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Krzysztof Ciesielski,* Department of Mathematics, West Virginia University, Morgantown, WV 26506-6310, USA (KCies@wvnvms.wvnet.edu)
Tomasz Natkaniec, Department of Mathematics, Gdańsk University, Wita Stwosza 57, 80-952 Gdańsk, Poland (mattn@ksinet.univ.gda.pl)

DARBOUX LIKE FUNCTIONS THAT ARE CHARACTERIZABLE BY IMAGES, PREIMAGES AND ASSOCIATIED SETS†

Abstract

For $\mathcal{A},\mathcal{B}\subset\mathcal{P}(\mathbb{R})$ let $\mathcal{C}_{\mathcal{A},\mathcal{B}}=\{f\in\mathbb{R}^{\mathbb{R}}\colon(\forall A\in\mathcal{A})\,(f(A)\in\mathcal{B})\}$ and $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}=\{f\in\mathbb{R}^{\mathbb{R}}\colon(\forall B\in\mathcal{B})\,(f^{-1}(B)\in\mathcal{A})\}$. A family $\mathcal{F}$ of real functions is characterizable by images (preimages) of sets if $\mathcal{F}=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ ($\mathcal{F}=\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, respectively) for some $\mathcal{A},\mathcal{B}\subset\mathcal{P}(\mathbb{R})$. We study which of the classes of Darboux like functions can be characterized in this way. Moreover, we prove that the class of all Sierpiński-Zygmund functions can be characterized by neither images nor preimages of sets.

1 Definitions and Preliminary Results

Our terminology is standard and follows [9]. We consider only real-valued functions of one real variable. No distinction is made between a function and its graph. By $\mathbb{R}$ and $\mathbb{I}$ we denote the set of all reals and the interval $[0,1]$, respectively. The family of all subsets of a set X is denoted by $\mathcal{P}(X)$. The family of all functions from a set X into Y is denoted by Y^X. By C and Const we denote the families of all continuous functions and all constant functions. The symbol $|X|$ stands for the cardinality of a set X. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. For the cardinal number κ we write $[X]^\kappa$ to denote the family

Key Words: Darboux functions, extendable functions, almost continuous functions, connectivity functions, functions with perfect road, peripherally continuous functions, DIVP-functions, CIVP-functions, SCIVP-functions, WCIVP-functions, Sierpiński- Zygmund functions, associated sets

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of all subsets Y of X with $|Y| = \kappa$. In particular, $[X]^1$ stands for the family of all singletons in X and $[X]^2$ for the family of all doubletons in X. By a Cantor set we mean any non-empty perfect nowhere dense subset of $\mathbb{R}$. Moreover, we say that a set $A \subset \mathbb{R}$ is Cantor dense in a set $X \subset \mathbb{R}$, if $A \cap J$ contains a Cantor set whenever J is a non-empty open interval J with $J \cap X \neq \emptyset$. By (a, b) we denote an open interval with endpoints a and b; i.e., the set of all $x \in \mathbb{R}$ such that $\min\{a, b\} < x < \max\{a, b\}$.

For families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$ we put

$$\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} \colon (\forall A \in \mathcal{A})\, (f(A) \in \mathcal{B})\},$$

and

$$\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \{f \in \mathbb{R}^{\mathbb{R}} \colon (\forall B \in \mathcal{B})\, (f^{-1}(B) \in \mathcal{A})\}.$$

Also, for a family $\mathcal{F}$ of real functions we will consider the following properties.

- $\mathcal{F}$ is *characterizable by images* of sets when $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$.
- $\mathcal{F}$ is *characterizable by preimages* of sets if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$.
- $\mathcal{F}$ is *topologized* if $\mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some <u>topologies</u> $\mathcal{A}, \mathcal{B}$ on $\mathbb{R}$; while,
- $\mathcal{F}$ is *characterizable by associated sets* if there exists an $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ such that

  > $f \in \mathcal{F}$ if and only if for every $\alpha \in \mathbb{R}$, the "associated" sets $E^\alpha(f) = \{x \colon f(x) < \alpha\}$ and $E_\alpha(f) = \{x \colon f(x) > \alpha\}$ belong to $\mathcal{A}$.

Clearly the class C can be defined by preimages of open sets; so it can be topologized and characterized by associated sets. On the other hand, this class cannot be characterized by images of sets [29, 12]. Nevertheless, some classes of functions, often considered in real analysis, have such characterizations. For example, the family D of all Darboux functions can be defined as the class of functions which map connected sets to connected sets. We will study which of the other classes of Darboux like functions can be characterized by images of sets. We consider also the analogous problem: which of the classes of Darboux like functions can be characterized by preimages of sets. Note that the problem of characterization of $\mathcal{F}$ by preimages is strongly connected with the problem of characterization of $\mathcal{F}$ by associated sets. In fact, if $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is characterizable by associated sets, then it is also characterizable by preimages. On the other hand, there exist families of functions that are characterizable by preimages but not by associated sets. (For example, the class of all quasi-continuous functions has this property [14]. Also, under GCH the class of all derivatives can be characterized by preimages [10], while it is not characterizable by associated sets [6].) Note also that those problems are connected with the problem of topologizing $\mathcal{F}$ that was studied recently in several papers. (See, e.g., [8].)

