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CHARACTERIZING DERIVATIVES BY PREIMAGES OF SETS

Abstract

In this note we will show that many classes $\mathcal{F}$ of real functions $f\colon \mathbb{R} \to \mathbb{R}$ can be characterized by preimages of sets in a sense that there exist families $\mathcal{A}$ and $\mathcal{D}$ of subsets of $\mathbb{R}$ such that $\mathcal{F} = \mathcal{C}(\mathcal{D}, \mathcal{A})$, where $\mathcal{C}(\mathcal{D}, \mathcal{A}) = \{f \in \mathbb{R}^{\mathbb{R}}\colon f^{-1}(A) \in \mathcal{D} \text{ for every } A \in \mathcal{A}\}$. In particular, we will show that there exists a Bernstein $B \subset \mathbb{R}$ such that the family Δ of all derivatives can be represented as $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$, where $\mathcal{A} = \bigcup_{c\in\mathbb{R}}\{(-\infty, c), (c, \infty), B + c\}$ and $\mathcal{D} = \{g^{-1}(A)\colon A \in \mathcal{A} \ \& \ g \in \Delta\}$.

1 Introduction

Our terminology is standard and follows [4]. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the set of all real and rational numbers, respectively. The symbol $\mathcal{P}(X)$ will stand for the family of all subsets of X. The family of all functions from a set X into Y is denoted by Y^X. In particular, $\mathbb{R}^{\mathbb{R}}$ will stand for the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. For a set $S \subset \mathbb{R}$ the symbol S^c will denote the complement of S, i.e., $S^c = \mathbb{R} \setminus S$. We will write $\mathcal{B}$or for the family of all Borel functions $f\colon \mathbb{R} \to \mathbb{R}$ and $\mathcal{B}$ for the family of all Borel subsets of $\mathbb{R}$. The ordinal numbers will be identified with the sets of all their predecessors, and cardinals with the initial ordinals. The cardinality of a set X will be denoted by $|X|$. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$ and referred as the *continuum.*

The problem of characterizing the real functions $f\colon \mathbb{R} \to \mathbb{R}$ that are derivatives of some function $F\colon \mathbb{R} \to \mathbb{R}$ preoccupied many authors for most of this century. The development led, for example, to a characterization of associated sets (i.e., sets of the form $\{x \in \mathbb{R}\colon f(x) < b\}$) for the derivatives ([14, 10])

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and many other results in this direction ([1, 2]). However, it is known already from the 1936 paper [9] of Mazurkiewicz that a "simple" characterization of derivatives might not exist, since the set of all differentiable functions is a true co-analytic set. Also, Freiling in his recent article [8] gives a convincing argument that any nice structural characterization of derivatives is circular in a sense that it allows us to solve, to some extend, the problem of finding the primitive of a derivative.

The main goal of this article is to show that many classes $\mathcal{F}$ of real functions, including the family Δ of all derivatives, can be characterized by means of preimages of some sets as can the class of all continuous functions; that is, as a family of the form

$$\mathcal{C}(\mathcal{D},\mathcal{A})=\{f\in\mathbb{R}^{\mathbb{R}}\colon f^{-1}(A)\in\mathcal{D}\ \text{ for every } A\in\mathcal{A}\},$$

where $\mathcal{A}$ is a family of subsets of $\mathbb{R}$ and $\mathcal{D}=\{f^{-1}(A)\colon f\in\mathcal{F}\ \&\ A\in\mathcal{A}\}$.

The general theorem in this direction proved here is the following.

Theorem 1.1. *Let $\mathcal{F},\mathcal{R}\subset\mathbb{R}^{\mathbb{R}}$ be such that $|\mathcal{R}|\le\mathfrak{c}^+$, $|\mathcal{F}|\le\mathfrak{c}$, $\mathcal{F}$ contains all constant functions, and $|g[\mathbb{R}]|=\mathfrak{c}$ for any non-constant function g which is a difference of two functions from $\mathcal{F}$. Then there exists a family $\mathcal{A}\subset\mathcal{P}(\mathbb{R})$ of cardinality less than or equal to $|\mathcal{R}|$ such that*

$$\mathcal{F}\cap\mathcal{R}=\mathcal{R}\cap\mathcal{C}(\mathcal{D},\mathcal{A})$$

where $\mathcal{D}=\{f^{-1}(A)\colon f\in\mathcal{F}\ \&\ A\in\mathcal{A}\}$.

Applying this theorem to $\mathcal{F}=\Delta$, $\mathcal{R}=\mathcal{B}\mathrm{or}$ and $\mathbb{R}^{\mathbb{R}}$ we immediately obtain the following two corollaries.

Corollary 1.2. *There exists a family $\mathcal{A}\subset\mathcal{P}(\mathbb{R})$ such that $|\mathcal{A}|\le\mathfrak{c}$ and*

$$\Delta=\mathcal{B}\mathrm{or}\cap\mathcal{C}(\mathcal{D},\mathcal{A}),$$

where $\mathcal{D}=\{f^{-1}(A)\colon f\in\Delta\ \&\ A\in\mathcal{A}\}$.

Corollary 1.3. *If the Generalized Continuum Hypothesis holds (more specifically, if $2^{\mathfrak{c}}=\mathfrak{c}^+$), then there exists a family $\mathcal{A}\subset\mathcal{P}(\mathbb{R})$ such that*

$$\Delta=\mathcal{C}(\mathcal{D},\mathcal{A}),$$

where $\mathcal{D}=\{f^{-1}(A)\colon f\in\Delta\ \&\ A\in\mathcal{A}\}$.

