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DARBOUX QUASICONTINUOUS FUNCTIONS

Abstract

Let $C(f)$ denote the set of points at which a function $f : I \to I$ is continuous, where $I = [0, 1]$. We show that if a Darboux quasicontinuous function f has a graph whose closure is bilaterally dense in itself, then f is extendable to a connectivity function $F : I^2 \to I$ and the set $I \setminus C(f)$ of points of discontinuity of f is f-negligible. We also show that although the family of Baire class 1 quasicontinuous functions can be characterized by preimages of sets, the family of Darboux quasicontinuous functions cannot. An example is found of an extendable function $f : I \to \mathbb{R}$ which is not of Cesaro type and not quasicontinuous.

1 Extensions

We begin with the following definitions of classes of functions which could be stated for $\mathbb{R}$ instead of I or $\mathbb{R}^2$ instead of I^2.

D: A Darboux function $f : I \to I$ maps connected sets to connected sets, and so it has the intermediate value property.

Conn: A connectivity function $F : I^2 \to I$ has the graph of its restriction $F|C$ connected for each connected subset $C \subset I^2$. According to [15], [10], and [16], this is equivalent to

PC: $F : I^2 \to I$ is peripherally continuous if for each $x \in I^2$ and all open sets U with $x \in U$ and V with $F(x) \in V$, there exists an open set W containing x such that $W \subset U$ and $F(\operatorname{bd} W) \subset V$.

Ext: A function $g : I \to I$ is said to be extendable if there exists a connectivity function $G : I^2 \to I$ such that $G(x, 0) = g(x)$ for all $x \in I$. For such

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an extendable function g, we then say that a set $A \subset I$ is g-negligible if whenever $f : I \to I$ is such that $f = g$ on $I \setminus A$ and the graph of $f|A$ is a subset of the closure, $\overline{g}$, of the graph of g, then f is extendable, too.

AC: Every open neighborhood in I^2 of the graph of an almost continuous function $f : I \to I$ contains the graph of a continuous function $g : I \to I$.

QC: We say $f : I \to I$ is quasicontinuous if for each $x \in I$ and open sets U containing x and V containing $f(x)$, there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$. That is, $f|C(f)$ is dense in the graph of f.

DIVP: An $f : I \to I$ has the dense intermediate value property if $f(A) \in \mathcal{D}_0 = \{D \cap J : D$ is dense in I and J is a nonempty interval or singleton$\}$ for every $A \in \mathcal{D}_0$.

CT: A function $f : I \to I$ is of the Cesaro type if there exist nonempty open sets U and V in I such that for each $y \in V$, $f^{-1}(y)$ is dense in U. Note that this implies the graph of f is somewhere dense in I^2.

Let $\ell_x = \{x\} \times I$. A function $f : I \to I$ has a closure that is *bilaterally dense in itself* if for each $x \in (0,1)$, $\mathrm{cl}(f|(0,x)) \cap \ell_x = \mathrm{cl}(f|(x,1)) \cap \ell_x$. It follows from [11] that for a Darboux function $f : I \to I$, $\overline{f} \cap \ell_x$ is a connected set for each $x \in I$, and $C(f)$ is a dense G_δ subset of I if f also has a G_δ graph. Of course, a function f equals its graph $\{(x, f(x)) : x \in I\}$. Π_1 and Π_2 denote the x-projection and y-projection, respectively, of I^2 onto I. In [9], Gibson and Reclaw give an example of a Darboux quasicontinuous function $f : I \to I$ whose graph is not connected, and in [8], Gibson and Natkaniec give an example of an almost continuous quasicontinuous function $f : I \to I$ which is not extendable. Examination of many other examples in the literature revealed that whenever a Darboux quasicontinuous function f was not extendable, then the closure of its graph failed to be bilaterally dense in itself. The first theorem shows that this is always the case. Example 1 in [9] is quasicontinuous with closure bilaterally dense in itself, but is not Darboux. Kellum and Garrett's function $f : I \to [-1,1]$ in Example 1 of [12] is in AC, a G_δ set, but not of Baire class 1. Letting K denote the Cantor ternary set in I and $J = \{e_1, e_2, e_3, \dots\}$ the set of endpoints of the complementary intervals of K, they define