Following Gibson and Natkaniec [15], by "Darboux like" functions we understand the following classes of functions (from $\mathbb{R}$ into $\mathbb{R}$, unless otherwise specified).

D — the family of *Darboux* functions; i.e., such that map connected sets onto connected sets.

AC — the class of *almost continuous* functions in the sense of Stallings; i.e., such that every open neighborhood of f in $\mathbb{R} \times \mathbb{R}$ contains a continuous function from $\mathbb{R}$ into $\mathbb{R}$.

Conn(X) — the class of *connectivity* functions from a topological space X into $\mathbb{R}$; i.e., functions $f\colon X \to \mathbb{R}$ such that the restriction $f{\restriction}C$ is a connected subset of $X \times \mathbb{R}$ whenever C is a connected subset of $\mathbb{R}$. We will write Conn for Conn($\mathbb{R}$).

Ext — the family of *extendable* functions; i.e., functions $f\colon \mathbb{R} \to \mathbb{R}$ for which there exists a connectivity function $F\colon \mathbb{R} \times \mathbb{I} \to \mathbb{R}$ with the property that $F(x, 0) = f(x)$ for every $x \in \mathbb{R}$.

PR — the class of functions with *perfect road*; i.e., such that for every $x \in \mathbb{R}$ there exists a perfect set $P \subset \mathbb{R}$ having x as a bilateral limit point for which the restriction $f{\restriction}P$ of f to P is continuous at x.

PC — the class of *peripherally continuous* functions; i.e., functions $f\colon \mathbb{R} \to \mathbb{R}$ which satisfy *Young's condition* at every $x \in \mathbb{R}$; that is, such that there are monotone sequences $a_n \nearrow x$ and $b_n \searrow x$ with the property that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x)$.

CIVP — the family of functions f having the *Cantor intermediate value property*; i.e., such that for every x, $y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f(C) \subset K$.

SCIVP — the family of functions f having the *strong Cantor intermediate value property*; i.e., such that for every x, $y \in \mathbb{R}$ and for each Cantor set

K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f(C) \subset K$ and $f{\restriction}C$ is continuous.

WCIVP — the family of functions f having the *weak Cantor intermediate value property*; i.e., such that for every $x,\, y \in \mathbb{R}$ with $f(x) \neq f(y)$ there is a Cantor set C between x and y such that $f(C) \subset (f(x), f(y))$.

An excellent description of the properties of these families is presented in a survey paper of Gibson and Natkaniec [15]. In particular, the following inclusions $\subset$, denoted by $\longrightarrow$, hold.

$$\begin{tikzcd}
 & & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{C} \arrow[r] & \mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] \arrow[dr] & \mathrm{PR} \arrow[ur] & \\
 & & & & \mathrm{WCIVP} &
\end{tikzcd}$$

Chart 1

Recall also that, generally, all those classes are different. However in the first class of Baire B_1, all of them, except for C and WCIVP, are equal [4]. Recall also that for the functions from $\mathbb{R}^2$ to $\mathbb{R}$ the notions of peripherally continuous and of connectivity are equivalent [19].

The following remarks are proved in [12, Fact 1.2].

Remark 1.1. *Assume that $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and $\mathcal{F} \neq \mathbb{R}^{\mathbb{R}}$. Then*

(1) *if* Const $\subset \mathcal{F}$, *then* $[\mathbb{R}]^1 \subset \mathcal{B}$*;*

(2) *if the identity function* id *belongs to $\mathcal{F}$, then $\mathcal{A} \subset \mathcal{B}$;*

(3) *$\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{A}^*}$, where $\mathcal{A}^* = \{f(A)\colon f \in \mathcal{F} \;\&\; A \in \mathcal{A}\}$;*

(4) *if $[\mathbb{R}]^1 \subset \mathcal{B}$ and $B \in \mathcal{B} \cap [\mathbb{R}]^2$, then $B^{\mathbb{R}} \subset \mathcal{F}$.* □

Corollary 1.1. *Assume that $\mathcal{F}$ satisfies the following conditions:*

(1) Const $\subset \mathcal{F}$*;*

(2) *for every distinct $a, b \in \mathbb{R}$ there exists $f \in \mathcal{F}$ with $f(\mathbb{R}) = \{f(a), f(b)\} \in [\mathbb{R}]^2$;*

(3) *there exists $Z \subset \mathbb{R}$ such that any distinct $a, b \in \mathbb{R}$ the "characteristic" function*

$$\varphi^Z_{a,b} = \begin{cases} a & \text{if } x \in Z, \\ b & \text{if } x \notin Z \end{cases}$$

does not belong to $\mathcal{F}$.

