For the sake of completeness we will repeat here an easy construction of such a function.

Proposition 4.4. *There exists a $\mathfrak{c}$-to-1 SCIVP function $f\colon \mathbb{R} \to \mathbb{R}$ that is nowhere continuous.*

Proof. Let $\mathcal{P}$ be a family of pairwise disjoint perfect sets with the property that $|\mathbb{R} \setminus \bigcup \mathcal{P}| = \mathfrak{c}$ and $|\{P \in \mathcal{P}\colon P \subset (a, b)\}| = \mathfrak{c}$ for every $a < b$. Let $\{\langle J_\xi, r_\xi \rangle\colon \xi < \mathfrak{c}\}$ be an enumeration of $\{(a, b)\colon a < b\} \times \mathbb{R}$. Choose pairwise disjoint sets $P_\xi \in \mathcal{P}$ such that $P_\xi \subset J_\xi$ for every $\xi < \mathfrak{c}$ and define f_0 on $\bigcup_{\xi<\mathfrak{c}} P_\xi$ by making $f_0|P_\xi \equiv r_\xi$. It is easy to see that any extension $f\colon \mathbb{R} \to \mathbb{R}$ of f_0 has the SCIVP and is nowhere continuous. $\square$

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