Certainly, we can obtain similar corollaries for a wide variety of classes $\mathcal{F}$. Moreover, specifically for the class Δ the following stronger characterization will be proved, where $\mathcal{DB}_1$ stands for the class of Darboux Baire one functions. Recall also that $B\subset\mathbb{R}$ is *Bernstein* if B and its complement intersect every non empty perfect subset of $\mathbb{R}$.

Theorem 1.4. *There exists a Bernstein set $B \subset \mathbb{R}$ such that*

$$\Delta = \mathcal{DB}_1 \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\}) = \mathcal{C}(\mathcal{D}, \mathcal{A}),$$

where $\mathcal{A} = \bigcup_{c\in\mathbb{R}}\{(-\infty, c), (c, \infty), B + c\}$, $\mathcal{D}_0 = \{f^{-1}(B + c) \colon f \in \Delta \ \& \ c \in \mathbb{R}\}$, *and* $\mathcal{D} = \{f^{-1}(A) \colon f \in \Delta \ \& \ A \in \mathcal{A}\}$.

Note that Theorem 1.4 and Corollary 1.3 (under the assumption $2^{\mathfrak{c}} = \mathfrak{c}^+$) generalize the following theorem of Preiss and Tartaglia [11], which was a motivation for this paper.

Proposition 1.5. *For every subset E of $\mathbb{R}$ there exists a family $\mathcal{D}_E$ (equal to $\{f^{-1}(E) \colon f \in \Delta\}$) such that Δ is equal to*

$$\mathcal{C}(\{\mathcal{D}_E\}_{E\in\mathcal{P}(\mathbb{R})}, \mathcal{P}(\mathbb{R})) = \{f \colon \mathbb{R} \to \mathbb{R} \colon f^{-1}(E) \in \mathcal{D}_E \ \text{ for every } E \in \mathcal{P}(\mathbb{R})\}.$$

The obvious disadvantage of the characterization of Δ as in Theorem 1.4 (and Corollaries 1.2 and 1.3) is its circular character: the family $\mathcal{D}$ is defined with the use of Δ as a kind of weak “topology” for the family Δ generated by a “topology” $\mathcal{A}$. However, by an argument of Freiling [8], any characterization of Δ will be, to some extend, circular.

Another disadvantage of the characterization from Theorem 1.4 is that it uses a Bernstein set which is highly nonconstructive. (It is non-measurable, does not have the Baire property, and its existence cannot be proved without the Axiom of Choice. In fact, even the Dependent Choice Axiom, which is a part of the Axiom of Choice that implies the classical induction theorem, is not sufficient for deducing its existence.) It would be nicer to have a similar characterization with $\mathcal{A}$ being a subfamily of the Borel sets. However, the existence of such a characterization is not clear at this point.

Despite all of these reservations, the characterization from Theorem 1.4 really says something. If a Darboux Baire one function $f \colon \mathbb{R} \to \mathbb{R}$ fails to be a derivative, then it is prevented from being so solely because of the form of its preimage $f^{-1}(B + c)$ of a translation of a single set B.

Notice also that although in the characterizations $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$ the family $\mathcal{D}$ is a weak “topology” for a family Δ generated by a “topology” $\mathcal{A}$, the family $\mathcal{A}$ cannot be a topology. This follows from the next proposition, which was proved by the author [3, Corollary 3] and, independently, by Tartaglia [12].[1]

Proposition 1.6. *There are no topologies τ_0 and τ on $\mathbb{R}$ with the property that $\Delta = \mathcal{C}(\tau_0, \tau)$.*

[1] The information on Tartaglia's comes from [11] since the preprint [12] is not available to the author.

Also, if $\mathcal{F} = \mathcal{C}(\mathcal{B}, \mathcal{A})$ for some families $\mathcal{A}$ and $\mathcal{B}$ of subsets of $\mathbb{R}$ then we also have $\mathcal{F} = \mathcal{C}(\mathcal{D}, \mathcal{A})$, where $\mathcal{D} = \{f^{-1}(A)\colon A \in \mathcal{A} \text{ and } f \in \mathcal{F}\}$, since $\mathcal{F} \subset \mathcal{C}(\mathcal{D}, \mathcal{A}) \subset \mathcal{C}(\mathcal{B}, \mathcal{A}) = \mathcal{F}$. Thus, the form of the family $\mathcal{D}$ in the above characterizations is, in a sense, forced on us. Also

$$\text{if } \Delta = \mathcal{C}(\mathcal{D}, \mathcal{A}) \text{ or } \Delta = \mathcal{B}\text{or} \cap \mathcal{C}(\mathcal{D}, \mathcal{A}) \text{ then } \mathcal{A} \subset \mathcal{D} \tag{1}$$

since the identity function belongs to Δ. However in Theorem 1.4 and Corollary 1.3 we cannot have $\mathcal{A} = \mathcal{D}$, since the class $\mathcal{C}(\mathcal{A}, \mathcal{A})$ is closed under composition, while there exist a derivative f and a homeomorphism h (which is also a derivative) such that $h \circ f$ is not a derivative. (See e.g. [1].)

Notice also that in the characterization $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$ neither $\mathcal{A}$ nor $\mathcal{D}$ can be an algebra, as follows from the following fact.

Proposition 1.7. *If $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$ for some families $\mathcal{D}$ and $\mathcal{A}$ of subsets of $\mathbb{R}$ then neither $\mathcal{A}$ nor $\mathcal{D}$ contain simultaneously a non-empty proper subset S of $\mathbb{R}$ and its complement S^c.*

In particular, neither $\mathcal{A}$ nor $\mathcal{D}$ is an algebra.