Then $\mathcal{F}$ cannot be characterized by images of sets. □

In particular, none of the following classes of functions can be characterized by images of sets. (See also [12, Section 4].)

- The class of all Lebesgue measurable functions.
- The class of all functions having the Baire property.
- The class of all Borel functions.
- The class of all quasi-continuous functions.
- The class of all cliquish functions.

(For more on quasi-continuous and cliquish functions see [21] and [28], respectively.)

The next theorem shows that there is a class $\mathcal{F}$ of functions with the Baire property such that $\mathcal{F}$ contains all continuous functions and it can be characterized by images of sets. This stands in contrast to a theorem of Ciesielski, Dikranjan and Watson from [12] in which the authors show that every class $\mathcal{F}$ of real functions which contains all continuous functions and can be characterized by images of sets must contain a non-measurable function.

Let

$$\mathcal{D}_0 = \{D \cap I \colon D \text{ is dense in } \mathbb{R} \text{ and } I \neq \emptyset \text{ is an interval}\}.$$

We say that $f \colon \mathbb{R} \to \mathbb{R}$ has a *Dense Intermediate Value Property (*DIVP*)* if $f[A] \in \mathcal{D}_0$ for every $A \in \mathcal{D}_0$. Clearly every continuous function is DIVP.

Theorem 1.1. *If f is* DIVP*, then f is continuous on a dense set. In particular f has the Baire property.*

PROOF. Let $C(f)$ be the set of points of continuity of f. So, $C(f)$ is a G_δ set. By way of contradiction assume that $C(f)$ is not dense. Then there exists a non-empty open interval U such that f is discontinuous at every point of U.

For every $x \in U$ let $n_x \in \{1, 2, 3, \ldots\}$ be the smallest number n such that the oscillation of f at x is greater than $1/n$. Then, by the Baire Category

Theorem, there exists n such that $S_n = \{x \in U : n_x = n\}$ is of second category. Once again by the Baire Category Theorem we can find a rational number q such that $T_0 = \{x \in S_n : |f(x) - q| < 1/3n\}$ is of the second category. Let W be a non-empty open interval such that the set $T = T_0 \cap W$ is dense in W. For every $x \in T$ choose $z_x \in W$ such that $|f(x) - f(z_x)| > 1/n$. This can be done by the oscillation requirement. Let $T' = \{z_x : x \in T\}$ and $A = T \cup T'$. Then $A \in \mathcal{D}_0$. However, $f[A] \notin \mathcal{D}_0$, since $f[A] = f[T] \cup f[T]$ and

$$f[T] \subset \{y : |y - q| < 1/3n\} \text{ while } f[T'] \subset \{y : |y - q| > 2/3n\}.$$

Consequently, $f \notin$ DIVP. □

It is worth noting that the class DIVP can be defined in a "natural" way as the uniform limit of the family of quasi-continuous Darboux functions. This class is usually denoted QU. It has been studied in [23] and, recently, in [22]. Recall that $f \in$ QU if and only if f satisfies the following conditions:

- f is quasi-continuous ($f \in$ Q); i.e., $f \restriction C(f)$ is dense in f;
- f belongs to the class $\mathcal{U}_0$ (see [7]); i.e., for all $a < b$ the set $f[(a, b)]$ is dense in $(f(a), f(b))$.

Theorem 1.2. DIVP = QU.

PROOF. DIVP $\subset$ QU. It is clear that DIVP $\subset \mathcal{U}_0$. Thus it is enough to verify that DIVP $\subset$ Q. Suppose that $f \in$ DIVP is not quasi-continuous; i.e., $f(x_0) \notin \mathrm{cl}(f \restriction C(f))$ for some $x_0 \in \mathbb{R}$. Thus there exist open intervals U containing x_0 and V containing $f(x_0)$ such that $f(x) \notin V$ for each $x \in C(f) \cap U$. Since $f \in$ DIVP, the set $A = (C(f) \cap U) \cup \{x_0\}$ is dense in U; so $A \in \mathcal{D}_0$. On the other hand, $f[A] \notin \mathcal{D}_0$, a contradiction.

QU $\subset$ DIVP. Fix $f \in$ QU and $A \in \mathcal{D}_0$. Then A is dense in $[a, b]$, where $a = \inf(A)$ and $b = \sup(A)$. We will prove that $f[A]$ is dense in $[c, d]$, for $c = \inf(f[a, b])$, $d = \sup(f[a, b])$. Fix $y \in (c, d)$ and a neighborhood V of y. Since $f \in \mathcal{U}_0$, there is $x \in (a, b)$ such that $f(x) \in V$ (cf., [7]). Because $f \in$ Q, we can assume that $x \in C(f)$. Thus there exists a neighborhood U of x such that $U \subset (a, b)$ and $f[U] \subset V$. Since A is dense in U, there is $x_0 \in A \cap U$. Therefore, $f[A] \cap V \neq \emptyset$. Thus $f[A]$ is dense in $[c, d]$. □