$$f(x) = \begin{cases} \sin \frac{1}{(m-x)(n-x)} & \text{if } x \text{ belongs to the component } (m,n) \text{ of } I \setminus K \\ 1 & \text{if } x \in K \setminus J \\ \frac{1}{r} & \text{if } x = e_r. \end{cases}$$

In [6], Gibson asks if this function is extendable. Since f is Darboux, quasicontinuous, and has a closure bilaterally dense in itself, then according to the following result, f is extendable and K is f-negligible.

Theorem 1. *If $g : I \to I$ is a Darboux quasicontinuous function whose graph has a closure that is bilaterally dense in itself, then g is extendable, and $I \backslash C(g)$ is g-negligible.*

Proof. We identify I with the subset $I \times \{0\}$ of I^2. By a "triangle" t, we mean $t = \operatorname{int}(s^2)$ (the set theoretic interior of s^2 in the space I^2), where $s^2$is a closed 2-simplex in I^2 with a 1-dimensional face lying in $I \times \{0\}$. The "base" b of t is $b = t \cap (I \times \{0\})$. For each positive integer n and $0 \leq i \leq 2^n - 1$, define $H\Big(\frac{i}{2^n}, \frac{i+1}{2^n}\Big) = \{x \in I : \overline{g} \cap \ell_x$ meets both $I \times \Big\{\frac{i}{2^n}\Big\}$ and $I \times \Big\{\frac{i+1}{2^n}\Big\}\}$. Each $H\Big(\frac{i}{2^n}, \frac{i+1}{2^n}\Big)$ is closed and nowhere dense in I, and

$$\bigcup\{H\Big(\frac{i}{2^n}, \frac{i+1}{2^n}\Big) : 0 \leq i \leq 2^n - 1\} \subset \bigcup\{H\Big(\frac{j}{2^{n+1}}, \frac{j+1}{2^{n+1}}\Big) : 0 \leq j \leq 2^{n+1} - 1\}.$$

For $n = 1, 2, 3, \ldots$ and $0 \leq i \leq 2^n - 1$, we let $T_{n,i}$ denote a finite collection of disjoint triangles t_j of diameter $< \frac{1}{n}$ in I^2 whose bases b_j form a finite collection $B_{n,i}$ of disjoint open intervals of $I \times \{0\}$ with endpoints denoted $\operatorname{endpts}(b_j) \subset C(g) \cup \{0, 1\}$ such that

$$I \times \{0\} = \bigcup\{\operatorname{cl}(b_j) : b_j \in B_{n,i}\}, \tag{1}$$

$$T_{n,k} \text{ is a refinement of } T_{n,i} \text{ for } k > i, \tag{2}$$

$$T_{n+1,0} \text{ is a refinement of } T_{n,2^n-1} \tag{3}$$

$$B_{n,k} \text{ is a refinement of } B_{n,i} \text{ for } k > i, \tag{4}$$

$$B_{n+1,0} \text{ is a refinement of } B_{n,2^n-1}, \text{ and} \tag{5}$$

$$\begin{gathered}\text{if 0 or 1 is an endpoint of } b_j, \text{ then } \operatorname{cl}(t_j) \\ \text{is a neighborhood of } (0,0) \text{ or } (1,0), \text{ respectively , in } I^2.\end{gathered} \tag{6}$$

Picture the elements of each $T_{n,i}$ arranged like adjacent teeth of a handsaw and the sawteeth of the next collection, which is either $T_{n,i+1}$ or $T_{n+1,0}$, constructed inside the sawteeth of $T_{n,i}$.

