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ON THE ALMOST CONTINUITY OF THE DIAGONAL OF FUNCTIONS

Abstract

In this paper we consider the following questions: Under what additional assumptions is the diagonal $f \triangle g$ of almost continuous functions vanishing on the set of discontinuity points an almost continuous function, too. Moreover, one can show that the facts considered (diagonals and ac-homotopies) can be applied to the characterization of continuity or the investigations of the algebraic operations on almost continuous functions.

Almost continuous functions, introduced by Stalling, ([8]) (in order to generalize the Brouwer fixed point theorem) have been intensively studied by many authors. There are various areas of interest in this study. Among them are the problems connected with the operations on almost continuous functions. It has been well known for several years that if f is an almost continuous function and g is a continuous function, then the diagonal function $f \triangle g$ is an almost continuous function ([6], [4]). In [6] the author proved the following theorem: *Suppose that Y_0 and Z_0 are convex subsets of normed spaces Y and Z, respectively, $f_1 : \mathbb{I} \to Y_0$, $f_2 : \mathbb{I} \to Z_0$ are almost continuous functions and D is the set of points at which f_1 is discontinuous. If the restriction of $f_{1|\overline{D}}$ is continuous and each point of $\overline{D}$ is a point of continuity of f_2, then the diagonal $f_1 \triangle f_2$ is an almost continuous function.*

In our paper we investigate the following question: *Under what additional assumptions is the diagonal of almost continuous functions also an almost continuous function*, even if the sets of discontinuity points are not disjoint.

We use the standard notions and notation. The open ball with center at x and radius $r > 0$ will be denoted by $B(x, r)$. The symbols $\overline{A}$, $\operatorname{Fr} A$ and $\operatorname{Int}(A)$ stand for the closure, the boundary and the interior of A, respectively.

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By $\mathbb{R}$ ($\mathbb{I}$) we denote the set of all real numbers (the segment $[0,1]$) with the natural topology τ_o. By C_f (D_f) we shall denote the set of all continuity (discontinuity) points of f. The symbol $\Gamma(f)$ stands for the graph of f. The restriction of f to the set A is denoted by $f_{|A}$. For a function $f : X \to \mathbb{R}^n$, we put $\mathcal{Z}(f) = \{x \in X : f(x) = (0, 0, \ldots, 0)\}$.

A function $f : X \to Y$ where X, Y are topological spaces is *almost continuous* if, for each open set $U \subset X \times Y$ containing $\Gamma(f)$, U contains the graph of some continuous functions $g : X \to Y$. The set of all almost continuous functions mapping X into Y (as well as the space [1] with the metric of uniform convergence ϱ) will be denoted by $\mathcal{A}(X,Y)$.

In this paper we consider functions belonging to the set

$$\mathcal{A}_o(X,\mathbb{R}^n) = \{f \in \mathcal{A}(X,\mathbb{R}^n) : D_f \subset \mathcal{Z}(f)\}$$

and

$$\mathcal{D}_o(X,\mathbb{R}^n) = \{f \in \mathcal{D}(X,\mathbb{R}^n) : D_f \subset \mathcal{Z}(f)\}$$

where $n \in 1, 2)$ and $\mathcal{D}(X,\mathbb{R}^n)$ is a family of Darboux functions[2] mapping X into $\mathbb{R}$.

For functions $f : X \to Y$ and $g : X \to Z$, we define their *diagonal function* $f \triangle g : X \to Y \times Z$ by $(f \triangle g)(x) = (f(x), g(x))$.

Let $f \in \mathcal{A}(X,Y)$ ($f \in \mathcal{A}_o(X,Y)$). Then we define $\Delta_{\mathcal{A}}(f)$ ($\Delta_{\mathcal{A}_o}(f)$) by

$$\Delta_{\mathcal{A}}(f) = \{g \in \mathcal{A}(X,Y) : f \triangle g \in \mathcal{A}(X, Y \times Y)\}$$
$$(\Delta_{\mathcal{A}_o}(f) = \{g \in \mathcal{A}_o(X,Y) : f \triangle g \in \mathcal{A}_o(X, Y \times Y)\}).$$

Let (X,τ), (X,τ') be topological spaces. We say that the topology τ' is *s-finer* than τ if τ' is finer than τ and $\{U \in \tau : U \text{ is } \tau\text{-closed}\} = \{U \in \tau : U \text{ is } \tau'\text{-closed}\}$. In a similar way as in [7] we can define ac-homotopies: The almost continuous functions $f, g : (X,\tau) \to (X,\tau^*)$ are called *ac-homotopic* if there exists a topology τ' s-finer than the topology τ, such that $f, g : (X,\tau') \to (Y,\tau^*)$ are continuous and there exists a homotopy $\xi : (X,\tau') \times \mathbb{I} \to (Y,\tau^*)$ between f and g such that $\xi : (X,\tau) \times \mathbb{I} \to (Y,\tau^*)$ is an almost continuous function. (The fact that f and g are ac-homotopic and ξ is an ac-homotopy between f and g is written down as $f \frac{ac}{\xi} g$.)