Corollary 1.2. *The relation of the class* DIVP *to the classes from Chart 1 is as follows.*

(1) C $\subset$ DIVP $\subset$ PR $\cap$ WCIVP, *and the inclusions are proper.*

Moreover, these are the only inclusions between the class DIVP *to the classes from Chart 1; i.e.,*

(2) Ext $\not\subset$ DIVP*,*

(3) DIVP $\not\subset$ CIVP*, and*

(4) DIVP $\not\subset$ D*.*

Proof. (1). The proper inclusion C $\subset$ DIVP is obvious. To prove the other inclusion assume that $f \in$ DIVP. To prove DIVP $\subset$ WCIVP, take $a < b$ with $f(a) \neq f(b)$. Since $C(f)$ is dense in $\mathbb{R}$, by the definition of the class DIVP we can choose $x_0 \in (a,b) \cap C(f)$ such that $f(x_0) \in (f(a), f(b))$. Then, by the continuity of f at x_0, there exists a Cantor set $C \subset (a,b)$ such that $f[C] \subset (f(a), f(b))$. Thus f has WCIVP.

Next, to show that DIVP $\subset$ PR fix an $x \in \mathbb{R}$. Because $f \in$ QU, there exists a sequence $\{x_n\}_{n=0}^{\infty}$ of points at which f is continuous such that $\{x_{2n}\}_{n=0}^{\infty}$ is increasing to x, $\{x_{2n+1}\}_{n=0}^{\infty}$ is decreasing to x and $\lim_{n\to\infty} f(x_n) = f(x)$. (This follows easily from the definition of the class DIVP and the fact that $C(f)$ is dense in $\mathbb{R}$, by the argument similar to that of the previous paragraph. But see also [23, Lemma 2].) Now, as in the previous paragraph, for each $n \in \mathbb{N}$ we can choose a perfect set C_n such that

- x_n is a bilateral limit point of C_n;
- $f[C_n] \subset (f(x_n) - 1/n, f(x_n) + 1/n)$;
- $C = \bigcup_n C_n \cup \{x\}$ is a perfect set.

Then $f \restriction C$ is continuous at x; so C is a perfect road of f at x.

The fact that the inclusion DIVP $\subset$ PR $\cap$ WCIVP is proper follows from Theorem 1.1, since there are functions in Ext $\subset$ PR $\cap$ WCIVP without the Baire property. (In fact, every real function $f\colon \mathbb{R} \to \mathbb{R}$ is a sum of two extendable functions [11, 27], which clearly implies that there are many extendable functions without the Baire property.)

(2). Ext $\setminus$ DIVP $\neq \emptyset$, since $C(f) \neq \emptyset$ for every $f \in$ DIVP and there are $f \in$ Ext with $C(f) = \emptyset$ [3, 16, 26, 11]. (Also, every $f \in$ DIVP is Baire, while it is not the case for the functions from Ext.)

(3). To see that DIVP $\not\subset$ CIVP let C be the Cantor ternary set and $\mathcal{J}_n$ be the union of all components of $\mathbb{I} \setminus C$ with length 3^{-n-1}. Choose an enumeration $\{q_n \colon n \in \mathbb{N}\}$ of $\mathbb{Q}$ and define $f\colon \mathbb{R} \to \mathbb{R}$ by putting $f(x) = q_n$ for $x \in \mathcal{J}_n$, $n \in \mathbb{N}$, and $f(x) = 0$ otherwise. Then $f \in$ DIVP and $f[\mathbb{R}] = \mathbb{Q}$. So, $f[\mathbb{R}] \not\subset K$ for any Cantor set $K \subset \mathbb{R} \setminus \mathbb{Q}$, and $f \notin$ CIVP.

(4). To see that DIVP $\not\subset$ D let $\{X, A, B\}$ be a partition of $\mathbb{R}$ onto $\mathfrak{c}$-dense sets. Define $f\colon \mathbb{R} \to \mathbb{R}$ such that $f[A] = \{0\}$, $f[B] = \{1\}$, and $f[(a,b) \cap X] = \mathbb{R}$ for every $a < b$. Then $f \in$ D and $f \notin$ DIVP, since $f[A \cup B] = \{0,1\} \notin \mathcal{D}_0$ while $A \cup B \in \mathcal{D}_0$. $\square$

Theorem 1.3. *The class* DIVP *cannot be characterized by preimages.*

PROOF.[1] By way of contradiction suppose that there exist $\mathcal{A},\mathcal{B}\in\mathcal{P}(\mathbb{R})$ such that $\mathrm{DIVP}=\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$. We may assume that $\mathcal{A}=\{f^{-1}(B)\colon\ f\in\mathrm{DIVP},B\in\mathcal{B}\}$ and $\mathcal{B}\not\subset\{\emptyset,\mathbb{R}\}$. So, fix $B\in\mathcal{B}\setminus\{\emptyset,\mathbb{R}\}$. Let $(d_n)_n$ be a sequence of reals such that

- the set $D=\{d_n\colon n=0,1,\ldots\}$ is dense;
- if n is even, then $d_n\in B$;
- if n is odd, then $d_n\in\mathbb{R}\setminus B$.