Proof. First note that $\mathcal{A} \not\subset \{\emptyset, \mathbb{R}\}$, since this and the inclusion $\mathcal{A} \subset \mathcal{D}$ would imply that $C(\mathcal{D}, \mathcal{A})$ consists of all real functions, contradicting $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$. In particular, $\mathbb{R} \in \mathcal{D}$, since Δ contains all constant functions. Now, if $S, S^c \in \mathcal{D}$ then the characteristic function χ_S of S belongs to $\mathcal{C}(\mathcal{D}, \mathcal{A}) = \Delta$. So $S \in \{\emptyset, \mathbb{R}\}$, since otherwise χ_S would not belong to Δ. (Derivatives have the Darboux property [1].) This implies the main part of the proposition, as $\mathcal{A} \subset \mathcal{D}$.

The additional part follows immediately from the first part, the inclusion $\mathcal{A} \subset \mathcal{D}$, and the fact that $\mathcal{A} \not\subset \{\emptyset, \mathbb{R}\}$. $\square$

Propositions 1.6 and 1.7 and condition (1) show, in particular, that we cannot expect to improve in any essential way the structure of the families $\mathcal{D}$ and $\mathcal{A}$ in $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$.

It is also worth mentioning that the class Δ cannot be characterized by images of sets in a way similar to $\Delta = \mathcal{C}(\mathcal{D}, \mathcal{A})$ in the sense that

$$\Delta \neq \{f \in \mathbb{R}^{\mathbb{R}}\colon f[A] \in \mathcal{B} \text{ for every } A \in \mathcal{A}\}$$

for any families $\mathcal{A}$ and $\mathcal{B}$ of subsets $\mathbb{R}$. This follows immediately from the following theorem [5, Thm. 4.1]. (The proof of the theorem is a modification of the proof of a theorem of Velleman from [13]. Compare also [6].)

Theorem 1.8. *If $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of $\mathbb{R}$ with the property that $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}}\colon f[A] \in \mathcal{B} \text{ for every } A \in \mathcal{A}\}$ contains all continuous functions, then there is a non-measurable function $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.*

2 Proof of Theorem 1.4

The results presented in this section are a modification of an argument sent to the author by an anonymous referee of a previous version of the paper which consisted mainly of the results presented in the next section.

The proof of the theorem presented below will be based on the following two lemmas. Recall that a set $T \subset \mathbb{R}^n$ is analytic, if it is the continuous image of a Borel subset of $\mathbb{R}^m$.

Lemma 2.1. *There exists a Bernstein set B such that for every analytic set $A \subset \mathbb{R}^2$*

(A) *if $T \cap (B \times B) = \emptyset$, then $T \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times C)$ for some countable set $C \subset B^c$;*

(B) *if $T \cap (B \times B^c) = \emptyset$, then $T \setminus \{\langle x, x\rangle : x \in \mathbb{R}\} \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times D)$ for some countable sets $C \subset B^c$ and $D \subset B$.*

(C) *if $T \cap (B^c \times B) = \emptyset$, then $T \setminus \{\langle x, x\rangle : x \in \mathbb{R}\} \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times D)$ for some countable sets $C \subset B$ and $D \subset B^c$.*

Proof. Let $\{A_0, A_1, A_2\}$ be a partition of $\mathfrak{c}$ onto the sets of cardinality $\mathfrak{c}$ and for $i < 3$ let $\langle T_\xi : \xi \in A_i\rangle$ be an enumeration of all analytic subsets of $\mathbb{R}^2$. By transfinite induction on $\xi < \mathfrak{c}$ we will choose disjoint four-element sets $D_\xi = \{a_\xi, b_\xi, c_\xi, d_\xi\}$ aiming for $B = \bigcup_{\xi<\mathfrak{c}} \{a_\xi, b_\xi\}$ (thus, also for $B^c \supset \bigcup_{\xi<\mathfrak{c}} \{c_\xi, d_\xi\}$). The construction is done maintaining the following conditions for every $\xi < \mathfrak{c}$, where $D_\xi = \bigcup_{\zeta<\xi} \{a_\zeta, b_\zeta\}$ and $C_\xi = \bigcup_{\zeta<\xi} \{c_\zeta, d_\zeta\}$.

Choose different $a, b, c, d \in \mathbb{R} \setminus (C_\xi \cup D_\xi)$.

For $\xi \in A_0$: If $T_\xi \subset (C_\xi \times \mathbb{R}) \cup (\mathbb{R} \times C_\xi)$, put $a_\xi = a$, $b_\xi = b$, $c_\xi = c$, and $d_\xi = d$. Otherwise choose $\langle z, d_\xi\rangle \in T_\xi \setminus (C_\xi \times \mathbb{R}) \cup (\mathbb{R} \times C_\xi)$. If $z \neq d_\xi$, we put $c_\xi = z$ and choose different $a_\xi, b_\xi \in \{a, b, c, d\} \setminus \{c_\xi, d_\xi\}$. Otherwise we choose different $a_\xi, b_\xi, c_\xi \in \{a, b, c, d\} \setminus \{d_\xi\}$.

For $\xi \in A_1$: If $T_\xi \setminus \{\langle x, x\rangle : x \in \mathbb{R}\} \subset (C_\xi \times \mathbb{R}) \cup (\mathbb{R} \times D_\xi)$, put $a_\xi = a$, $b_\xi = b$, $c_\xi = c$, and $d_\xi = d$. Otherwise choose $\langle c_\xi, a_\xi\rangle \in T_\xi \setminus (C_\xi \times \mathbb{R}) \cup (\mathbb{R} \times D_\xi)$ with $a_\xi \neq c_\xi$. Then choose different $b_\xi, d_\xi \in \{a, b, c, d\} \setminus \{a_\xi, c_\xi\}$.

