Real Analysis Exchange
Vol. 23(2), 1997–1998, pp. 641–651

F. S. Cater, Department of Mathematics, Portland State University, Portland, Oregon 97207, USA

ON DERIVATIVES VANISHING ALMOST EVERYWHERE ON CERTAIN SETS

Abstract

Let g be a measurable real valued function on a bounded, measurable subset of the real line. We prove that if $g(E)$ has measure 0, then 0 is one of the derived numbers of g at almost every point in E. We find a function H on the real line that is nondecreasing and closely associated with G, such that if $g(E)$ has measure 0, the H' vanishes almost everywhere. Moreover, if g is an N-function on E and if H' vanishes almost everywhere, then $g(E)$ has measure 0.

1

In this paper g is a measurable function on a bounded measurable set E of real numbers. We let m denote Lebesgue measure and m_e denote Lebesgue exterior measure. From [K] or [SV] we deduce that if g is differentiable almost everywhere on E and if $m(g(E)) = 0$, then $g' = 0$ almost everywhere on E. Moreover, if g is an N-function (this means g maps subsets of E of measure zero to sets of measure zero) and if g has zero derivative almost everywhere on E, then $m(g(E)) = 0$. These results have application, for example, to variations on the chain rule of differentiation and the change of variables formula of integration (consult [F] and [SV]).

Approximate differentiation [S, chapters VII and IX] is important in real analysis. In section 2, we prove that these results hold when derivatives are replaced by approximate derivatives. We offer (See also [F, Lemma K] and [El, page 489] the following theorem.

Theorem 2.1. *Let g be approximately differentiable almost everywhere on E. We have:*

(1) if $m(g(E)) = 0$, then $g'_{ap} = 0$ almost everywhere on E,

Key Words: derivative, approximate derivative, derived number, N-function, measure, bounded variation
Mathematical Reviews subject classification: 26A24, 28A20
Received by the editors December 17, 1997

(2) if g is an N-function on E, and if $g'_{ap} = 0$ almost everywhere on E, then $m(g(E)) = 0$.

We say that a point $x_o \in E$ is a *knot* point of g if $D^+g(x_o) = D^-g(x_o) = \infty$ and $D_+g(x_o) = D_-g(x_o) = -\infty$ where D^+g denotes the upper right Dini derivative of g relative to E, etc.

We deduce from [S, Theorem 10.1, chapter IX] that for almost every $x \in E$, either x is a knot point of g or g is approximately differentiable at x. Immediately from Theorem 2.1 we obtain:

Corollary 2.2. *Let $m(g(E)) = 0$. Then almost every $x \in E$ is either a knot point of g or $g'_{ap}(x) = 0$.*

Corollary 2.3. *Let $g'_{ap} \neq 0$ almost everywhere on E, and let g be a one-to-one function on E. Then g^{-1} is an N-function on $g(E)$.*

When we use derived numbers [N, chapter VIII, p. 207] relative to E, we can delete the differentiation hypothesis altogether. In section 3, we offer:

Theorem 3.1. *Let $m(g(E)) = 0$. Then 0 is a derived number of g at almost every $x \in E$.*

An immediate consequence of this is:

Corollary 3.2. *Let g be one-to-one on E, and let all the derived numbers of g be nonzero at almost every $x \in E$. Then g^{-1} is an N-function on $g(E)$.*

Apparently neither Theorem 2.1(1) nor Theorem 3.1 implies the other, although they each imply part of the result cited in [SV].

In section 4 we try to link zero derivatives with $m_e(g(E))$ when g is measurable. The obvious problem is that g need not be differentiable, so we use derivatives of a function closely associated with g. For each real number y, let $H(y) = m\big(\{t \in E : g(t) < y\}\big)$. Then $H(y)$ is a nondecreasing function of y mapping $\mathbb{R}$ into the interval $[0, m(E)]$. We offer:

Theorem 4.1. *We have:*

(1) if $m(g(E)) = 0$, then $H' = 0$ almost everywhere on $\mathbb{R}$;

(2) if $H' = 0$ almost everywhere on $\mathbb{R}$, and if g is an N-function on E, then $m(g(E)) = 0$.