Since $B_{n,i}$ is an open cover of $H\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ with mesh $< \frac{1}{n}$,

$$\text{(7)}\quad \begin{array}{l} \text{if } x \in I \setminus C(g), \text{ then there exist } n \text{ and } i \text{ such that } x \in H\left(\dfrac{i}{2^n}, \dfrac{i+1}{2^n}\right) \\ \text{in which case } x \in b_j \text{ or } x = 0 \text{ or } x = 1 \text{ and } x \text{ is an endpoint of some} \\ b_j \in B_{n,i}, \text{ and we may assume } B_{n,i} \text{ is constructed so that} \\ g(\text{endpts}\,(b_j) \setminus \{0,1\}) \in \left[\dfrac{i}{2^n}, \dfrac{i+1}{2^n}\right] \text{ because } g \text{ has a closure that is} \\ \text{bilaterally dense in itself and } g|C(g) \text{ is dense in } g. \end{array}$$

We can define an extension $G : I^2 \to I$ of g so that for each n and i,

$$\text{(8)}\quad \begin{array}{l} \text{the variation of } G \text{ on } \text{bd}(t_j) \text{ (the set theoretic boundary in } I^2) \\ \text{is } < \dfrac{1}{n} \text{ for each } t_j \in T_{n,i}, \end{array}$$

$$\text{(9)}\quad \begin{array}{l} \text{if } x \in (I \setminus C(g) \cap H\left(\dfrac{i}{2^n}, \dfrac{i+1}{2^n}\right) \text{ and } x \in b_j \in B_{n,i}, \text{ then} \\ G(\text{bd}(t_j)) \subset [\min g(\text{endpts}\,(bj)), \max g(\text{endpts}\,(b_j))], \text{ but if } x = 0 \text{ or } 1 \\ \text{and } x = \text{ endpt}\,(b_j), \text{ then } G(\text{bd}(t_j)) = g(\text{endpts}\,(b_j) \setminus \{0,1\}). \\ \text{If } x \in C(g) \text{ and } x = 0 \text{ or } 1 \text{ and } x = \text{endpt}\,(b_j) \text{ for some } b_j \in B_{n,i}, \\ \text{then } G(\text{bd}(t_j)) \subset [\min g(\text{endpts}\,(b_j)), \max g(\text{endpts}\,(b_j))], \text{ and} \end{array}$$

$$\text{(10)}\qquad G \text{ is continuous on } I^2 \cup \{\text{cl}(t_j) : t_j \in T_{n,i}\}.$$

Here is how to obtain condition (8). Suppose E denotes the set of endpoints of all intervals belonging to $B_{n,i-1}$ along with the endpoints of just those members b_j of $B_{n,i}$ constructed as described in (7) which each contain at least one point of $H\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ and which together cover $H\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$. Suppose c and d are consecutive points of E such that no point of $H\left(\frac{i}{2^n}, \frac{i+1}{2^n}\right)$ lies between c and d. Because of (7) and (9), we may as well suppose that if $c = 0$, then $c \in C(g)$. Even though $|g(d) - g(c)|$ might not be a small value, a finite number of triangles of diameter $< \frac{1}{n}$ belonging to $T_{n,i}$ can be constructed as follows forming sawteeth from c to d so that the total variation of G on the slanted sides of each of the triangles will be less than $\frac{1}{n}$. First choose a partition $P = \{x_0 = c, x_1, x_2, \ldots, x_k = d\}$ of $[c,d]$ in $C(g)$ with norm less than $\frac{1}{n}$. Next, since g is Darboux, for $m = 1, 2, \ldots, k$, there exists a finite (possibly very irregular) partition P_m of $[x_{m-1}, x_m]$ such that $g|P_m$ is monotone and $|g(y) - g(x)| < \frac{1}{n}$ for each pair of consecutive points x and y in P_m. $P \cup \bigcup_{m=1}^{k} P_m$ partitions $[c, d]$ into subintervals whose interiors are to belong to $B_{n,i}$ and are bases of a sawtooth collection of triangles of diameter $< \frac{1}{n}$ that are to belong to $T_{n,i}$. Then the extension G of g can be defined to be piecewise linear and of total variation $< \frac{1}{n}$ on the slanted sides of each triangle in this collection. We now show that $G : I^2 \to I$ is in PC and hence in Conn. We only have to check peripheral continuity on $I \times \{0\}$ because according to (10), G is continuous on $I^2 \setminus (I \times \{0\})$. Let $\epsilon > 0$.