The notions and symbols we use, connected with porosity, come from papers [9] and [10]. Let X be a metric space. Let $M \subset X$, $x \in X$ and

[1] Of course, in this case we consider only bounded functions.

[2] A function $f : X \to Y$ is *a Darboux function* if the image of any connected set is a connected set, too.

$R > 0$. Then we denote by $\gamma(x, R, M)$ the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. If $p(M, x) = 2 \cdot \limsup_{R\to 0+} \frac{\gamma(x,R,M)}{R} > 0$, then we say that M is porous at x.

Before giving the main results, we make the following observation.

Remark. If $f : \mathbb{R} \to \mathbb{R}$ is a function such that $D_f \subset \mathcal{Z}(f)$, then D_f is nowhere dense set.

The example given below[3] shows that there exist two functions $f, g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$ such that $D_f = \{0\} = D_g$ and $f(0) = 0 = g(0)$ but $f\triangle g$ is not an almost continuous function.

Example. Let $f(x) = 0$ for $x \in (-\infty, 0] \cup [1, +\infty) \cup \{\frac{1}{2n+1} : n = 0, 1, 2, \dots\}$; $f(\frac{1}{2n}) = 1$ $(n = 1, 2, \dots)$ and let f be linear in each interval $[\frac{1}{n+1}, \frac{1}{n}]$ $(n = 1, 2, \dots)$. Then $D_f = \{0\}$ and, consequently (f is a Darboux, Baire one function), $f \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$ ([2]). In each interval $[\frac{1}{2n+2}, \frac{1}{2n+1}]$ ($[\frac{1}{2n+3}, \frac{1}{2n+2}]$) (for $n = 0, 1, 2, \dots$) we can choose a point p_n (q_n) such that $f(p_n) = \frac{3}{4}$ ($f(q_n) = \frac{3}{4}$) $(n = 0, 1, 2, \dots)$.

Now, we define a function $g : \mathbb{R} \to \mathbb{R}$ in the following way: $g(x) = 0$ for $x \in (-\infty, 0] \cup [1, +\infty) \cup \{\frac{1}{2n} : n = 1, 2, \dots\}$; $g(\frac{1}{2n+1}) = 1$ $(n = 1, 2, \dots)$; $g(p_n) = g(q_n) = \frac{3}{4}$ for $n = 0, 1, 2, \dots$; let g be a linear function in each interval $[p_n, \frac{1}{2n+1}]$, $[\frac{1}{2n+2}, p_n]$, $[q_n, \frac{1}{2n+2}]$, $[\frac{1}{2n+3}, q_n]$ for $n = 0, 1, 2, \dots$.

It is easy to see that $f, g \in \mathcal{A}_o$; $f\triangle g(0) = (0, 0)$ and $f\triangle g(x) \notin (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ for $x \in (0, \frac{1}{2}]$ and so, $f\triangle g([0, \frac{1}{2}])$ is not a connected set, which means that $f\triangle g \notin \mathcal{D}(\mathbb{R}, \mathbb{R}^2)$ and, consequently (cf. [6], Theorem 1.1.11), $f\triangle g \notin \mathcal{A}(\mathbb{R}, \mathbb{R}^2)$. □

The problem raised at the beginning can be formulated in the following way: *Given a fixed function $f \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$, under what assumptions on $g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$ do we have $g \in \Delta_{\mathcal{A}_o}(f)$?* The answer is contained in the next theorem which we precede by the following lemma.

Lemma 1. $\mathcal{A}_o(\mathbb{R}, \mathbb{R}^n) = \mathcal{D}_o(\mathbb{R}, \mathbb{R}^n)$.

PROOF. The inclusion $\mathcal{A}_o(\mathbb{R}, \mathbb{R}^n) \subset \mathcal{D}_o(\mathbb{R}, \mathbb{R}^n)$ follows from Theorem 1.1.11 of paper [6].