Let C be the Cantor ternary set and let J_n be the union of closures of all components of $\mathbb{I}\setminus C$ with the length 3^{-n-1}. Now, define $A_0=\mathbb{R}\setminus\bigcup_{n=0}^{\infty}J_{2n+1}$ and $A_1=\bigcup_{n=0}^{\infty}J_{2n+1}$. Note that $A_0\cup A_1=\mathbb{R}$ and $A_0\cap A_1=\emptyset$. Put

$$f_0(x)=\begin{cases} d_0 & \text{for } x\notin\bigcup_{n=0}^{\infty}J_n\\ d_n & \text{for } x\in J_n\end{cases}\quad\text{and}\quad f_1(x)=\begin{cases} d_1 & \text{for } x\notin\bigcup_{n=0}^{\infty}J_n\\ d_{n+1} & \text{for } x\in J_n\end{cases}$$

It is easy to observe that $f_0,f_1\in\mathrm{DIVP}$ and $A_0=f_0^{-1}(B)$, $A_1=f_1^{-1}(B)$; so $A_0,A_1\in\mathcal{A}$. Moreover, $\{\mathbb{R},\emptyset\}\subset\mathcal{A}$, because all constant functions are in DIVP. Now, define $h\in\mathbb{R}^{\mathbb{R}}$ by $h(x)=i$ for $x\in A_i$, $i=0,1$. Then $h\in\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}\setminus\mathrm{DIVP}$. □

2 Classes of Functions from Chart 1

In the next part of this paper we will use the following lemma.

Lemma 2.1. *If* $\mathrm{DB}_1\subset\mathcal{C}_{\mathcal{A},\mathcal{B}}$ *and* $\mathcal{A}$ *contains a non-degenerate interval, then every interval belongs to* $\mathcal{B}$.

PROOF. It is well-known (and easy to verify) that for all intervals I and J, if $|I|=|J|$, then there exists a Darboux, Baire one function $f\in\mathbb{R}^{\mathbb{R}}$ such that $f(I)=J$. □

Theorem 2.1. *The following classes of Darboux like functions can be characterized by images of sets:*

Ext	AC	Conn	D	PC	SCIVP	CIVP	WCIVP	PR
$-$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$

In this table the symbol "+" ("−") means that the given class can (respectively, cannot) be characterized by images.

[1] The authors would like to thank Professor Havrey Rosen for pointing out a mistake in the first version of this proof.

Proof. We will use sets $A \subset \mathbb{R}$ that have the following properties:

(C_1) A is an interval; i.e., if $a, b \in A$, then $(a, b) \subset A$;

(C_2) for every $a, b \in A$ if $C \subset (a, b)$ is a Cantor set, then $C \cap A \neq \emptyset$;

(C_3) for every $a, b \in A$ with $a < b$ and for every F_σ set $E \subset (a, b)$ if E is Cantor dense in (a, b), then $E \cap A \neq \emptyset$;

(C_4) A is ordinarily dense in itself; i.e., if $a, b \in A$ and $a < b$, then there is $c \in A$ with $a < c < b$.

Note that for each $i = 1, 2, 3$ property (C_i) implies (C_{i+1}).

1. $\mathrm{D} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, where $\mathcal{A} = \mathcal{B}$ is the family of all intervals in $\mathbb{R}$ (i.e., all sets that satisfy the condition (C_1)). Thus D is characterizable by images of sets.

2. Neither PR nor PC can be characterized by images of sets.

Indeed, let $(Z, \mathbb{R} \setminus Z)$ be a partition of $\mathbb{R}$ onto sets that are Cantor dense in $\mathbb{R}$. Then for each $a, b \in \mathbb{R}$ with $a \neq b$, the function $\varphi^Z_{a,b}$ belongs to PR; thus also to PC. On the other hand, characteristic function of no singleton belongs to PC and therefore to PR. Hence by Corollary 1.1, the classes PC and PR are not characterizable by images of sets.

3. None of the classes Ext, AC, and Conn can be characterized by images of sets.

Indeed, suppose that $\mathrm{Ext} \subset \mathcal{F} \subset \mathrm{D}$ and $\mathcal{F} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. We can assume that $\mathcal{B} = \{f[A] \colon A \in \mathcal{A} \ \& \ f \in \mathcal{F}\}$. We will show that $\mathcal{F} = \mathrm{D}$. Note that $[\mathbb{R}]^1 \subset \mathcal{B}$, $[\mathbb{R}]^1 \neq \mathcal{B}$ and $\mathcal{A} \subset \mathcal{B}$.

Claim. Every $A \in \mathcal{A}$ has property C_3.