For $\xi \in A_2$: If $T_\xi \setminus \{\langle x, x\rangle : x \in \mathbb{R}\} \subset (D_\xi \times \mathbb{R}) \cup (\mathbb{R} \times C_\xi)$, put $a_\xi = a$, $b_\xi = b$, $c_\xi = c$, and $d_\xi = d$. Otherwise choose $\langle a_\xi, c_\xi\rangle \in T_\xi \setminus (D_\xi \times \mathbb{R}) \cup (\mathbb{R} \times C_\xi)$ with $a_\xi \neq c_\xi$. Then choose different $b_\xi, d_\xi \in \{a, b, c, d\} \setminus \{a_\xi, c_\xi\}$.

This finishes the construction.

The construction immediately gives us (A)-(C) with sets C and D having cardinality less than $\mathfrak{c}$. But this implies that the appropriate analytic set is covered by less than $\mathfrak{c}$ many horizontal and vertical lines, and hence it is covered by countably many of these lines [7].

To see that B is Bernstein, take an arbitrary non empty perfect set $P \subset \mathbb{R}$ and notice that $P \times \mathbb{R}$ must be intersected by $B \times B$ and $B^c \times B$. $\square$

In what follows we will use the following notation. We will write $\mathcal{J}$ for the family of all intervals in the form $(-\infty, c)$ and (c, ∞) with $c \in \mathbb{R}$, and M_0 for the family of all F_σ subsets E of $\mathbb{R}$ such that every point of E is a point of bilateral accumulation of E. Recall also (see e.g. [1, page 62]) that $\mathcal{C}(M_0, \mathcal{J})$ is equal to the family $\mathcal{DB}_1$ of Darboux Baire class one functions, and that $\Delta \subset \mathcal{DB}_1$.

The following lemma has been proved by Preiss and Tartaglia [11, Lemma 2]. (The lemma in the paper is stated there only for the family Δ. However, at the end of the paper the authors remark that it is true for much wider classes of functions, including the case presented below.)

Lemma 2.2. *Let $\mathcal{F} \subset \mathbb{R}^\mathbb{R}$ be such that it contains all constant functions and that $|g[\mathbb{R}]| = \mathfrak{c}$ for every non-constant g which is the difference of two functions from $\mathcal{F}$. Then for every $h \in \mathbb{R}^\mathbb{R}$ there exists at most one non-constant $f \in \mathcal{F}$ such that for some $Z \subset \mathbb{R}$, $|Z| < \mathfrak{c}$,*

$$f(x) = h(x) \text{ for every } x \in \mathbb{R} \text{ such that } \{f(x), h(x)\} \not\subset Z.$$

Theorem 2.3. *Assume that $\mathcal{F} \subset \mathcal{DB}_1$ contains all constant functions, is closed under constant addition, and that any non-constant g which is the difference of two functions from $\mathcal{F}$ has uncountable range. Then*

$$\mathcal{F} = \mathcal{C}(M_0, \mathcal{J}) \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\}) = \mathcal{C}(\mathcal{D}, \mathcal{A}),$$

where $B \subset \mathbb{R}$ is a Bernstein set from Lemma 2.1, $\mathcal{A} = \mathcal{J} \cup \{B + c \colon c \in \mathbb{R}\}$, $\mathcal{D}_0 = \{f^{-1}(B + c) \colon f \in \mathcal{F} \ \& \ c \in \mathbb{R}\}$, and $\mathcal{D} = \{f^{-1}(A) \colon f \in \mathcal{F} \ \& \ A \in \mathcal{A}\}$.

Proof. Note that $\mathcal{D}_0 = \{f^{-1}(B) \colon f \in \mathcal{F}\}$, since $f^{-1}(B + c) = (f - c)^{-1}(B)$ and $\mathcal{F}$ is closed under constant addition. Also $\mathcal{D} \subset M_0 \cup \mathcal{D}_0$.

Clearly $\mathcal{F} \subset \mathcal{C}(\mathcal{D}, \mathcal{A})$. We will show that

$$\mathcal{C}(\mathcal{D}, \mathcal{A}) \subset \mathcal{C}(M_0, \mathcal{J}) \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\}) \subset \mathcal{F}.$$

To argue for the first inclusion fix an $h \in \mathcal{C}(\mathcal{D}, \mathcal{A})$. Our first goal will be to show that $h \in \mathcal{B}$or. For this is enough to prove that

$$\text{for every } c \in \mathbb{R} \text{ there is } E \in \mathcal{B} \text{ with } h^{-1}((-\infty, c)) \subset E \subset h^{-1}((-\infty, c]). \quad (2)$$

To see (2) fix $c \in \mathbb{R}$. If either $h^{-1}((-\infty,c)) \in \mathcal{B}$ or $h^{-1}((-\infty,c]) \in \mathcal{B}$, then (2) clearly holds. So, we can assume that it is not the case. Then $h^{-1}((-\infty,c)), h^{-1}((c,\infty)) \in \mathcal{D}_0$. Therefore there exist $f,g \in \mathcal{F}$ such that $h^{-1}((-\infty,c)) = f^{-1}(B)$ and $h^{-1}((c,\infty)) = g^{-1}(B)$. But then

$$T = \{\langle f(x), g(x)\rangle \colon x \in \mathbb{R}\}$$

does not intersect $B \times B$, since $f(x) \in B$ implies $h(x) \in (-\infty,c)$, and $g(x) \in B$ implies $h(x) \in (c,\infty)$. However, T is analytic as an image of $\mathbb{R}$ under a Borel function $\langle f,g\rangle$. So, by Lemma 2.1(A), there exists a countable set $C \subset B^c$ such that $T \subset (C \times \mathbb{R}) \cup (\mathbb{R} \times C)$. But then

$$h^{-1}((-\infty,c)) = f^{-1}(B) \subset g^{-1}(C) \subset g^{-1}(B^c) = h^{-1}((-\infty,c]),$$

where the first inclusion follows from the fact that $x \in f^{-1}(B)$ implies that $f(x) \in B$; so $f(x) \notin C$, and $g(x) \in C$. Therefore (2) is satisfied by a Borel set $E = g^{-1}(C)$.