Now, we shall prove the opposite inclusion. So, let $f \in \mathcal{D}_o(\mathbb{R}, \mathbb{R}^n)$. Note that $\overline{D_f}$ is closed and nowhere dense set and f is continuous on every component of $\mathbb{R} \setminus \overline{D_f}$. Since f is Darboux function, f is almost continuous on the closure of every component of $\mathbb{R} \setminus \overline{D_f}$ (see [6], Theorem 1.5.2) and according to ([6], Lemma 1.3.2), f is almost continuous. □

[3]This example is very simple. However, the functions constructed within the framework of it will be useful in the explanation of certain questions in the remainder of the paper.

Theorem 1. *Let $f \in \mathcal{A}_o(\mathbb{R},\mathbb{R})$. Then $g \in \Delta_{\mathcal{A}_o}(f)$ if and only if $f \frac{ac}{\xi} g$ and $(\overline{D_f} \cup \overline{D_g}) \times \mathbb{I} \subset \mathcal{Z}(\xi)$, for an ac-homotopy ξ.*

PROOF. *Necessity.* Put $D = D_f \cup D_g$. It is not hard to verify that

$$D = D_{f\triangle g} \text{ and } \overline{D} \subset \mathcal{Z}(f\triangle g). \tag{1}$$

Set $D' = \mathbb{R} \setminus \overline{D}$ and let $\{(p_n,q_n)\}_n$ be the sequence of all components of the set D'. (It is possible that $p_n = -\infty$ or $q_n = +\infty$ for some n.) Fix $x \in D'$. Then there exists a positive integer n_x such that $x \in (p_{n_x}, q_{n_x})$ $(p_{n_x} < q_{n_x})$. Put

$$\mathcal{B}(x) = \{(x-\delta, x+\delta) : \ \delta \in (0, \min(x - p_{n_x}, q_{n_x} - x))\}.$$

Now, let $x \in \overline{D}$ and let k be a fixed positive integer. Consider the interval $(x, x+\frac{1}{k})$, and let $U_x^k = (x, x+\frac{1}{k}) \cap D'$. According to the Remark above, U_x^k is an open and dense set in $[x, x+\frac{1}{k}]$.

Now, we shall show that

$$\text{there exists } z_x^k \in U_x^k \text{ such that } f(z_x^k), g(z_x^k) \in (-\frac{1}{k}, \frac{1}{k}). \tag{2}$$

Indeed, suppose to the contrary that, for each $y \in U_x^k$, $f(y) \notin (-\frac{1}{k}, \frac{1}{k})$ or $g(y) \notin (-\frac{1}{k}, \frac{1}{k})$. Therefore $f\triangle g(U_x^k) \cap ((-\frac{1}{k}, \frac{1}{k}) \times (-\frac{1}{k}, \frac{1}{k})) = \emptyset$ and, consequently, $f\triangle g([x, x+\frac{1}{k}]) \subset \{(0,0)\} \cup \big(\mathbb{R}^2 \setminus ((-\frac{1}{k}, \frac{1}{k}) \times (-\frac{1}{k}, \frac{1}{k}))\big)$ and, moreover (according to (1)),

$$(0,0) \in f\triangle g([x, x+\frac{1}{k}]) \text{ and } \left(\mathbb{R}^2 \setminus ((-\frac{1}{k}, \frac{1}{k}) \times (-\frac{1}{k}, \frac{1}{k}))\right) \cap f\triangle g([x, x+\frac{1}{k}]) \neq \emptyset.$$

This means that $f\triangle g$ is not a Darboux function and, consequently (cf. Theorem 1.1.11 of [6]), $f\triangle g \notin \mathcal{A}(\mathbb{R},\mathbb{R}^2)$, which contradicts our assumption. The proof of (2) is finished.

Of course, $z_x^k \in D'$. Let $t_{z_x^k}$ be a positive integer such that $(z_x^k - \frac{1}{t_{z_x^k}}, z_x^k + \frac{1}{t_{z_x^k}})$ is a subset of the component of U_x^k, containing z_x^k and $f((z_x^k - \frac{1}{t_{z_x^k}}, z_x^k + \frac{1}{t_{z_x^k}})) \subset (-\frac{1}{k}, \frac{1}{k}) \supset g((z_x^k - \frac{1}{t_{z_x^k}}, z_x^k + \frac{1}{t_{z_x^k}}))$. By the arbitrariness of k and $x \in \overline{D}$, we can construct the set U_x^k for any positive integer k and $x \in \overline{D}$. Additionally, we may assume that

$$\text{if } k_1 < k_2, \text{ then } z_x^{k_1} > z_x^{k_2} \text{ for any} x \in \overline{D}.$$