Indeed, suppose that there are $a, b \in A$ and an F_σ set $E \in (a, b)$ that is Cantor dense in (a, b) and $E \cap A = \emptyset$. We will construct an extendable function $f \colon \mathbb{R} \to \mathbb{I}$ such that $f[\mathbb{R} \setminus E] = \{0, 1\}$.

Let $g \colon \mathbb{I} \to \mathbb{I}$ be an extendable function whose graph is dense in $\mathbb{I}^2$. (See [3, 16]. Compare also [11] and [26].) Then there exists a Cantor dense F_σ set $F \subset (0, 1)$ such that $\mathbb{I} \setminus F$ is g-negligible [25]. (This means that every function $\tilde{g} \colon \mathbb{I} \to \mathbb{I}$ with $\tilde{g} \restriction F = g \restriction F$ is still extendable). Let $h \colon [a, b] \to \mathbb{I}$ be a homeomorphism such that $h[E] = F$. (See [18, Lemma 3].) Then the composition $g \circ h \colon [a, b] \to \mathbb{I}$ is an extendable function and the set $[a, b] \setminus E$ is $(g \circ h)$-negligible [24]. Thus $f_0 \colon [a, b] \to \mathbb{I}$ defined by

$$f_0(x) = \begin{cases} 0 & \text{if } x = a, \\ g \circ h(x) & \text{if } x \in E, \\ 1 & \text{otherwise} \end{cases}$$

is an extendable function. Let $f\colon \mathbb{R} \to \mathbb{I}$ be the extension of f_0 such that $f(x) = 0$ for $x < a$ and $f(x) = 1$ for $x > b$. Observe that $f \in \text{Ext}$. Indeed, according to [17], there exists a peripherally continuous function $F_0\colon [a,b]\times\mathbb{I} \to \mathbb{I}$ such that $F_0{\restriction}([a,b]\times\{0\}) = f_0$. Moreover, we can assume that $F_0{\restriction}(\{a,b\}\times\mathbb{I})$ is continuous. (Actually, F_0 can be constant on intervals $\{a\}\times\mathbb{I}$ and $\{b\}\times\mathbb{I}$. While this is not mentioned explicitly in [17], it can be achieved by a minimal modification[2] of the proof presented there.) Then $F\colon \mathbb{R}\times\mathbb{I} \to \mathbb{I}$ defined by

$$F(x,y) = \begin{cases} F_0(x,y) & \text{if } x \in [a,b], \\ F_0(a,y) & \text{if } x < a, \\ F_0(b,y) & \text{if } x > b, \end{cases}$$

also is peripherally continuous; so $f = F{\restriction}\mathbb{R}\times\{0\}$ is extendable.

Let f be such a function. Then $f[A] = \{0,1\}$. So $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ contains characteristic functions of all subsets of $\mathbb{R}$, contrary to $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{F} \subset \text{D}$. The Claim has been proved.

Now we will prove that every $A \in \mathcal{A}$ is an interval. Indeed, suppose that there is $B \in \mathcal{A}$ that is not an interval. Then $B \in \mathcal{B}$, since $\mathcal{A} \subset \mathcal{B}$. Let $f\colon \mathbb{R} \to B$ be a surjection such that for every $y \in B$ the level set $f^{-1}(y)$ is Cantor dense in $\mathbb{R}$. Then $f[A] = B$ for $A \in \mathcal{A} \setminus [\mathbb{R}]^1$. Hence $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$. On the other hand, $f[\mathbb{R}]$ is not connected. Thus $f \notin \text{D}$, contrary to $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{F} \subset \text{D}$. Since $\mathcal{A} \neq [\mathbb{R}]^1$, there is $A \in \mathcal{A}$ that is not a non-degenerate interval. Thus, by Lemma 2.1, every interval I belongs to $\mathcal{B}$ and consequently, $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \text{D}$.

4. The class CIVP is characterizable by images of sets.

Let $\mathcal{A}$ be the family of all sets $A \subset \mathbb{R}$ that satisfy the condition (C_2) and let $\mathcal{B} = \mathcal{A}$. We will prove that $\text{CIVP} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Fix $f \in \text{CIVP}$, $A \in \mathcal{A}$, $a,b \in A$ and a Cantor set $C \subset (f(a), f(b))$. Then there exists a Cantor set $K \subset (a,b)$ such that $f[K] \subset C$. Then $C \cap f[A] \neq \emptyset$, since $K \cap A \neq \emptyset$, and so $f[A] \in \mathcal{B}$. Hence $\text{CIVP} \subset \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Thus, $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ proving $\text{CIVP} \subset \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Now fix $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and by way of contradiction suppose that $f \notin \text{CIVP}$. So there exist $a,b \in \mathbb{R}$ and a Cantor set $C \subset (f(a), f(b))$ such that $f[K] \subset C$ for no Cantor set $K \subset (a,b)$. Thus $A = [a,b] \setminus f^{-1}(C) \in \mathcal{A}$ and $f[A] \notin \mathcal{B}$, contrary to $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

5. The class SCIVP cannot be characterized by images of sets.

[2] Let $\mathcal{J}$ be a family of peripheral intervals as defined in [17] and let $\mathcal{J}_0$ be the set of all $\langle I, J\rangle \in \mathcal{J}$ such that if $0 \in I$ $(1 \in I)$, then $f(0) \in J$ $(f(1) \in J)$. Then $\mathcal{J}_0$ is also a family of peripheral intervals. Now, in the definition of g (described in [17]) we can additionally assume that $g(\{0\}\times I) = \{f(0)\}$ and $g(\{1\}\times I) = \{f(1)\}$. Then $F_0 = g$ has the desired properties.