To show that $h \in \mathcal{C}(M_0,\mathcal{J}) = \mathcal{DB}_1$ notice first that

$$g^{-1}(B) \notin \mathcal{B} \text{ for every non-constant } g \in \mathcal{DB}_1. \tag{3}$$

Indeed, if $g^{-1}(B) \in \mathcal{B}$ then $g[g^{-1}(B)]$ is an analytic subset of a Bernstein set B, so it is countable. Similarly, $g[g^{-1}(B^c)] \subset B^c$ is analytic, thus countable. Therefore

$$g[g^{-1}(B)] \cup g[g^{-1}(B^c)] = g[g^{-1}(B \cup B^c)] = g[\mathbb{R}]$$

is countable as well. So g, being Darboux, must be constant.

Now, to prove that $h \in \mathcal{DB}_1 = \mathcal{C}(M_0,\mathcal{J})$ fix $J \in \mathcal{J}$. We have to show that

$$h^{-1}(J) \in M_0.$$

Indeed, we know that $h^{-1}(J) \in \mathcal{D} \subset M_0 \cup \mathcal{D}_0$. If $h^{-1}(J) \in M_0$, there is nothing to prove. But if $h^{-1}(J) \in \mathcal{D}_0$, then $h^{-1}(J) = g^{-1}(B)$ for some $g \in \mathcal{F}$. In particular $g^{-1}(B) \in \mathcal{B}$, since h is Borel. So, by (3), g is constant. Therefore, $h^{-1}(J) = g^{-1}(B) \in \{\emptyset, \mathbb{R}\} \subset M_0$.

Next we will show that $h \in \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\})$. Indeed, it is obvious if h is constant. So assume that h is not constant. Then, by (3),

$$h^{-1}(B+c) = (h-c)^{-1}(B) \in \mathcal{D} \setminus \mathcal{B} \subset \mathcal{D}_0 \quad \text{for every} \quad c \in \mathbb{R}, \tag{4}$$

since $g = h - c$ is a Darboux non-constant function. The proof of the inclusion $\mathcal{C}(\mathcal{D},\mathcal{A}) \subset \mathcal{C}(M_0,\mathcal{J}) \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\})$ has been completed.

To show that $\mathcal{C}(M_0,\mathcal{J}) \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\}) \subset \mathcal{F}$ fix an arbitrary $h \in \mathcal{C}(M_0,\mathcal{J}) \cap \mathcal{C}(\mathcal{D}_0, \{B + c \colon c \in \mathbb{R}\})$. We will show that $h \in \mathcal{F}$. Clearly

$h^{-1}(B+c) \in \mathcal{D}_0$ for every $c \in \mathbb{R}$, since $h \in \mathcal{C}(\mathcal{D}_0, \{B+c\colon c \in \mathbb{R}\})$. In particular, for every $c\in\mathbb{R}$ there exists $f_c \in \mathcal{F}$ such that

$$h^{-1}(B+c) = f_c^{-1}(B). \tag{5}$$

We claim that $h = f_0$, which will finish the proof, since $f_0 \in \mathcal{F}$.

To see this, let us first note that $f_c = f_0 - c$ for every $c \in \mathbb{R}$. Indeed, $(h-c)^{-1}(B) = f_c^{-1}(B)$, so $U = \{\langle h(x)-c, f_c(x)\rangle\colon h(x)-c \neq f_c(x)\}$ is an analytic subset of $[(B\times B)\cup(B^c\times B^c)]\setminus\{\langle x,x\rangle\colon x\in\mathbb{R}\}$. Thus, by parts (B) and (C) of Lemma 2.1, there exist countable sets $C \subset B^c$, $D \subset B$, $C_1 \subset B$, and $D_1 \subset B^c$ such that U is a subset of a countable set

$$[(C\times\mathbb{R})\cup(\mathbb{R}\times D)]\cap[(C_1\times\mathbb{R})\cup(\mathbb{R}\times D_1)] = (C\times D_1)\cup(C_1\times D).$$

Thus the set

$$U + \langle c,c\rangle = \{\langle h(x),(f_c+c)(x)\rangle\colon h(x)\neq f_c(x)+c\}$$

is countable too. Similarly we show that the set $\{\langle h(x), f_0(x)\rangle\colon h(x)\neq f_0(x)\}$ is countable. Thus, by Lemma 2.2, $f_0 = f_c + c$.

Now, to prove that $h = f_0$ assume, by way of contradiction, that there exists an $x\in\mathbb{R}$ such that $h(x)\neq f_0(x)$. Then $b = f_0(x)-h(x)\neq 0$. Applying Lemma 2.1(B) to $T = \{\langle y, y+b\rangle\colon y\in\mathbb{R}\}$ we may find $y\in\mathbb{R}$ with the property that $\langle y, y+b\rangle \in B\times B^c$. But then $x\notin f_0^{-1}(B+h(x)-y) = f_{h(x)-y}^{-1}(B)$ while $x\in h^{-1}(B+h(x)-y)$, contradicting (5). $\square$

Since for $\mathcal{F} = \Delta$ the assumptions of Theorem 2.3 are clearly satisfied, Theorem 1.4 can be easily deduced.