Put

$$U_s^+(x) = \{x\} \cup \bigcup_{k=s}^{\infty} (z_x^k - \frac{1}{t_{z_x^k} + s}, z_x^k + \frac{1}{t_{z_x^k} + s}) \text{ for } s = 1, 2, \ldots \text{ and } x \in \overline{D}.$$

In a similar way we can define the sets $U_s^-(x)$. Let

$$\mathcal{B}(x) = \{U_s(x) = U_s^-(x) \cup U_s^+(x) : \; s = 1, 2, \dots\} \; for \; x \in \overline{D}.$$

Let τ be a topology generated by the local system $\{\mathcal{B}(x)\}_{x \in \mathbb{R}}$. It is easy to see that τ is finer than τ_o. Now, we shall show that

$$\tau \text{ is } s\text{-finer than } \tau_o.$$

To prove this fact, it is sufficient to show that $\{V \in \tau_o : \; V \text{ is } \tau\text{-closed}\} = \{\emptyset, \mathbb{R}\}$. Suppose to the contrary that there exists a τ-closed set $V_o \in \tau_o$ such that $\emptyset \neq V_o \neq \mathbb{R}$. Let (u, v) $(u < v)$ be a component of V_o (in the topology τ_o). Suppose, for instance, that $u > -\infty$. From the construction of $\mathcal{B}(u)$ (in the case when $u \in D'$, as well as in the case when $u \in \overline{D}$) we may deduce that $u \in \mathrm{cl}_\tau((u, v))$ where cl_τ denotes the closure in the space $(\mathbb{R}, \tau)$. Then $u \in \mathrm{cl}_\tau(V_o)$ and, consequently (V_o is a τ-closed set), $u \in V_o$, which is impossible because (u, v) is a component of V_o (in the topology τ_o).

Let us observe that

$$f, g : (\mathbb{R}, \tau) \to \mathbb{R} \text{ are continuous functions.}$$

Now, let $\xi : (\mathbb{R}, \tau) \times \mathbb{I} \to \mathbb{R}$ be a natural homotopy between f and g, which means that

$$\xi(x, r) = (1 - r) f(x) + r g(x) \; for \; x \in \mathbb{R}, r \in \mathbb{I}.$$

From (1) we conclude that

$$(\overline{D_f} \cup \overline{D_g}) \times \mathbb{I} \subset \mathcal{Z}(\xi). \tag{3}$$

The proof of necessity will be completed by showing that

$$\xi : \mathbb{R} \times \mathbb{I} \to \mathbb{R} \text{ is an almost continuous; function} \tag{4}$$

(in the natural topology of $\mathbb{R} \times \mathbb{I}$). Let $\{a_m\}_{-\infty}^{+\infty} \subset Int(D')$ be a sequence of real numbers such that

$$-\infty \leftarrow \dots < a_{-2} < a_{-1} < a_o < a_1 < a_2 < \dots \to +\infty.$$

Let V be an arbitrary open set containing $\Gamma(\xi)$. Fix a positive integer m and put $I_m = [a_m, a_{m+1}]$. First, we assume that $I_m \cap \overline{D} \neq \emptyset$. It is not hard to verify that

$$\forall_{x \in \overline{D} \cap I_m} \exists_{\delta_x > 0} \, ([x - \delta_x, x + \delta_x] \times \mathbb{I}) \times [-\delta_x, \delta_x] \subset V \quad \text{and } [x - \delta_x, x + \delta_x] \subset \mathrm{Int}\, I_m. \tag{5}$$

In the rest of the paper, δ_x always denotes fixed positive numbers for $x \in \overline{D} \cap I_m$, such that relations (5) take place.

