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ON CONTINUOUS ONE-TO-ONE FUNCTIONS ON SETS OF REAL NUMBERS

1 Introduction

For the purpose of this paper, we say that two topological spaces X and Y are a *special pair* if there are continuous one-to-one mappings of X onto Y and Y onto X, but X and Y are not homeomorphic. In the standard example of a special pair, X is the union of countably many open intervals in the real line, $\mathbb{R}$, together with countably many isolated points, and Y is the union of countably many half open intervals together with countably many isolated points. In [P] Priestley posed the question: *is there a special pair such that both members are countable subspaces of* $\mathbb{R}$*?* In a private correspondence he proved the existence of such a special pair X, Y. But his proof was not constructive. He did not define explicitly the points in the spaces X and Y.

In this paper we construct a countable set E_1 of real numbers and a point $p \in E_1$ such that $X_1 = E_1$, $Y_1 = E_1 \setminus \{p\}$ are a special pair under the Euclidean topology.

Neither member of a special pair can be a closed bounded subspace of $\mathbb{R}$, because a continuous one-to-one mapping from a compact space is necessarily a homeomorphism. But we will construct a special pair X_2, Y_2 such that both X_2 and Y_2 are closed subspaces of $\mathbb{R}$. (They are uncountable however.) We will construct a closed subset E_3 of $\mathbb{R}$ and a point $p \in E_3$ such that $X_3 = E_3$, $Y_3 = E_3 \setminus \{p\}$ are a special pair.

There are special pairs each member of which is the union of mutually disjoint compact intervals in $\mathbb{R}$. We will construct a sequence $J_1, J_2, J_3, \ldots$ of mutually disjoint (nontrivial) compact intervals such that

$$X_4 = \cup_{i=1}^{\infty} J_i, \quad Y_4 = \cup_{i=2}^{\infty} J_i$$

are a special pair.

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Finally, let X be the union of a set of mutually disjoint compact intervals $\{I_1, I_2, I_3, \ldots\}$. Endow the countable family $\{I_i\}$ with the metric topology using Euclidean distance. Now by [S], there is a homeomorphism mapping the metric space $\{I_i\}$ into $\mathbb{R}$. The range of this homeomorphism is of course countable.

Thus, for example, if Y is another union of mutually disjoint compact intervals, if Φ_1 is the homeomorphism mentioned in the preceding paragraph, and if Φ_2 is the corresponding homeomorphism for Y, then X cannot be homeomorphic to Y if the ranges of Φ_1 and Φ_2 are not homeomorphic.

2 Results

Let $E(0,1)$ denote the countable set

$$\left\{0, 1, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{6}, \tfrac{5}{6}, \ldots\right\},$$

and if $a < b$, let $E(a,b) = \left\{bx + a(1-x) \,:\, x \in E(0,1)\right\}$. Then $E(a,b)$ is a countable closed subset of $\mathbb{R}$.

Let C denote Cantor's ternary set, let P denote the (countable) set of all left endpoints of complementary intervals of C of finite length, and let P^+ denote the set of all right endpoints of these intervals. Let X_0 denote the set $C \setminus \{1\}$ and X^* denote

$$X_0 \cup \Big(\cup_{a \in P} E(a, a^+)\Big) \cup E(-1, 0),$$

where a^+ is the point in P^+ that is the immediate successor of a in C. Let $Y^* = X^* \setminus \{-1\}$ and $Z^* = \{x \in X^* \,:\, x > -1/2\}$.

Now for $a, b \in P$, $a < b$ and $c, d \in P$, $c < d$, there is an order preserving mapping of $P \cap [a,b]$ onto $P \cap [c,d]$. (Note that each ordered set of this kind has a first and a last element and no immediate successors or predecessors.) It is easy to see that this mapping is a homeomorphism of $P \cap [a,b]$ onto $P \cap [c,d]$. Extend this mapping to a homeomorphism of $(P \cup P^+) \cap [a,b]$ onto $(P \cup P^+) \cap [c,d]$ by preserving order in the obvious way. Finally, extend this mapping to a homeomorphism of $X^* \cap [a,b]$ onto $X^* \cap [c,d]$ by preserving order in the obvious way.