3 Proof of Theorem 1.1

Clearly, we can assume that $\mathcal{F}$ contains non-constant functions, since otherwise $\mathcal{F}$ equals the class of all constant functions, and for such $\mathcal{F}$ the theorem is obvious. Also, independently of the choice of the family $\mathcal{A}$, we will have $\mathcal{F}\subset\mathcal{C}(\mathcal{D},\mathcal{A})$, by the definition of family $\mathcal{D}$. So, we do not have to worry about the inclusion $\subset$.

To prove the converse inclusion we have to find $\mathcal{A}$ such that no function $h\in\mathcal{R}\setminus\mathcal{F}$ belongs to $\mathcal{C}(\mathcal{D},\mathcal{A})$. This will be done by choosing for every $h\in\mathcal{R}\setminus\mathcal{F}$ a non-empty "witness set" $A_h\subset\mathbb{R}$ such that

$$h^{-1}(A_h)\notin\mathcal{D}, \tag{6}$$

where $\mathcal{A} = \{A_h \colon h \in \mathcal{R} \setminus \mathcal{F}\}$. Evidently such an $\mathcal{A}$ will have all desired properties.

The construction of sets A_h will be done by transfinite induction. More precisely, let $\kappa = |\mathcal{R}| \leq \mathfrak{c}^+$ and let $\{h_\alpha \colon \alpha < \kappa\}$ be an enumeration of $\mathcal{R} \setminus \mathcal{F}$. The induction will be on the length κ and at stage $\alpha < \kappa$ we will choose a set A_α playing the role of A_{h_α}, that is, satisfying (6).

There is a technical problem choosing an A_α at stage α satisfying (6) arising from the fact that we do not yet know the entire set $\mathcal{D}$, which will be equal to $\{f^{-1}(A_\beta) \colon f \in \mathcal{F} \text{ and } \beta < \kappa\}$. The best we can do at this point is to choose A_α with $h_\alpha^{-1}(A_\alpha) \notin \{f^{-1}(A_\beta) \colon f \in \mathcal{F} \;\&\; \beta < \alpha\}$, i.e., such that

(I$_\alpha$) $h_\alpha^{-1}(A_\alpha) \neq f^{-1}(A_\beta)$ for every $f \in \mathcal{F}$ and $\beta < \alpha$,

and with $h_\alpha^{-1}(A_\alpha) \notin \{f^{-1}(A_\alpha) \colon f \in \mathcal{F}\}$, that is,

(II$_\alpha$) $h_\alpha^{-1}(A_\alpha) \neq f^{-1}(A_\alpha)$ for every $f \in \mathcal{F}$.

In order to have $h_\alpha^{-1}(A_\alpha) \notin \{f^{-1}(A_\beta) \colon f \in \mathcal{F} \;\&\; \alpha < \beta < \kappa\}$ we will ensure that $h_\alpha^{-1}(A_\alpha) \neq f^{-1}(A_\beta)$ for every $f \in \mathcal{F}$ and $\alpha < \beta < \kappa$. Thus, we will be choosing "future" A_β's ($\alpha < \beta < \kappa$) to satisfy this requirements. But, by interchanging β with α, it is the same as choosing the set A_α at each step α satisfying

(III$_\alpha$) $f^{-1}(A_\alpha) \neq h_\beta^{-1}(A_\beta)$ for every $f \in \mathcal{F}$ and $\beta < \alpha$.

Thus, the choice of A_α satisfying conditions (I$_\alpha$)-(III$_\alpha$) will result in ensuring (6) to be satisfied.

In addition to these conditions, however, in order to be assured that the construction can continue to completion we still have to make sure that the conclusion of Proposition 1.7 is satisfied by $\mathcal{D}$. To this end, our induction will satisfy the following inductive condition

($\triangle_\alpha$) $\{A, \mathbb{R} \setminus A\} \not\subset \{f^{-1}(A_\alpha) \colon f \in \mathcal{F} \;\&\; \beta < \alpha\}$ for every $A \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}$.

In order to preserve this condition while choosing A_α we require that

(IV$_\alpha$) $f^{-1}(A_\alpha) \neq \mathbb{R} \setminus g^{-1}(A_\beta)$ for every non-constant $f, g \in \mathcal{F}$ and $\beta < \alpha$

and

(V$_\alpha$) $f^{-1}(A_\alpha) \neq \mathbb{R} \setminus g^{-1}(A_\alpha)$ for every distinct non-constant $f, g \in \mathcal{F}$.

At this point of the proof the reader should be convinced that choosing A_α satisfying (I$_\alpha$)–(V$_\alpha$) and ($\triangle_{\alpha+1}$) will finish the proof, as long as we have already chosen a sequence $\langle A_\beta \colon \beta < \alpha \rangle$ of non-empty subsets of $\mathbb{R}$ satisfying $(\triangle_\alpha)$ and (I_β)–(V_β) for all $\beta < \alpha$.

Constructing of A_α to satisfy (I_α)–(V_α) and $(\triangle_{\alpha+1})$ will be done by yet another transfinite induction argument. For this, let $\{\langle f_\xi, g_\xi, \beta_\xi \rangle \colon 0 < \xi < \mathfrak{c}\}$ be an enumeration of the set $\mathcal{F} \times \mathcal{F} \times \{\beta \colon \beta < \alpha\}$. This can be found, since $|\mathcal{F}| \leq \mathfrak{c}$ and $\alpha < \kappa \leq \mathfrak{c}^+$. We will construct increasing sequences $\langle Y_\xi \subset \mathbb{R} \colon \xi < \mathfrak{c} \rangle$ and $\langle Z_\xi \subset \mathbb{R} \colon \xi < \mathfrak{c} \rangle$ inductively with sets Y_0 and Z_0 having cardinality less than $\mathfrak{c}$ and such that

($\star_\xi$) $Y_\xi \cap Z_\xi = \emptyset$ for every $\xi < \mathfrak{c}$, and

($\star\star_\xi$) the sets $Y_\xi \setminus \bigcup_{\zeta<\xi} Y_\zeta$ and $Z_\xi \setminus \bigcup_{\zeta<\xi} Z_\zeta$ are finite for every $0 < \xi < \mathfrak{c}$.