Now, for $x \in \overline{D} \cap I_m$, let k_x denote positive integers such that $\overline{U_{k_x}} \subset (x-\delta_x, x+\delta_x)$ and $f(U_{k_x}) \subset (-\frac{\delta_x}{2}, \frac{\delta_x}{2}) \supset g(U_{k_x})$; $U_{k_x} = U_{k_x}(x) \in \mathcal{B}(x)$. This means that

$$\forall_{x \in \overline{D} \cap I_m} \xi(U_{k_x} \times \mathbb{I}) \subset (-\delta_x, +\delta_x). \tag{6}$$

We let $x' = \inf U_{k_x} \in \operatorname{Int}(I_m)$, $x" = \sup U_{k_x} \in \operatorname{Int}(I_m)$ for $x \in \overline{D} \cap I_m$. Now, we consider the family $\{D' \cap I_m\} \cup \{(x', x"); \ x \in \overline{D} \cap I_m\}$. Let $x_1, x_2, \ldots, x_q$ be a finite sequence of numbers belonging to $\overline{D} \cap I_m$ such that

$$I_m = (D' \cap I_m) \cup \bigcup_{n=1}^{q} (x'_n, x"_n). \tag{7}$$

Without loss of generality we may assume that

$$x'_1 \leq x'_2 \leq \ldots \, . \tag{8}$$

Put $\delta_o = \min(\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_q})$. First, we consider x_1 and components (x'_1, s_1) and $(z_1, x"_1)$ of the set $U_{k_{x_1}}$. Let $x^1_* \in (x'_1, s_1)$. From (6) we infer that $\xi(\{x^1_*\} \times \mathbb{I}) \subset (-\delta_{x_1}, \delta_{x_1})$. Let $x^*_1 \in (z_1, x"_1)$. Then $\xi(\{x^*_1\} \times \mathbb{I}) \subset (-\delta_{x_1}, \delta_{x_1})$. If $x"_1$ belongs to the component of D' which contains a_{m+1}, then we end our considerations. If not, we shall consider x_2. If $x"_2 \leq x"_1$, then we are passing to the next step dealing with the considerations of x_3. So, we suppose that $x"_2 > x"_1$. Then, according to (7), (8), $x'_2 < x"_1$ or $x"_2$ belongs to the component of D' which contains $x"_1$.

Let $(d'_2, d"_2)$ be a component of D' such that $x"_1 \in [d'_2, d"_2]$. Since $x"_1$ does not belong to the component of D' which contains a_{m+1}, we may infer that $d"_2 < a_{m+1}$. Denote by x^2_* a point from the interval $\big(\max(x^*_1, x'_2), d"_2\big)$ such that $f(x^2_*) \in (-\frac{\delta_o}{2}, \frac{\delta_o}{2})$ and $g(x^2_*) \in (-\frac{\delta_o}{2}, \frac{\delta_o}{2})$. Then $\xi(\{x^2_*\} \times \mathbb{I}) \subset (-\delta_o, \delta_o) \subset (-\delta_{x_2}, \delta_{x_2})$.

Now, we consider the component $(b_2, x"_2)$ of $U_{k_{x_2}}$. Let $x^*_2 \in (z_2, x"_2)$. Then $\xi(\{x^*_2\} \times \mathbb{I}) \subset (-\delta_{x_2}, \delta_{x_2})$. Now, if x^*_2 belongs to the component of D' which contains a_{m+1}, then we end our considerations. In the opposite case, we shall consider x_3. There is no loss of generality in assuming that we obtain the sequence

$$x^*_o = a_m < x^1_* < x^*_1 < x^2_* < x^*_2 < \cdots < x^q_* < x^*_q < a_{m+1} = x^{q+1}_*.$$

It is easy to see that

$$\forall_{i=1,2,..,q} [x^i_*, x^*_i] \subset (x_i - \delta_{x_i}, x_i + \delta_{x_i}) \text{ and } \xi(\{x^i_*, x^*_i\} \times \mathbb{I}) \subset (-\delta_{x_i}, \delta_{x_i}). \tag{9}$$

Let $\xi_i = \xi_{|\{x_*^i, x_i^*\}\times\mathbb{I}} :\ \{x_*^i, x_i^*\} \times \mathbb{I} \to [-\delta_{x_i}, \delta_{x_i}]$ for $i = 1, 2, \ldots, q$. Then ξ_i is a continuous function ($i = 1, 2, \ldots, q$) and so there exists a continuous extension $\xi_i^* :\ [x_*^i, x_i^*] \times \mathbb{I} \to [-\delta_{x_i}, \delta_{x_i}]$ of ξ_i ($i = 1, 2, \ldots, q$). Let $\zeta_m :\ I_m \times \mathbb{I} \to \mathbb{R}$ be a function defined by the formula

$$\zeta_m(z) = \begin{cases} \xi(z) & \text{for } z \in \bigcup_{i=0}^{q} [x_i^*, x_*^{i+1}] \times \mathbb{I} \\ \xi_i^*(z) & \text{for } z \in [x_*^i, x_i^*] \times \mathbb{I}\ (i = 1, 2, \ldots, q). \end{cases}$$

If $I_m \cap \overline{D} = \emptyset$, then we put $\zeta_m = \xi_{|I_m \times \mathbb{I}}$. Let $\hat{\xi} = \nabla_m \zeta_m$ be a combination of compatible functions ζ_m (cf. [3]). Of course, $\hat{\xi}$ is a continuous function and, according to (5) and (9), $\Gamma(\hat{\xi}) \subset V$.