We also find use for infinite derivatives. Put

$$T = \left\{t \in E : m(g^{-1}(g(t))) = 0\right\}.$$

We offer:

Theorem 4.2. *We have:*

(1) if $m(g(E)) = 0$, then $H'(g(t)) = \infty$ for almost every $t \in T$;

(2) if $H'(g(t)) = \infty$ for almost every $t \in T$, and if g is an N-function on E, then $m(g(E)) = 0$.

Now for $t \in E$, put

$$K(t) = \limsup_{r \downarrow 0} r\Big(H\big(g(t) + r\big) - H\big(g(t)\big)\Big)^{-1}.$$

We offer:

Theorem 4.3. *Let g be an N-function on E. Then K is a measurable extended real valued function that is finite almost everywhere on E. Moreover,*

$$m(g(E)) = \int_E K(t)\, dt. \qquad (*)$$

Thus we found a function K closely associated with g for which equation $(*)$ holds.

2

To prove Theorem 2.1, let g be approximately differentiable almost everywhere on E. Say $A = \{x \in E : g \text{ is approximately differentiable at } x\}$ where $m(E \setminus A) = 0$. From [S, Theorem 10.8, chapter VII] we deduce that there is a sequence of sets $A_1, A_2, A_3, \ldots$ where $A = \cup_n A_n$ and g is of bounded variation on each A_n. Fix n and let f denote the restriction of g to A_n. It follows that $f'_{A_n}(x) = g'_{ap}(x)$ at any point of density x of A_n, and thus $f'_{A_n} = g'_{ap}$ almost everywhere on A_n.

To prove part (1) assume $m(g(E)) = 0$. Then $m(f(A_n)) = 0$ and by [SV] we have $f'_{A_n} = 0$ almost everywhere on A_n. It follows that $g'_{ap} = 0$ almost everywhere on A_n. But n was arbitrary, so $g'_{ap} = 0$ almost everywhere on A and on E. □

To prove part (2) assume g is an N-function on E and $g'_{ap} = 0$ almost everywhere on E. Let f and A_n be as in the preceding paragraph. Then f is an N-function on A_n and $f'_{A_n} = 0$ almost everywhere on A_n. By [SV] we have $m(f(A_n)) = 0$ and hence $m(g(A_n)) = 0$. But n is arbitrary, so $m(g(A)) = 0$. Finally, $m(E \setminus A) = m(g(E \setminus A)) = 0$ because g is an N-function on E. It follows that $m(g(E)) = 0$. □

The proof of Corollary 2.2 was essentially given in section 1, so we omit it here.

To prove Corollary 2.3, let B be a subset of $g(E)$ with $m(B)=0$. There is a set C that is the intersection of countably many open sets in $\mathbb{R}$ such that $B \subset C$ and $m(C)=0$. Then $g^{-1}(C)$ is measurable because g is a measurable function on E. By Theorem 2.1, $g'_{ap}=0$ almost everywhere on $g^{-1}(C)$ and from the hypothesis we deduce that $m\big(g^{-1}(C)\big)=0$. But $B \subset C$ so $m\big(g^{-1}(B)\big)=0$. □

3

We begin this section with a lemma that may be of some interest in its own right.

Lemma I. *Let $p \geq 1$ and let $m(E) > p^2 m_e(g(E))$. Then there is a measurable subset A of E such that $m(A) > (1-p^{-1})m(E)$ and for each $x \in A$ there is a $u \in E$ (depending on x) with $\big|g(x)-g(u)\big| < 2p^{-1}|x-u|$.*