Suppose that SCIVP $= \mathcal{C}_{\mathcal{A},\mathcal{B}}$, where $\mathcal{B} = \{f[A]\colon f \in \text{SCIVP} \,\&\, A \in \mathcal{A}\}$. Observe that $[\mathbb{R}]^1 \subset \mathcal{B}$, $[\mathbb{R}]^1 \neq \mathcal{B}$, and moreover, if $A \in \mathcal{A}$, then A satisfies the condition (C_2). Indeed, suppose that there exist $A \in \mathcal{A}$, $a, b \in A$ and a Cantor set $C \subset (a, b)$ such that $C \cap A = \emptyset$. Decompose C into $\mathfrak{c}$ many sets $\{C_\alpha\colon \alpha < \mathfrak{c}\}$, where each C_α is Cantor dense in C. Let $\mathbb{R} = \{r_\alpha\colon \alpha < \mathfrak{c}\}$ and put

$$f(x) = \begin{cases} 0 & \text{if } x \text{ and } a \text{ belong to the same component of } \mathbb{R} \setminus C, \\ r_\alpha & \text{if } x \in C_\alpha,\ \alpha < \mathfrak{c}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $f \in$ SCIVP and $f[A] = \{0, 1\} \in \mathcal{B}$. Thus $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ contains characteristic functions of all subsets of $\mathbb{R}$, contrary to $\mathcal{C}_{\mathcal{A},\mathcal{B}} =$ SCIVP.

Since SCIVP $\subset$ CIVP, by case **4** each $B \in \mathcal{B}$ has property (C_2). On the other hand, each $B \subset \mathbb{R}$ that satisfies (C_2) belongs to $\mathcal{B}$. Indeed, fix such a B. Let $f\colon \mathbb{R} \to B$ be a function such that for each $y \in B$ the level set $f^{-1}(y)$ is Cantor dense in $\mathbb{R}$. Then $f \in$ SCIVP and $f[A] = B$ for each $A \in \mathcal{A} \setminus [\mathbb{R}]^1$. Thus $B \in \mathcal{B}$. Consequently, CIVP $\subset \mathcal{C}_{\mathcal{A},\mathcal{B}}$, a contradiction.

6. The class WCIVP is characterizable by images of sets.

Let $\mathcal{A}$ be the family of all sets that satisfy the condition (C_2) and let $\mathcal{B}$ be the family of all $B \subset \mathbb{R}$ that satisfy statement (C_4). We will verify that WCIVP $= \mathcal{C}_{\mathcal{A},\mathcal{B}}$. The inclusion WCIVP $\subset \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is obvious. Now assume that $f \notin$ WCIVP. Then there are $a, b \in \mathbb{R}$ such that $a < b$ and $f[C] \not\subset (f(a), f(b))$ for each Cantor set $C \subset (a, b)$. Put $A = [a, b] \setminus f^{-1}(f(a), f(b))$. Then $A \in \mathcal{A}$ and $f[A] \notin \mathcal{B}$; thus $f \notin \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Hence $\mathcal{C}_{\mathcal{A},\mathcal{B}} \subset$ WCIVP, and consequently, we have the equality WCIVP $= \mathcal{C}_{\mathcal{A},\mathcal{B}}$. $\square$

Now we will consider the problem of determining which of the classes of Darboux like functions from Chart 1 can be characterized by preimages or by associated sets. The question whether the class $\mathcal{F}$ is characterizable by associated sets have been studied for the following classes of Darboux like functions: D [5], Conn [13], AC [20] and Ext [27]. Recall that none of those classes can be characterized by associated sets. The next theorem generalizes these results.

Theorem 2.2. *The following classes of Darboux like functions can be characterized by preimages.*

Ext	AC	Conn	D	PC	SCIVP	CIVP	WCIVP	PR
$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$+$

In this table the symbol "+" ("−") means that the given class can (respectively, cannot) be characterized by preimages.

PROOF. The argument will be split into three cases.

1. None of the classes, Ext, AC, Conn, D, SCIVP, CIVP, WCIVP, can be characterized by preimages of sets.

Assume that $\mathrm{Ext} \subset \mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$. We will prove that $\mathcal{F} \setminus (\mathrm{D} \cup \mathrm{WCIVP}) \neq \emptyset$. We can assume that:

- $\mathcal{A} = \{f^{-1}(B)\colon f \in \mathcal{F} \ \&\ B \in \mathcal{B}\}$;
- $\mathcal{B} \neq \emptyset$ and $\{\emptyset, \mathbb{R}\} \cap \mathcal{B} = \emptyset$;
- $\{\emptyset, \mathbb{R}\} \subset \mathcal{A}$.