Sufficiency. If g is a continuous function, then (Theorem 1.4.6 of [6]) the proof of sufficiency is trivial. Assume that $D_g \neq \emptyset$. Since $(\overline{D_f} \cup \overline{D_g}) \times \mathbb{I} \subset \mathcal{Z}(\xi)$, we infer that

$$D_{f\triangle g} \subset \mathcal{Z}(f\triangle g). \tag{10}$$

By $g(x) = \xi(x, 1)$ and (Theorem 1.1.11 of [6]), the image $\xi(L)$ is a connected set for each arc $L \subset \mathbb{R}\times\mathbb{I}$. Therefore g is a Darboux function such that $D_g \subset \mathcal{Z}(g)$ and so, according to Lemma 1, $g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$. So, it is sufficient to show that $f\triangle g$ is an almost continuous function and, at the same time, according to Lemma 1, it is sufficient to show that $f\triangle g$ is a Darboux function.

We suppose to the contrary that $f\triangle g$ does not possess the Darboux property. This means that there exists an interval $[a, b] \subset \mathbb{R}$ ($a < b$) such that $f\triangle g([a, b]) = A \cup B$ where A and B are nonempty, disjoint separated sets. Put $A' = [a, b] \cap (f\triangle g)^{-1}(A)$ and $B' = [a, b] \cap (f\triangle g)^{-1}(B)$. Then A', B' are not separated sets. Suppose, for instance, that there exists $a_0 \in A' \cap \overline{B'}$. Let $\{b_n\}_{n=1}^{\infty} \subset B' \cap (-\infty, b)$ be a sequence such that $b_n \searrow a_0$. Then $b_n \neq a_0$ and $a_0 \in \overline{D_f} \cup \overline{D_g}$. Consequently, by applying (10), $f\triangle g(a_0) = (0, 0) \in A$. According to the separateness of A and B, there exists $\varepsilon > 0$ such that

$$K_\varepsilon \cap B = \emptyset \text{ where } K_\varepsilon = (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon). \tag{11}$$

It is easy to check ((10)) that $b_n \notin \overline{D_f} \cup \overline{D_g}$ ($n = 1, 2, \ldots$). Fix n_o. Let (p, q) be a component of $D' = \mathbb{R} \setminus \overline{(\overline{D_f} \cup \overline{D_g})}$ such that $b_{n_o} \in (p, q)$. Clearly, $p \in [a_0, b_{n_o}] \subset [a, b)$ and $p \in (\overline{D_f} \cup \overline{D_g}) = \overline{D_{f\triangle g}}$ and, consequently, by applying (10), $f\triangle g(p) = (0, 0)$. By our assumptions, there exists a topology τ s-finer than the topology τ_o, such that $f, g : (\mathbb{R}, \tau_o) \to \mathbb{R}$ are continuous. Let U be a τ-neighborhood of p such that

$$U \subset (-\infty, b) \text{ and } f(U),\ g(U) \subset (-\varepsilon, \varepsilon). \tag{12}$$

We shall show that

$$U \cap (p, q) \neq \emptyset. \tag{13}$$

Suppose to the contrary that $U \cap (p,q) = \emptyset$. Then $p \in U \cap (-\infty, q) \subset (-\infty, p]$, which means that $(-\infty, p] = (-\infty, p) \cup (U \cap (-\infty, q))$ is a τ-open set and, consequently, $(p, +\infty)$ is a τ closed set, which is impossible. From (13) we infer that there exists $y \in U \cap (p, q)$ and, according to (11) and (12),

$$f \triangle g(y) \in A.$$

Since $f \triangle g_{|(p,q)}$ is a continuous function, $f \triangle g([b_{n_o}, y])$ is a connected set contained in $f \triangle g([a,b])$ such that $f \triangle g([b_{n_o}, y]) \cap A \neq \emptyset$ and $f \triangle g([b_{n_o}, y]) \cap B \neq \emptyset$, which contradicts the separateness of A and B. □