Proof. Let $I_1, I_2, I_3, \dots$ be a sequence of mutually disjoint open intervals covering $g(E)$ such that $\sum_n m(I_n) < p^{-2}m(E)$. Let $J_1, J_2, J_3, \dots$ be those intervals I_n for which $m\big(g^{-1}(I_n)\big) > p \cdot m(I_n)$, and let $K_1, K_2, K_3, \dots$ be the remaining I_n. Now g is measurable, so

$$m\Big(\cup_j\, g^{-1}(K_j)\Big) = \sum_j m\big(g^{-1}(K_j)\big) \leq p \cdot \sum_j m(K_j)$$

by the choice of the K_j. But $\sum_j m(K_j) \leq \sum_n m(I_n) < p^{-2}m(E)$, so

$$m\Big(\cup_j\, g^{-1}(K_j)\Big) < p^{-1}m(E)\,. \tag{1}$$

Also by (1),

$$m(E) = m\Big(\cup_n\, g^{-1}(I_n)\Big) = m\Big(\cup_j\, g^{-1}(J_j)\Big) + m\Big(\cup_j\, g^{-1}(K_j)\Big) <$$

$$< m\Big(\cup_j\, g^{-1}(J_j)\Big) + p^{-1}m(E)\,,$$

so

$$m\Big(\cup_j\, g^{-1}(J_j)\Big) > \big(1-p^{-1}\big)m(E)\,. \tag{2}$$

It remains to prove that $\cup_j g^{-1}(J_j)$ suffices for A. Let $x \in \cup_j g^{-1}(J_j)$. Say $x \in g^{-1}(J_N)$ and $g(x) \in J_N$. Recall that by the choice of J_N,

$$m\big(g^{-1}(J_N)\big) > p \cdot m(E)\,. \tag{3}$$

There are points $u, v \in g^{-1}(J_N)$ such that

$$|u - v| > p \cdot m(J_N)\,. \tag{4}$$

Moreover, $g(u)$, $g(v)$ $g(x) \in J_N$ and because J_N is an open interval,

$$\big|g(x) - g(u)\big| < m(J_N) \quad \text{and} \quad \big|g(x) - g(v)\big| < m(J_N)\,. \tag{5}$$

Now by (4), $|x-u|+|x-v| \geq |u-v| > p \cdot m(J_N)$, so either $|x-u| > p \cdot m(J_N)/2$ or $|x - v| > p \cdot m(J_N)/2$. Then by (5), either $|x - u| > p|g(x) - g(u)|/2$ or $|x - v| > p|g(x) - g(v)|/2$. □

To prove Theorem 3.1, for each positive integer i partition E into finitely many mutually disjoint measurable sets $E_{i1}, E_{i2}, E_{i3}, \ldots$, each of diameter $< 2^{-i}$. By hypothesis, $m(g(E_{ij})) = 0$ for all i and j. We deduce from Lemma I, there is measurable set $A_{ij} \subset E_{ij}$ such that $m(E_{ij} \setminus A_{ij}) < 2^{-i-j} m(E_{ij})$ and for each $x \in A_{ij}$ there is a $u \in E_{ij}$ with $|g(x) - g(u)| < 2^{-i}|x - u|$. We leave the proof that 0 is a derived number of g at each point in $B = \cap_{k=1}^{\infty} \cup_{i=k}^{\infty} \cup_j A_{ij}$ and $m(E \setminus B) = 0$. □

The proof of Corollary 3.2 is analogous to the proof of Corollary 2.3, so we leave it.

4

We begin with a lemma to dispose of certain details.

Lemma II. *If $S \subset T$, and $m(H(g(S))) = 0$, then $m(S) = 0$. Moreover, if at each $x \in S$ either $H'(g(x)) = 0$ or H does not have a finite or infinite derivative at $g(x)$, then $m(S) = 0$.*

Proof. Let $m(H(g(S))) = 0$. Choose $\epsilon > 0$. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ be a sequence of open intervals covering $H(g(S))$ with $\sum_i (b_i - a_i) < \epsilon$. Thus

$$H\big(g(S)\big) \subset \cup_i (a_i, b_i)\,. \tag{1}$$

For each index i, put $S_i = \{s \in S : H(g(s)) \in (a_i, b_i)\}$. Let $u_1, u_2 \in S_i$ for some index i, where $H(g(u_1)) \leq H(g(u_2))$. Then $a_i < H(g(u_1)) \leq H(g(u_2)) < b_i$. So

$$a_i < m\big\{t \in E : g(t) < g(u_1)\big\} \leq m\big\{t \in E : g(t) < g(u_2)\big\} < b_i\,.$$