Case 1: $x \in I \setminus C(g)$. Then by (7) and (9), G is peripherally continuous at $(x, 0)$.

Case 2: $x \in C(g)$ and x is an endpoint of an interval belonging to some $B_{m,i}$. If $x = 0$ or 1, then by (9), G is peripherally continuous at $(x, 0)$. Therefore suppose $x \neq 0, 1$. Then x is an endpoint of adjacent intervals b_j and b_k in $B_{n,p}$ for each $B_{n,p} \in \{B_{m,i}, B_{m,i+1}, \cdots, B_{m+1,0}, B_{m+1,1}, \cdots\}$. There exists an $n \geq m$ such that $\frac{2}{n} < \epsilon$, $t_j \cup t_k$ has diameter $< \frac{2}{n}$, and the variation of G on $\mathrm{bd}(t_j \cup t_k)$ is $< \frac{2}{n}$. Since by (10), G restricted to $I^2 \setminus \{\mathrm{cl}(t_j) : t_j \in T_{n,p}\}$ is continuous at $(x, 0)$, there exists an open semicircular disk D in I^2 having $(x, 0)$ at the center of its diameter and not containing the other vertices of t_j and t_k such that the diameter of the open neighborhood $W = t_j \cup t_k \cup D$ of $(x, 0)$ in I^2 is $< \frac{2}{n}$ and $\mathrm{diam}(\{G(x, 0)\} \cup G(bd(W))) < \frac{2}{n} < \epsilon$. This shows G is peripherally continuous at $(x, 0)$.

Case 3: $x \in C(g)$ and x is not an endpoint of any b_j in any $B_{n,i}$. For each n and i, there exists $b_j \in B_{n,i}$ such that $(x, 0) \in b_j$. Let $\{a_j\}$ be a sequence whose jth term a_j is an endpoint of b_j in $C(g)$. Then $a_j \to x$ and $G(a_j, 0) \to G(x, 0)$. Since the variation of G on $\mathrm{bd}(t_j)$ is $< \frac{1}{n}$, G is peripherally continuous at $(x, 0)$.

To show $I \setminus C(g)$ is g-negligible, suppose $f : I \to I$ with $f = g$ on $C(g)$ and $f|(I \setminus C(g)) \subset \overline{g}$. Since $g|C(g)$ is dense in g and since $f = g$ on $C(g)$, $\overline{f} = \overline{g}$. We show that

$$F(x,t) = \begin{cases} G(x,t) & \text{on } I^2 \setminus ((I \setminus C(g)) \times \{0\}) \\ f(x) & \text{on } (I \setminus C(g)) \times \{0\} \end{cases}$$

is a peripherally continuous extension of f. F is peripherally continuous at each point of $I^2 \setminus ((I \setminus C(g)) \times \{0\})$ because $F = G$ on this set, which contains $\mathrm{bd}(W)$ in case 2 and contains $\mathrm{bd}(t_j)$ in case 3. Let $x \in I \setminus C(g)$, $\epsilon > 0$ and $\delta > 0$. There exist an n and i such that $\frac{1}{n} < \delta$ and

$$\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \subset (F(x,0) - \epsilon, F(x,0) + \epsilon) \cap \Pi_2(\overline{g} \cap \ell_x).$$