Before going on, we introduce a bit more notation which we shall use. For an arbitrary function $f \in \mathcal{A}_o(X, Y)$,

$$acH(f) = \{g \in \mathcal{A}_o(X,Y) : \ f \frac{ac}{\xi} g, \text{ for some homotopy } \xi\}.$$

It is not hard to verify that if f, g, $h \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$, then $f \in acH(f)$ and if $g \in acH(f)$, then $f \in acH(g)$. But, in the general case, the ac-homotopies are not transitive relations. In fact, let f, g be functions from the example given at the beginning and let h be an arbitrary continuous function. Since (([4]),([6])) $f \triangle h, g \triangle h \in \mathcal{A}(\mathbb{R}, \mathbb{R}^2)$, Theorem 1 shows that $f \frac{ac}{\xi_1} h$ and $h \frac{ac}{\xi_2} g$ by the symmetry of ac-homotopies. Of course, f and g are not ac-homotopic.

The following theorems show that the facts considered (diagonals and ac-homotopies) can be applied to the characterization of continuity or investigations of operations on almost continuous functions.

Theorem 2. *Let $f : \mathbb{R} \to \mathbb{I}$ be an arbitrary function. Then the following conditions are equivalent:*

(i) *f is a continuous function;*

(ii) *$f \in \mathcal{A}_o(\mathbb{R}, \mathbb{I})$ and $\Delta_{\mathcal{A}}(f)$ is not a porous set at f in the space $\mathcal{A}(\mathbb{R}, \mathbb{I})$;*

(iii) *$f \in \mathcal{A}_o(\mathbb{R}, \mathbb{I})$ and, in the subspace $\mathcal{A}_o^f(\mathbb{R}, \mathbb{I}) = \{\eta \in \mathcal{A}_o(\mathbb{R}, \mathbb{I});\ \mathcal{Z}(\eta) \subset \mathcal{Z}(f)\}$ of $\mathcal{A}_o(\mathbb{R}, \mathbb{I})$, $acH(f) \cap \mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$ is not a porous set at f.*

PROOF. First we shall show that $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$. Let $f : \mathbb{R} \to \mathbb{I}$ be an arbitrary continuous function. According to Theorem 1.4.4 from [6], $f \triangle g \in \mathcal{A}(\mathbb{R}, \mathbb{I}^2)$ for an arbitrary $g \in \mathcal{A}(\mathbb{R}, \mathbb{I})$ and, consequently, $\Delta_{\mathcal{A}}(f) = \mathcal{A}(\mathbb{R}, \mathbb{I})$, which ends the proof of the first implication. Moreover, let $t \in \mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$. Then $D_{f \triangle t} \subset \mathcal{Z}(t) = \mathcal{Z}(f \triangle t)$ and so (according to the above considerations),

$t \in \Delta_{\mathcal{A}_o}(f)$. Theorem 1 now leads to the relation $f \frac{ac}{\xi} t$ and so, $acH(f) \cap \mathcal{A}_o^f(\mathbb{R}, \mathbb{I}) = \mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$.

Now, we shall show that $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$. Suppose to the contrary that there exists $x_0 \in D_f$. Then let $\varepsilon > 0$ be a number such that $\frac{\varepsilon}{2}$ is a (for instance, right–hand side) cluster number of f at x_0. Since x_0 is a right–hand side Darboux point of f ([1], [5]) for a fixed $\alpha \in (\frac{\varepsilon}{4}, \frac{\varepsilon}{2})$, there exists a sequence $\{x_n\}$ such that $x_n \searrow x_0$ and $f(x_n) = \alpha$. Since $x_n \in C_f$ $(n = 1, 2, \dots)$, for every n, there exist x'_n, x''_n such that $x''_{n+1} < x'_n < x_n < x''_n$ and

$$f([x'_n, x''_n]) \subset (\frac{\varepsilon}{4}, \frac{\varepsilon}{2}). \tag{14}$$

Moreover, let $p_n \in (x'_n, x_n)$, $q_n \in (x_n, x''_n)$ for $n = 1, 2, \dots$. Now, we define a function $h : \mathbb{R} \to \mathbb{I}$ in the following way:
$h(x) = 0$ for $x = x_n$ $(n = 1, 2, \dots)$;
$h(x) = f(x)$ for $x \leq x_0$, $x = p_n$, $x = q_n$ $(n = 1, 2, \dots)$ and $x \geq q_1$;
$h(x) = \frac{\varepsilon}{2}$ for $x = x'_n$ or $x = x''_n$ $(n = 1, 2, \dots)$;
$h(x)$ is linear in the segments $[x'_n, p_n]$, $[p_n, x_n]$, $[x_n, q_n]$ $(n = 1, 2, \dots)$ and $[q_n, x''_n]$ $(n = 2, 3, \dots)$;
$h(x) = \max(f(x), \frac{\varepsilon}{2})$ for $x \in [x''_{n+1}, x'_n]$ $(n = 1, 2, \dots)$.