Because g is measurable,

$$\begin{aligned} &m\big\{t \in E : g(u_1) \leq g(t) < g(u_2)\big\} \\ =&m\big\{t \in E : g(t) < g(u_2)\big\} - m\big\{t \in E : g(t) < g(u_1)\big\}\,. \end{aligned}$$

So

$$\{t \in E : g(u_1) \le g(t) < g(u_2)\} < b_i - a_i\,. \tag{2}$$

But $u_2 \in T$, so $m\{t \in E : g(t) = g(u_2)\} = 0$, and by (2)

$$m\{t \in E : g(u_1) \le g(t) \le g(u_2)\} < b_i - a_i\,. \tag{3}$$

It is not difficult to see that

$$m(S_i) \le (b_i - a_i)\,. \tag{4}$$

(Just let $g(u_1)$ tend to $\inf g(S_i)$ and $g(u_2)$ tend to $\sup g(S_i)$, etc.)
It follows from (1) and (4) that

$$m(S) \le \sum_i m(S_i) \le \sum_i (b_i - a_i) < \epsilon\,. \tag{5}$$

Finally, ϵ is arbitrary, so $m(S) = 0$. Put

$$S' = \{s \in S : H'(g(s)) = 0\},$$

$$S'' = \{s \in S : H \text{ has no finite or infinite derivative at } g(s)\}.$$

Then $m(H(g(S'))) = 0$. By de la Vallée Poussin's Theorem (see for example [S, Theorem (9.1), chapter IV]), we see that $m(H(g(S''))) = 0$.

To prove the second statement in Lemma II, assume $S = S' \cup S''$. Hence $m(H(g(S))) \le m(H(g(S'))) + m(H(g(S''))) = 0$. So $m(H(g(S))) = 0$. By the previous part, $m(S) = 0$. □

We turn now to the theorems in section 4.

Proof of Theorem 4.1(1). Let $m(g(E)) = 0$. Let $\epsilon > 0$. Let $I_1, I_2, I_3, \ldots$ be mutually disjoint open intervals covering $g(E)$ such that $\sum_j m(I_j) < \epsilon$. Select an index N so that $\sum_{j=N+1}^{\infty} m(g^{-1}(I_j)) < \epsilon^2$. Then

$$m\Big(\cup_{j=N+1}^{\infty} g^{-1}(I_j)\Big) < \epsilon^2\,. \tag{6}$$

Let $[a_1, b_1], [a_2, b_2], [a_3, b_3], \ldots$ be mutually disjoint closed intervals, each disjoint from $\cup_{j=1}^{N} I_j$. By (6) and the definition of H, we have $\sum_j \big(H(b_j) - H(a_j)\big) < \epsilon^2$. By [HS, Theorem (18.14), chapter V],

$$\sum_j \int_{a_j}^{b_j} H'(x)\,dx \le \sum_j \big(H(b_j) - H(a_j)\big) < \epsilon^2\,. \tag{7}$$

Let $D = \{x : H'(x) > \epsilon\}$. We deduce from (7) that

$$m\Big(D \cap \big(\cup_j [a_j, b_j]\big)\Big) < \epsilon . \tag{8}$$

From (8) we deduce that $m\big(D \setminus (\cup_{j=1}^{N} I_j)\big) \le \epsilon$. But $m\big(\cup_{j=1}^{N} I_j\big) < \epsilon$, so

$$m(D) < 2\epsilon . \tag{9}$$

Because ϵ is arbitrary, we conclude that $m\big\{x : H'(x) > 0\big\} = 0$. □

Proof of Theorem 4.1(2). Let $H' = 0$ almost everywhere, and let g be an N-function on E. Let $S = \{s \in T : H'(g(s)) = 0\}$. By Lemma II, $m(S) = 0$. Because g is an N-function on E, $m(g(S)) = 0$. Now $H' = 0$ almost everywhere, so $m(g(T \setminus S)) = 0$, and hence $m(g(T)) = 0$.