Then for some j, $x \in b_j \in B_{n,i}$ and $F(\mathrm{bd}(t_j)) = G(\mathrm{bd}(t_j)) \subset \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right] \subset (F(x,0)-\epsilon, F(x,0)+\epsilon)$. Therefore F is peripherally continuous at each point of $(I \setminus C(g)) \times \{0\}$, too, and so $I \setminus C(g)$ is g-negligible. □

A function $f : I \to \mathbb{R}$ belongs to B_1^*, the class Baire*1, if each perfect set in I contains a portion on which the restriction of f is continuous. In the space DB_1 of Darboux Baire class 1 functions $f : I \to \mathbb{R}$ with the metric $d(f,g) = \min\{1, \sup|f(x) - g(x)|\}$ of uniform convergence, let $\mathcal{G}$ be the subspace of quasicontinuous functions, and let $\mathcal{G}_0$ be the subspace of quasicontinuous functions having closures that are bilaterally dense in themselves. In [5], Darji, Evans, and O'Malley show that $\mathcal{G}$ is closed and nowhere dense in DB_1 and that DB_1^* is of the first category in $\mathcal{G}$. $\mathcal{G}_0$ is closed in DB_1 and a proof similar to theirs would show that the subspace of DB_1^* consisting of functions whose closures are bilaterally dense in themselves is of first category in $\mathcal{G}_0$.

2 Preimages

For $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$, the family of all subsets of $\mathbb{R}$, let $C_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} :$ for every $A \in \mathcal{A}, f(A) \in \mathcal{B}\}$ and $C^{-1}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} :$ for every $B \in \mathcal{B}, f^{-1}(B) \in \mathcal{A}\}$. A family $\mathcal{F}$ of real functions is *characterizable by images of sets* if $\mathcal{F} = C_{\mathcal{A},\mathcal{B}}$ and *by preimages of sets* if $\mathcal{F} = C^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$. The class QC of all quasicontinuous functions is characterizable by preimages [13] but not by images [3]. We examine the classes $\mathrm{QC} \cap \mathrm{B}_1$, $\mathrm{QC} \cap \mathrm{DIVP}$, and $\mathrm{QC} \cap \mathrm{D}$.

Theorem 2. *$QC \cap B_1$ is characterizable by preimages of sets.*

PROOF. Let

$$\mathcal{A} = \{A \subset \mathbb{R} : A \text{ is an } F_\sigma \text{ set and for every interval } (a,b) \text{ meeting } A, \\ (a,b) \cap A \text{ contains a somewhere dense } G_\delta \text{ subset of } \mathbb{R}\}$$

and let $\mathcal{B}$ be the family of all open intervals (c,d) in $\mathbb{R}$. If $f \in \mathrm{B}_1$, then $A = f^{-1}(c,d)$ is an F_σ set and if $f \in \mathrm{QC}$, then each nonempty set $(a,b) \cap f^{-1}(c,d)$ contains a somewhere dense G_δ subset (of continuities of f). Therefore $\mathrm{QC} \cap \mathrm{B}_1 \subset C^{-1}_{\mathcal{A},\mathcal{B}}$. Now suppose $f \in C^{-1}_{\mathcal{A},\mathcal{B}}$. Then for every (a,b) and (c,d), if $(a,b) \cap f^{-1}(c,d)$ is nonempty, then it contains a somewhere dense G_δ subset G of $\mathbb{R}$. Since $f^{-1}(c,d)$ is an F_σ set for each $(c,d) \in \mathcal{B}$, $f \in \mathrm{B}_1$ and therefore $C(f)$ is a dense G_δ set. By the Baire Category Theorem, $C(f) \cap G$ is a somewhere dense G_δ subset of $\mathbb{R}$. It follows that $f \in \mathrm{QC}$. □

In [4], Ciesielski and Natkaniec show that the class DIVP cannot be characterized by preimages. Their exact same proof verifies the next result because the functions f_0 and f_1 they construct there in DIVP are also in QC.