Let $a_1 < a_2 < a_3 < \ldots$ be an increasing sequence of points in P converging to 1, let $b_1 < b_1 < b_3 < \ldots$ be an increasing sequence of points in P converging to $1/3$, and let $c_1 < c_2 < c_3 < \ldots$ be an increasing sequence of points in P converging to 1, where $c_1 = 7/9$ and $a_1 = b_1 = 1/9$. Let F_0 be the identity mapping of

$$X^* \cap \left(-\frac{1}{2}, \frac{a_1 + a_1^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(-\frac{1}{2}, \frac{b_1 + b_1^+}{2}\right).$$

There is an obvious order preserving homeomorphism F_1 of $X^* \cap [-1, -1/2)$ onto $Z^* \cap [1/3, 1/2)$. By the preceding paragraph, there is an order preserving homeomorphism of $X^* \cap [a_1, a_2]$ onto $X^* \cap [1/3, c_1]$. From this we deduce that there is an order preserving homeomorphism F_2 of

$$X^* \cap \left(\frac{a_1 + a_1^+}{2}, \frac{a_2 + a_2^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(\frac{1}{2}, \frac{c_1 + c_1^+}{2}\right).$$

There is likewise an order preserving homeomorphism F_3 of

$$X^* \cap \left(\frac{a_2 + a_2^+}{2}, \frac{a_3 + a_3^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(\frac{b_1 + b_1^+}{2}, \frac{b_2 + b_2^+}{2}\right).$$

There is an order preserving homeomorphism F_4

$$X^* \cap \left(\frac{a_3 + a_3^+}{2}, \frac{a_4 + a_4^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(\frac{c_1 + c_1^+}{2}, \frac{c_2 + c_2^+}{2}\right).$$

In general for $n > 1$, there is an order preserving homeomorphism F_{2n} of

$$X^* \cap \left(\frac{a_{2n-1} + a_{2n-1}^+}{2}, \frac{a_{2n} + a_{2n}^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(\frac{c_{n-1} + c_{n-1}^+}{2}, \frac{c_n + c_n^+}{2}\right),$$

and an order preserving homeomorphism F_{2n+1} of

$$X^* \cap \left(\frac{a_{2n} + a_{2n}^+}{2}, \frac{a_{2n+1} + a_{2n+1}^+}{2}\right) \quad \text{onto} \quad Z^* \cap \left(\frac{b_n + b_n^+}{2}, \frac{b_{n+1} + b_{n+1}^+}{2}\right).$$

The domains of the functions F_j are mutually disjoint open and closed subsets of X^* whose union is X^*, and the ranges of the F_j are mutually disjoint subsets of Z^* whose union is Z^*. It follows that the common extension F of the functions F_j is a one-to-one continuous function of X^* to Z^*.

Let d_j be an isolated point in $E(c_j, c_j^+)$ for $j = 1, 2, 3, \ldots$. Let D denote the countable set $\{d_j\}$. There is an obvious order preserving homeomorphism of $E(c_j, c_j^+) \setminus \{d_j\}$ onto $E(c_j, c_j^+)$. So there is an obvious homeomorphism of $Z^* \setminus D$ onto Z^*. Thus we can extend this to a homeomorphism of Z^* onto Y^* by mapping the discrete set D onto the discrete set $Y^* \setminus Z^*$.

Likewise we construct a one-to-one continuous function of Y^* onto X^* where $Y^* \setminus \{d_1\}$ maps onto Y^* and d_1 maps to the point -1.