According to (14), Theorem 1.7.11 and Lemma 1.2.4 from [6], h is an almost continuous function such that $D_h \subset (D_f \cap (-\infty, x_0]) \cup (D_f \cap [q_1, +\infty)) \subset D_f$ and $\mathcal{Z}(h) \supset (\mathcal{Z}(f) \cap (-\infty, x_0]) \cup (\mathcal{Z}(f) \cap [q_1, +\infty))$, which means that $h \in \mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$. Note that $\varrho(h, f) \leq \frac{\varepsilon}{2}$. Now, we consider $B(h, \frac{\varepsilon}{8})$ (in the case of our considerations, the symbol $B(h, \frac{\varepsilon}{8})$ denotes the open ball in the space $\mathcal{A}(\mathbb{R}, \mathbb{I})$ as well as in the space $\mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$). It is obvious that $B(h, \frac{\varepsilon}{8}) \subset B(f, \varepsilon)$. The proof will be completed by showing that

$$B(h, \frac{\varepsilon}{8}) \cap \Delta_{\mathcal{A}}(f) = \emptyset \tag{15}$$

in the space $\mathcal{A}(\mathbb{R}, \mathbb{I})$ and

$$B(h, \frac{\varepsilon}{8}) \cap acH(f) = \emptyset \tag{16}$$

in the space $\mathcal{A}_o^f(\mathbb{R}, \mathbb{I})$.
Proof of (15). Let $g \in B(h, \frac{\varepsilon}{8})$. Then

$$f \triangle g([x_0, x_1]) \subset \{(0,0)\} \cup \Big(\mathbb{R}^2 \setminus ([0, \frac{\varepsilon}{5}] \times [0, \frac{\varepsilon}{5}])\Big),\ (0,0) \in f \triangle g([x_0, x_1])$$

and $f \triangle g([x_0, x_1]) \cap \big(\mathbb{R}^2 \setminus ([0, \frac{\varepsilon}{5}] \times [0, \frac{\varepsilon}{5}])\big) \neq \emptyset$, which (according to theorem 1.1.11 of [6]) means that $f \triangle g$ is not an almost continuous function.

PROOF OF (16). Suppose to the contrary that there exists $g \in B(h, \frac{\varepsilon}{8}) \cap acH(f)$. Then $g(x) > \frac{\varepsilon}{8}$ for each $x \in (x_0, x_1) \setminus \bigcup_{n=1}^{\infty}(p_n, q_n)$. Let τ be a topology s-finer than τ_o such that f, $g : (\mathbb{R}, \tau) \to \mathbb{I}$ are continuous and let V be a τ-neighborhood of x_0 such that $f(V) \subset [0, \frac{\varepsilon}{9}) \supset g(V)$. Since $g(V) \subset [0, \frac{\varepsilon}{9})$,

$$V \cap (x_0, x_1] \subset \bigcup_{n=1}^{\infty}(p_n, q_n).$$

On the other hand, $f(V) \subset [0, \frac{\varepsilon}{9})$ implies that

$$V \cap (x_0, x_1] \cap \bigcup_{n=1}^{\infty}(p_n, q_n) = \emptyset.$$

This means that $V \cap (x_0, x_1] = \emptyset$ and, consequently, $(x_0, +\infty)$ is a τ-closed set, which is impossible. □

Theorem 3. *Let $f, g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$. If $f \triangle g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R}^2)$, then $f + g$, $f \cdot g$, $\min(f, g)$, and $\max(f, g) \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$.*

PROOF. We give the proof only in the case $f + g$. Assume that $f \triangle g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R}^2)$. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by $h(x, y) = x + y$. Then h is continuous, so $f + g = h \circ (f \triangle g)$ is an almost continuous function (see [8] or [6]). Since $D_{f+g} \subset D_f \cup D_g = D_{f\triangle g}$, $(f + g)(x) = 0$ for each $x \in D_{f+g}$. Thus $f + g \in \mathcal{A}_o(\mathbb{R}, \mathbb{R})$. □

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