Theorem 3. *$QC\cap DIVP$ cannot be characterized by preimages of sets.*

Theorem 4. *$QC\cap D$ cannot be characterized by preimages.*

Proof. Assume, otherwise, that there exist $\mathcal{A},\mathcal{B}\subset\mathcal{P}(\mathbb{R})$ such that QC$\cap$D$=C^{-1}_{\mathcal{A},\mathcal{B}}$. We may suppose that $\mathcal{A}=\{f^{-1}(B):f\in$QC$\cap$D and $B\in\mathcal{B}\}$ and $\mathcal{B}\not\subset\{\emptyset,\mathbb{R}\}$. Let $B\in\mathcal{B}\setminus\{\emptyset,\mathbb{R}\}$. Let $\{d_n:n=0,1,2,\dots\}$ be a dense sequence in B and $\{e_n:n=0,1,2,\dots\}$ be a dense sequence in $\mathbb{R}\setminus B$. C denotes the Cantor ternary set, and J_n denotes the union of closures of all the components of $I\setminus C$ with length $\frac{1}{3^{n+1}}$. C is the union of disjoint c-dense subsets C_1 and C_2. Let C_0 be the set of the endpoints of all the intervals J_n. Put

$$f_0(x)=\begin{cases} e_n & \text{if } x\in J_{2n}\\ e_0 & \text{if } x\in(C_1\setminus C_0)\cup(\mathbb{R}\setminus(0,1))\\ d_n & \text{if } x\in J_{2n+1}\\ & \text{takes on every value of } \mathbb{R}\ c\text{-many times on every nonempty}\\ & \text{relative subinterval } (a,b)\cap(C_2\setminus C_0).\end{cases}$$

Let $C_3=\{x\in C_2\setminus C_0:f_0(x)\in B\}$. Then $f_0\in$ QC$\cap$D and $f_0^{-1}(B)=C_3\cup\bigcup_{n=0}^{\infty}J_{2n+1}\in\mathcal{A}$. Notice C_3 is c-dense in C_2. Now define

$$f_1(x)=\begin{cases} e_n & \text{if } x\in J_{2n+1}\\ d_0 & \text{if } x\in(C_2\setminus(C_3\cup C_0))\cup(\mathbb{R}\setminus(0,1))\\ d_n & \text{if } x\in J_{2n}\\ & \text{takes on every value of } \mathbb{R} \text{ on every nonempty relative sub-}\\ & \text{interval } (a,b)\cap((C_1\setminus C_0)\cup C_3) \text{ with } f_1((a,b)\cap(C_1\setminus C_0))=B\\ & \text{and } f_1((a,b)\cap C_3)=\mathbb{R}\setminus B.\end{cases}$$

Then $f_1\in$QC$\cap$D and $f_1^{-1}(B)=(\mathbb{R}\setminus(C_3\cup C_0))\cup_{n=0}^{\infty}J_{2n}\in\mathcal{A}$, $\mathbb{R}=f_0^{-1}(B)\cup f_1^{-1}(B)$ and $f_0^{-1}(B)\cap f_1^{-1}(B)=\emptyset$. $\{\emptyset,\mathbb{R}\}\subset\mathcal{A}$ because the constant functions are in QC$\cap$D. Define $h\in\mathbb{R}^{\mathbb{R}}$ by $h(x)=i$ if $x\in f_i^{-1}(B)$ and $i=0,1$. Therefore $h\in C^{-1}_{\mathcal{A},\mathcal{B}}\setminus$D, a contradiction. □

3 Ext$\setminus$(CT$\cup$QC)

Smital and Stanova showed that there exists an almost continuous function $f:I\to\mathbb{R}$ which is in neither CT nor QC [14]. In [7], Gibson asked if there exists an extendable function $f : I \to \mathbb{R}$ which is in neither CT nor QC. We see the answer is yes by applying the next theorem to either of the following examples.