Moreover, $g^{-1}(y)$ can have positive measure for at most countably many y, so $g(E \setminus T)$ is countable. Finally, $m(g(E)) = 0$. □

Proof of Theorem 4.2(1). Let $m(g(E)) = 0$. Put

$T_1 = \{t \in T : H$ has no finite or infinite derivative at $g(t)\}$,

$T_2 = \{t \in T : H$ has a finite derivative at $g(t)\}$.

By Lemma II, $m(T_1) = 0$. We deduce from [S, Theorem (4.5), chapter IX] and $m\big(g(T_2)\big) = 0$ that $m\big(H(g(T_2)\big) = 0$. By Lemma II, $m(T_2) = 0$. So $m(T_1 \cup T_2) = 0$, and $t \in T \setminus (T_1 \cup T_2)$ implies $H'(g(t)) = \infty$. □

Proof of Theorem 4.2(2). Let $H'(g(t)) = \infty$ for almost every $t \in T$ and let g be an N-function on E. Put $T_0 = \{t \in T : H'(g(t)) = \infty\}$. Then $m(g(T_0)) = 0$ by [S, Theorem (4.4), chapter IX]. But $m(T \setminus T_0) = 0$ by hypothesis and hence $m\big(g(T \setminus T_0)\big) = 0$ because g is an N-function on E. It follows that $m(g(T)) = 0$. We recall that $g(E \setminus T)$ is a countable set, so $m(g(E)) = 0$. □

For our last result, we need more lemmas.

Lemma III. *Let g be an N-function on E. Then there exists a measurable set $P \subset T$ such that*

(i) $K(t) = 0$ *for almost every* $t \in E \setminus P$,

(ii) $m\big(g(E \setminus P)\big) = 0$,

(iii) $0 < K(t) = 1/H'\big(g(t)\big) < \infty$ *for every* $t \in P$.

Proof. If $t \in E \setminus T$, it follows that

$$H\big(g(t)+h\big) - H\big(g(t)\big) \geq m\big(g^{-1}(g(t))\big) > 0$$

for any $h > 0$, and it follows from the definition of K that $K(t) = 0$. We recall that $g(E \setminus T)$ is countable, so $m(g(E \setminus T)) = 0$. Now put

$$T_3 = \{t \in T : H \text{ has no finite or infinite derivative at } g(t)\},$$

$$T_4 = \{t \in T : H'(g(t)) = 0\},$$

$$T_5 = \{t \in T : H'(g(t)) = \infty\}.$$

By Lemma II, $m(T_3) = m(T_4) = 0$, and because g is an N-function on E, $m(g(T_3)) = m(g(T_4)) = 0$. For $t \in T \setminus (T_3 \cup T_4)$ it follows that H has a positive finite or infinite derivative at $g(t)$, and it follows from the definition of K that $K(t) = 1/H'(g(t))$ (here $0 = 1/\infty$). But $H(g(t))$ and $H\big(g(t)+h\big)$ are measurable functions of t because g is measurable and H is monotonic. We deduce that K is measurable on $T \setminus (T_3 \cup T_4)$. Then T_5 is a measurable set. By [S, Theorem (4.4), chapter IX], $m(g(T_5)) = 0$. By the definition of K, $K(t) = 0$ for any $t \in T_5$.

Put $P = T \setminus (T_3 \cup T_4 \cup T_5)$. Then P is measurable because T and the T_i are measurable. Finally, (i), (ii) and (iii) follow from the preceding paragraph. $\square$

It is well-known that if g is a measurable N-function on E, then $g(E)$ is measurable. It follows that $g(P)$ is measurable in Lemma III.