The construction is aimed to ensure that $A_\alpha = \bigcup_{\xi<\mathfrak{c}} Y_\xi$ satisfies (I_α)–(V_α) and that $(\triangle_{\alpha+1})$ holds. Note that condition $(\star\star_\xi)$ together with the cardinality assumption on Y_0 and Z_0 guarantee that all sets Y_ξ and Z_ξ will have cardinalities less than $\mathfrak{c}$.

To construct Y_0 and Z_0 let $f_0 \in \mathcal{F}$ be the unique non-constant function from Lemma 2.2 for $h = h_\alpha$, if it exists or an arbitrary non-constant function from $\mathcal{F}$ otherwise.

If $|h_\alpha[\mathbb{R}]| = \mathfrak{c}$, choose $x_0 \in \mathbb{R}$ such that $h_\alpha(x_0) \neq f_0(x_0)$ and define $Y_0 = \{h_\alpha(x_0)\}$ and $Z_0 = \{f_0(x_0), z\}$, where $z \in h_\alpha[\mathbb{R}] \setminus \{h_\alpha(x_0)\}$. Notice that this implies that A_α will be non-empty and will have the following properties:

($\mathbf{R}_\alpha$) $h_\alpha^{-1}(A_\alpha) \neq \mathbb{R}$, and

($\mathbf{ii}_0$) $h_\alpha^{-1}(A_\alpha) \neq f_0^{-1}(A_\alpha)$,

as $x_0 \in h_\alpha^{-1}(A_\alpha) \setminus f_0^{-1}(A_\alpha)$.

If $|h_\alpha[\mathbb{R}]| < \mathfrak{c}$ let $\bar{Y}_0$ and $\bar{Z}_0$ be non-empty sets forming a partition of $h_\alpha[\mathbb{R}]$. Thus, $h_\alpha^{-1}(\bar{Y}_0)$ and $h_\alpha^{-1}(\bar{Z}_0)$ are non-empty, disjoint sets forming a partition of $\mathbb{R}$. But, by condition $(\triangle_\alpha)$, at least one of these sets does not belong to $\{f^{-1}(A_\beta) \colon f \in \mathcal{F} \ \&\ \beta < \alpha\}$. Without loss of generality we may assume that

$$h_\alpha^{-1}(\bar{Y}_0) \notin \{f^{-1}(A_\beta) \colon f \in \mathcal{F} \ \&\ \beta < \alpha\}. \tag{7}$$

Next choose $x_0 \in \mathbb{R}$ such that $f_0(x_0) \notin \bar{Y}_0 \cup \bar{Z}_0 = h_\alpha[\mathbb{R}]$. This can be done since $|f_0[\mathbb{R}]| = \mathfrak{c}$, as f_0 is non-constant and is the difference of two functions from $\mathcal{F}$. If $h_\alpha(x_0) \in \bar{Y}_0$, we put $Y_0 = \bar{Y}_0$ and $Z_0 = \bar{Z}_0 \cup \{f_0(x_0)\}$. Otherwise we put $Y_0 = \bar{Y}_0 \cup \{f_0(x_0)\}$ and $Z_0 = \bar{Z}_0$. Notice that the condition (R_α) is guaranteed and that x_0 distinguishes between $h_\alpha^{-1}(A_\alpha)$ and $f_0^{-1}(A_\alpha)$ implying (ii_0). Also $h_\alpha^{-1}(A_\alpha) = h_\alpha^{-1}(\bar{Y}_0)$. So, by (7), (I_α) holds.

To proceed farther assume that for some ordinal $0 < \xi < \mathfrak{c}$ the sequences $\langle Y_\zeta \colon \zeta < \xi \rangle$ and $\langle Z_\zeta \colon \zeta < \xi \rangle$ have already been constructed. So $Y^0 = \bigcup_{\zeta<\xi} Y_\zeta$

and $Z^0=\bigcup_{\zeta<\xi} Z_\zeta$ are disjoint and have cardinalities less than $\mathfrak{c}$. Sets Y_ξ and Z_ξ will be disjoint finite extensions of Y^0 and Z^0, respectively, and will imply the following properties:

(i$_\xi$) $h_\alpha^{-1}(A_\alpha)\neq f_\xi^{-1}(A_{\beta_\xi})$;

(ii$_\xi$) $h_\alpha^{-1}(A_\alpha)\neq f_\xi^{-1}(A_\alpha)$;

(iii$_\xi$) $f_\xi^{-1}(A_\alpha)\neq h_{\beta_\xi}^{-1}(A_{\beta_\xi})$;

(iv$_\xi$) $f_\xi^{-1}(A_\alpha)\neq \mathbb{R}\setminus g_\xi^{-1}(A_{\beta_\xi})$ provided f_ξ and g_ξ are non-constant; and

(v$_\xi$) $f_\xi^{-1}(A_\alpha)\neq \mathbb{R}\setminus g_\xi^{-1}(A_\alpha)$ provided f_ξ and g_ξ are non-constant.

This will finish the proof, since our choice of triples $\langle f_\xi,g_\xi,\beta_\xi\rangle$ guarantees that all of the conditions (i$_\xi$) imply (I$_\alpha$), all of the conditions (ii$_\xi$) imply (II$_\alpha$), and similarly for conditions (III$_\alpha$)–(V$_\alpha$).