Example 1. Let $f : I \to I$ be Croft's function, which has the properties that f is Darboux, Baire class 1, and $f = 0$ a.e. but not identically 0. (See p.12 in [2].) Let $E \neq \emptyset$ be that set of measure zero. Then $f^{-1}(0) = I \setminus E$ and $I \setminus E$ is dense in I.

Example 2. More generally, suppose $\emptyset \neq E \subset I$ with E an F_σ set bilaterally c-dense in itself and $I \setminus E$ dense in I. For example, E could be a certain union of countably many Cantor sets. By Theorem 2.4 on p. 13 in [2], there exists a Darboux Baire class 1 function $f : I \to I$ such that $f^{-1}(0) = I \setminus E$.

Theorem 5. *If $f : I \to I$ is a Darboux Baire class 1 function and E is a set obeying $\emptyset \neq E \subset I$, $f^{-1}(0) = I \setminus E$ and $I \setminus E$ is dense in I, then $f \in Ext \setminus (CT \cup QC)$.*

PROOF. According to Brown, Humke, and Laczkovich [1], a Baire class 1 function f is Darboux if and only if f is extendable. The graph of a Baire class 1 function f is nowhere dense in $I \times I$ and hence $f \notin$CT. Since $E \neq \emptyset$, there exists $a \in E$. Since $f^{-1}(0) = I \setminus E$, $f(a) > 0$. Therefore f is not quasicontinuous at a because $f(a) > 0$, $f(I \setminus E) = 0$, and $I \setminus E$ is dense in I. □

References

[1] J. B. Brown, P. Humke and M. Laczkovich, *Measurable Darboux functions*, Proc. Amer. Math. Soc., **102** (1988), 603–610.

[2] A. M. Bruckner, *Differentiation of real functions*, Lecture Notes in Math., vol. **659**, Springer-Verlag, Berlin, 1978.

[3] K. Ciesielski, D. Dikranjan, S. Watson, *Functions characterized by images of sets*, Colloq. Math., to appear.

[4] K. Ciesielski and T. Natkaniec, *Darboux like functions that are characterizable by images, preimages and associated sets*, Real Analysis Exch., **23**, this issue.

[5] U. B. Darji, M. J. Evans and R. J. O'Malley, *Some interesting small subclasses of the Darboux Baire 1 functions*, Real Analysis Exch. **19** (1993–94), 328–331.

[6] R. G. Gibson, *Darboux functions with a perfect road*, Real Analysis Exch. **15** (1989–90), 582–591.

[7] R. G. Gibson, *Concerning a characterization of continuity*, Real Analysis Exch. **22** (1996–97), 437–442.

[8] R. G. Gibson and T. Natkaniec, *Darboux like functions*, Real Analysis Exch. **22** (1996–97), 492–533.

[9] R. G. Gibson and I. Reclaw, *Concerning functions with a perfect road*, Real Analysis Exch. **19** (1993–94), 564–570.

[10] M. R. Hagan, *Equivalence of connectivity maps and peripherally continuous transformations*, Proc. Amer. Math. Soc. **17** (1966), 175–177.

[11] F. B. Jones and E. S. Thomas, Jr., *Connected G_δ graphs*, Duke Math. J. **33** (1966), 341–345.

[12] K. R. Kellum and B. D. Garrett, *Almost continuous real functions*, Proc. Amer. Math. Soc. **33** (1972), 181–184.

[13] A. Neubrunnova, *On certain generalizations of the notion of continuity*, Mat. Casopis Sloven. Akad. Vied **23** (1973), 374–380.

[14] J. Smital and E. Stanova, *On almost continuous functions*, Acta Math. Univ. Comen. **37** (1980), 147–155.

[15] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.

[16] G. T. Whyburn, *Connectivity of peripherally continuous functions*, Proc. Nat. Acad. Sci. USA **55** (1966), 1040–1041.