Lemma IV. *Let g be an N-function on E. Let c, d, u be real numbers such that $u > 0$ and $0 < c < d$. Let L be a closed set such that for every $x \in L$ and y satisfying $x < y < x + u$, we have $c(y - x) \leq H(y) - H(x) \leq d(y - x)$. Then*

$$c \cdot m(L) \leq m\big(g^{-1}(L)\big) \leq d \cdot m(L)\,.$$

Proof. Let n be an integer with $n^{-1} < u$. Cover L with countably many mutually disjoint half open intervals $[a_1, b), [a_2, b_2), [a_3, b_3), \ldots$ so that $b_i - a_i < n^{-1}$ and $a_i \in L$ for each i. Let $U_n = \cup_i [a_i, b_i)$. It follows that

$$c(b_i - a_i) \leq H(b_i) - H(a_i) \leq d(b_i - a_i) \quad \text{for each } i\,,$$

and hence

$$c(b_i - a_i) \leq m\big(g^{-1}[a_i, b_i)\big) \leq d(b_i - a_i)\,. \tag{10}$$

It follows that

$$c \cdot \sum_i (b_i - a_i) \leq \sum_i m\big(g^{-1}([a_i, b_i))\big) \leq d \cdot \sum_i (b_i - a_i)$$

and

$$c \cdot m(U_n) \leq m\big(g^{-1}(U_n)\big) \leq d \cdot m(U_n)\,. \tag{11}$$

By inductive construction, we choose U_n so that $U_n \subset U_{n-1}$ for all $n > 1+u^{-1}$. The distance from the closed set L to any point in U_n cannot exceed n^{-1}. Hence $\cap_n U_n = L$. From (11) we deduce $c \cdot m(L) \leq m\big(g^{-1}(L)\big) \leq d \cdot m(L)\,.$ □

Lemma V. *Let g be an N-function on E. Let c, d, u be real numbers such that $u > 0$ and $0 < c < d$. Let $L_1 = \{x \in g(P) :$ for any y such that $x < y < x+u$ we have $c(y-x) \leq H(y) - H(x) \leq d(y-x)\}$. Then*

$$c \cdot m(L_1) \leq m\big(g^{-1}(L_1)\big) \leq d \cdot m(L_1)\,.$$

Proof. If $x \in g(P)$ and x is in the closure of L_1 and if H is continuous at x, it is easy to see that $x \in L_1$. So we leave the proof that L_1 is measurable. Say $L_1 = M_0 \cup M_1 \cup M_2 \cup M_3 \cup \ldots$ where $m(M_0) = 0$, $M_1 \subset M_2 \subset M_3 \subset \ldots$, and each M_i $(i > 0)$ is closed. Now $M_0 \subset g(P)$ and because H is differentiable at each point of M_0, we have $m(H(M_0)) = 0$. Then by Lemma II, $m(g^{-1}(M_0)) = 0$. By Lemma IV, $c \cdot m(M_i) \leq m(g^{-1}(M_i)) \leq d \cdot m(M_i)$ for each $i > 0$. It follows that

$$c \cdot m\big(M_0 \cup M_1 \cup M_2 \cup \ldots\big) \leq m\big(g^{-1}(M_0 \cup M_1 \cup M_2 \cup \ldots)\big) \leq$$
$$\leq d \cdot m\big(M_0 \cup M_1 \cup M_2 \cup \ldots\big)\,,$$

or in other words $c \cdot m(L_1) \leq m\big(g^{-1}(L_1)\big) \leq d \cdot m(L_1)\,.$ □

In the next lemma, we can see the proof of Theorem 4.3 emerging.

Lemma VI. *Let g be an N-function on E. Let c, d be real numbers such that $0 < c < d$ and let $V = \{x \in g(P) : c < H'(x) < d\}$. Then*

$$c \cdot m(V) \leq m\big(g^{-1}(V)\big) \leq d \cdot m(V)\,.$$

Proof. For indices i, j, put $V_{ij} = \{x \in V :$ for any y such that $x < y < x + i^{-1}$, we have $(c + j^{-1})(y - x) \leq H(y) - H(x) \leq (d - j^{-1})(y - x)\}$. By Lemma V, we have for each i and j,

$$(c + j^{-1}) \cdot m(V_{ij}) \leq m\big(g^{-1}(V_{ij})\big) \leq (d - j^{-1}) \cdot m(V_{ij})\,. \tag{12}$$

For each j, $V_{1j} \subset V_{2j} \subset V_{3j} \subset \ldots$ and we deduce from (12) that for each j,

$$(c + j^{-1}) \cdot m(\cup_i V_{ij}) \leq m\big(g^{-1}(\cup_i V_{ij})\big) \leq (d - j^{-1}) \cdot m(\cup_i V_{ij})\,. \tag{13}$$