Thus there is a one-to-one continuous function mapping any one of the spaces X^*, Y^*, Z^* onto any other. But X^* is not homeomorphic to Y^* or Z^*. To see this, observe that -1 is an accumulation point of X^* and there is a neighborhood of -1 that contains no other accumulation point of X^*. On the other hand, Y^* and Z^* contain no point enjoying these properties. So X^*, Y^* are a special pair, and X^*, Z^* are a special pair. Note that X^* contains all the accumulation points of X^* in $\mathbb{R}$ except the point 1. Likewise Z^* contains all the accumulation points of Z^* in $\mathbb{R}$ except the point 1. For $w < 1$, put $\Phi(w) = w/(1-w)$. Then $\Phi(X^*)$, $\Phi(Z^*)$ are a special pair and both are closed subsets of $\mathbb{R}$. Put $X_2 = \Phi(X^*)$ and $Y_2 = \Phi(Z^*)$. Put $X_3 = E_3 = \Phi(X^*)$ and $p = -1/2$. Then $Y_3 = E_3 \setminus \{p\} = \Phi(Y^*)$.

Let us start again and redefine X^* to be

$$\Big(\cup_{a\in P} E(a, a^+)\Big) \cup E(-1, 0)\,.$$

All the arguments proving that X^*, Y^* are a special pair go through as before. However, now X^* and Y^* are countable sets. Put $X_1 = E_1 = X^*$ and $p = -1$. Then $Y_1 = E_1 \setminus \{p\} = Y^*$.

It remains to define X_4 and Y_4. For each positive integer j, put $u_j = 4^{-1} + 4^{-j}$ and $v_j = 1 - u_j$. Let $G(0, 1)$ denote the union of mutually disjoint compact intervals,

$$\big[0, \tfrac{1}{4}\big] \cup \Big(\cup_{j=1}^{\infty} [u_{2j+1}, u_{2j}]\Big) \cup \Big(\cup_{j=1}^{\infty} [v_{2j}, v_{2j+1}]\Big) \cup \big[\tfrac{3}{4}, 1\big]\,.$$

For $a < b$, let $G(a, b) = \{bx + a(1-x) \,:\, x \in G(0, 1)\}$, and put

$$X_4 = \Big(\cup_{a\in P} G(a, a^+)\Big) \cup G(-1, 0)\,, \quad Y_4 = X_4 \setminus \big[-1, -\tfrac{3}{4}\big]\,,$$

$$Z_4 = \Big\{x \in X_4 \,:\, x > -\tfrac{1}{2}\Big\}\,.$$

The proof that there is a one-to-one continuous function mapping any one of the spaces X_4, Y_4, Z_4 onto any other is just like the corresponding proof for X_1, Y_1, Z_1, so we leave it. The difference is that now the components are compact intervals instead of singleton sets. Use increasing homeomorphisms from component to component just as we mapped points to points for X_1, Y_1, Z_1. To prove that X_4 is not homeomorphic to Y_4 or Z_4 requires a slightly different argument. Say that a component of X_4 is a type 1 component if it is not an open set in X_4. For example, $[-1, -3/4]$ is a type 1 component, but it lies in a neighborhood $\{x \in X_4 \,:\, x < -1/2\}$ that contains no other type 1 component. On the other hand, Y_4 and Z_4 have no component with this property. So X_4, Y_4 are a special pair and X_4, Z_4 are a special pair.

Now put $U = X_4 \cup (C \setminus \{1\})$ and $V = Z_4 \cup (C \setminus \{1\})$. Then U, V are a special pair by the same argument used for X_4 and Z_4. Note also that U contains all the accumulation points of U (in $\mathbb{R}$) except the point 1. Likewise V contains all the accumulation points of V except the point 1. Neither U nor V contains an isolated point. It follows that $\Phi(U)$, $\Phi(V)$ are a special pair, and both are perfect subsets of $\mathbb{R}$. Put $X_5 = \Phi(U)$ and $Y_5 = \Phi(V)$. Then X_5, Y_5 are a special pair both of which are perfect subsets of $\mathbb{R}$, and moreover each is the closure of an open set (observe the union of all the interiors of all the interval components of X_5, etc.). On the other hand, X_2 and Y_2 are closed subsets of $\mathbb{R}$ with void interiors.

We conclude with some questions that could be the subject of further research. We can make E_1 either a countable or a closed uncountable subset of $\mathbb{R}$. Can we make E_1 a closed countable subset of $\mathbb{R}$? Likewise can we make X_2 and Y_2 closed countable subsets of $\mathbb{R}$? I conjecture no on both counts.

References

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