To fulfill these requirements we will construct increasing disjoint sequences $\langle Y^i\colon i=0,\ldots,5\rangle$ and $\langle Z^i\colon i=0,\ldots,5\rangle$, at each step taking care of one of the above conditions ensuring that $Y_\xi=Y^5$ and $Z_\xi=Z^5$ will have the desired properties.

Step (i). If $|h_\alpha[\mathbb{R}]|<\mathfrak{c}$ then the choice of $Y_0\subset Y^0$ and $Z_0\subset Z^0$ guarantee (I$_\alpha$) already and so, also (i$_\xi$). Then we can put $Y^1=Y^0$ and $Z^1=Z^0$. Otherwise, choose $x_1\in\mathbb{R}$ such that $h_\alpha(x_1)\notin Y^0\cup Z^0$. If $x_1\notin f_\xi^{-1}(A_{\beta_\xi})$, put $Y^1=Y^0\cup\{h_\alpha(x_1)\}$ and $Z^1=Z^0$. Otherwise put $Y^1=Y^0$ and $Z^1=Z^0\cup\{h_\alpha(x_1)\}$. It is easy to see that this guarantees (i$_\xi$), with x_1 distinguishing between $h_\alpha^{-1}(A_\alpha)$ and $f_\xi^{-1}(A_{\beta_\xi})$.

Step (ii). If $f_\xi=f_0$ or f_ξ is constant, then (ii$_\xi$) is already implied either by (ii$_0$) or by (R$_\alpha$) and we can define $Y^2=Y^1$ and $Z^2=Z^1$. Otherwise, by Lemma 2.2 and the choice of f_0, there is an $x_2\in\mathbb{R}$ such that $h_\alpha(x_2)\neq f_\xi(x_2)$ and $\{h_\alpha(x_2),f_\xi(x_2)\}\not\subset Y^1\cup Z^1$. This ensures that one can write $\{h_\alpha(x_2),f_\xi(x_2)\}$ as $\{y,z\}$ with $y\notin Z^1$ and $z\notin Y^1$. Then one can let $Y^2=Y^1\cup\{y\}$ and $Z^2=Z^1\cup\{z\}$. Then x_2 distinguishes between $h_\alpha^{-1}(A_\alpha)$ and $f_\xi^{-1}(A_\alpha)$, implying (ii$_\xi$).

Step (iii). If f_ξ is constant, then $Y^3=Y^2$ and $Z^3=Z^2$ imply (iii$_\xi$) by (II$_{\beta_\xi}$). Otherwise, there exists $x_3\in\mathbb{R}$ such that $f_\xi(x_3)\in f_\xi[\mathbb{R}]\setminus(Y^2\cup Z^2)$. If $x_3\notin h_{\beta_\xi}^{-1}(A_{\beta_\xi})$, put $Y^3=Y^2\cup\{f_\xi(x_3)\}$ and $Z^3=Z^2$. Otherwise define $Y^3=Y^2$ and $Z^3=Z^2\cup\{f_\xi(x_3)\}$. Then x_3 distinguishes between $h_{\beta_\xi}^{-1}(A_{\beta_\xi})$ and $f_\xi^{-1}(A_\alpha)$, implying (iii$_\xi$).

Step (iv). If f_ξ is constant, then $Y^4=Y^3$ and $Z^4=Z^3$ imply (iv$_\xi$). Otherwise, there exists $x_4\in\mathbb{R}$ such that $f_\xi(x_4)\in f_\xi[\mathbb{R}]\setminus(Y^3\cup Z^3)$. If

$x_4 \notin \mathbb{R} \setminus g_\xi^{-1}(A_{\beta_\xi})$, put $Y^4 = Y^3 \cup \{f_\xi(x_4)\}$ and $Z^4 = Z^3$. Otherwise define $Y^4 = Y^3$ and $Z^4 = Z^3 \cup \{f_\xi(x_4)\}$. Then x_4 distinguishes between $\mathbb{R} \setminus g_\xi^{-1}(A_{\beta_\xi})$ and $f_\xi^{-1}(A_\alpha)$, implying (iv$_\xi$).

Step (v). If f_ξ is constant, then $Y^5 = Y^4$ and $Z^5 = Z^4$ imply (v$_\xi$). So, assume that f_ξ is not constant. Then there exists $x_5 \in \mathbb{R}$ such that $f_\xi(x_5) \in f_\xi[\mathbb{R}] \setminus (Y^4 \cup Z^4)$. If $g_\xi(x_5) \in Y^4$, put $Y^5 = Y^4 \cup \{f_\xi(x_5)\}$ and $Z^5 = Z^4$. This implies that $x_5 \in g_\xi^{-1}(A_\alpha) \cap f_\xi^{-1}(A_\alpha)$, so (v$_\xi$) holds. If $g_\xi(x_5) \notin Y^4$, define $Y^5 = Y^4$ and $Z^5 = Z^4 \cup \{f_\xi(x_5), g_\xi(x_5)\}$. Then $x_5 \in (\mathbb{R} \setminus g_\xi^{-1}(A_\alpha)) \setminus f_\xi^{-1}(A_\alpha)$ again implying (v$_\xi$).

Since the construction clearly preserves ($\star_\alpha$), the construction and the proof are completed. □

The following problems seem to be interesting.

Problem 1. *Can the family $\mathcal{A}$ in either Theorem 1.4 or Corollaries 1.2 or 1.3 consist of any kind of regular sets like Lebesgue measurable, Borel, or sets with Baire property?*

Problem 2. *Can Corollary 1.3 be proved without extra set theoretic assumptions?*

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