Moreover $\cup_i V_{i1} \subset \cup_i V_{i2} \subset \cup_i V_{i3} \subset \cup \ldots$ and we deduce from (13) that

$$c \cdot m(\cup_i \cup_j V_{ij}) \leq m\big(g^{-1}(\cup_i \cup_j V_{ij})\big) \leq d \cdot m(\cup_i \cup_j V_{ij})\,.$$

Finally, $\cup_i \cup_j V_{ij} = V$. □

Proof of Theorem 4.3. Let g be an N-function on E. Choose $\epsilon > 0$. Let $y_0, y_1, y_{-1}, y_2, y_{-2}, y_2, y_{-3}, \ldots$ be positive numbers such that $0 < y_i - y_{i-1} < \epsilon$, $m\big(\{t \in P : K(t) = y_i\}\big) = 0$ for each index i, and

$$\lim_{i \to -\infty} y_i = 0\,, \qquad \lim_{i \to \infty} y_i = \infty\,.$$

Let $P_i = \{t \in P : y_{i-1} < K(t) < y_i\}$ for each i. By Lemma III, $y_i^{-1} < H'(g(t)) < y_{i-1}^{-1}$ for $t \in P_i$. By Lemma VI and the definition of P_i, we have $P_i = g^{-1}(g(P_i))$ and

$$y_i^{-1} \cdot m\big(g(P_i)\big) \le m(P_i) \le y_{i-1}^{-1} m\big(g(P_i)\big)\,.$$

This can be rewritten

$$y_{i-1} \cdot m\big(P_i\big) \le m\big(g(P_i)\big) \le y_i \cdot m\big(P_i\big)\,.$$

But also

$$y_{i-1} \cdot m(P_i) \le \int_{P_i} K(t)\,dt \le y_i \cdot m(P_i)$$

and we combine these inequalities to obtain

$$\left| m\big(g(P_i)\big) - \int_{P_i} K(t)\,dt \right| \le (y_i - y_{i-1}) \cdot m(P_i) < \epsilon \cdot m(P_i)\,. \tag{14}$$

We sum to obtain

$$\left| \sum_{i=-\infty}^{\infty} \Big(m\big(g(P_i)\big) - \int_{P_i} K(t)\,dt \Big) \right| \le \epsilon \cdot \sum_{i=-\infty}^{\infty} m(P_i) = \epsilon \cdot m(P)\,. \tag{15}$$

It follows from (15) that $\left|m(g(P)) - \int_P K(t)\,dt\right| \le \epsilon \cdot m(P)$ if $m(g(P)) < \infty$, and $\int_P K(t)\,dt = \infty$ if $m(g(P)) = \infty$. Because ϵ is arbitrary we conclude that in any case

$$m\big(g(P)\big) = \int_P K(t)\,ft\,. \tag{16}$$

In view of Lemma III, equation $(*)$ follows from (16). □

References

[El] H. W. Ellis, *Darboux properties and applications to non-absolute convergent integrals* , Canad. J. Math., **3** (1951), 471–485.

[En] V. Ene, *Real functions-current topics* , Lect. Notes in Math., vol. 1603, Springer-Verlag, 1995.

[F] J. Foran, *A chain rule for the approximate derivative and change of variables for the* (D)*-integral* , Real Analysis Exch., **8** (1982–83), 443–454.

[HS] E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer Verlag,

[K] K. Krzyzewski, *On change of variable in the Denjoy-Perron integral (I), (II)* , Coll. Math., **9** (1962), 99–104 and 317–323. 1965.

[N] I. P. Natanson, *Theory of functions of a real variable*, Ungar, New York, 1955.

[S] S. Saks, *Theory of the integral*, 2nd. rev. ed., Dover, New York, 1964.

[SV] J. Serrin and D. E. Varberg, *A general chain rule for derivatives and the change of variable formula for the Lebesgue integral*, Amer. Math. Monthly **76** (1969), 514–520.