Fix $B \in \mathcal{B}$, $y_0 \in \mathcal{B}$ and $y_1 \notin \mathcal{B}$. Let $f \in \mathrm{Ext}$ be dense in $\mathbb{R}^2$. (See [26] or [11] for examples of such functions.) By a result from [25], there exists an F_σ meager set $C \subset \mathbb{R}$ such that $\mathbb{R} \setminus C$ is f-negligible. As in [24] (and by an argument similar to that used in the proof of Claim from case **3** of Theorem 2.1), we can construct $f_1 \in \mathrm{Ext}$ and an F_σ meager set $D \subset \mathbb{R}$ such that f_1 is dense in $\mathbb{R}^2$, D is f_1-negligible, and $C \cap D = \emptyset$. Set $C_0 = C \cap f^{-1}(B)$, $C_1 = C \setminus C_0$, $D_0 = D \cap f^{-1}(B)$, $D_1 = D \setminus D_0$,

$$g(x) = \begin{cases} f(x) & \text{for } x \in C \\ y_0 & \text{for } x \in D_1 \\ y_1 & \text{otherwise} \end{cases} \quad \text{and} \quad g_1(x) = \begin{cases} f_1(x) & \text{for } x \in D \\ y_1 & \text{for } x \in C_0 \\ y_0 & \text{otherwise} \end{cases}$$

Then $g, g_1 \in \mathrm{Ext}$. Thus $E = C_0 \cup D_1 = g^{-1}(B)$ and $F = \mathbb{R} \setminus (C_0 \cup D_1) = g_1^{-1}(B)$ belong to $\mathcal{A}$. Note that E and F are dense in $\mathbb{R}$, $E \cup F = \mathbb{R}$ and $E \cap F = \emptyset$. Let $h \in \mathbb{R}^{\mathbb{R}}$ be the characteristic function of E. Then $h \in \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} \setminus (\mathrm{D} \cup \mathrm{WCIVP})$.

2. The class PC can be characterized by preimages.

Indeed, $\mathrm{PC} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, where $\mathcal{A}$ is the family of all bilaterally dense in itself subsets of $\mathbb{R}$ and $\mathcal{B}$ is the family of open intervals.

3. The class PR can be characterized by preimages.

Indeed, let $\mathcal{A}$ be the family of all bilaterally Cantor dense in itself subsets of $\mathbb{R}$ and $\mathcal{B}$ be the family of open intervals. Then $\mathrm{PR} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$.

(Notice that the last two equalities follow also from the fact that the classes PC and PR can be defined in terms of the continuity with respect to systems of paths. See [2].) □

Corollary 2.1. *None of the classes of Darboux like functions from Chart 1 can be topologized.*

PROOF. By Theorem 2.2 we need consider only two classes: PC and PR. Assume that $\mathcal{A}$ and $\mathcal{B}$ are topologies on $\mathbb{R}$ and $\mathrm{PR} \subset \mathcal{F} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$. We will prove that $\mathcal{F} = \mathbb{R}^{\mathbb{R}}$. This is obvious if $\mathcal{B} = \{\emptyset, \mathbb{R}\}$. Thus suppose that there exists $B \in \mathcal{B} \setminus \{\emptyset, \mathbb{R}\}$ and fix $y_0 \in B$, $y_1 \notin B$. We will prove that $[\mathbb{R}]^1 \subset \mathcal{A}$; so $\mathcal{A} = \mathcal{P}(\mathbb{R})$ since $\mathcal{A}$ is a topology. For an $x_0 \in \mathbb{R}$ divide the set $\mathbb{R} \setminus \{x_0\}$ into two sets C_0 and C_1, each Cantor dense in $\mathbb{R}$. Put

$$f_0(x) = \begin{cases} y_0 & \text{for } x \in \{x_0\} \cup C_0 \\ y_1 & \text{for } x \in C_1 \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} y_0 & \text{for } x \in \{x_0\} \cup C_1 \\ y_1 & \text{for } x \in C_0 \end{cases}$$

Then $f_0, f_1 \in \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, since $f_0, f_1 \in \mathrm{PR}$. Thus $\{x_0\} = f_0^{-1}(B) \cap f_1^{-1}(B) \in \mathcal{A}$. It follows that $\mathcal{A} = \mathcal{P}(\mathbb{R})$. □

Corollary 2.2. *None of the classes of Darboux like functions from Chart 1 can be defined by associated sets.*

PROOF. By Theorem 2.2 we need consider only two classes: PC and PR. Assume that $\mathrm{PR} \subset \mathcal{F}$ and $\mathcal{F}$ can be characterized by associated sets. We will prove that $\mathcal{F} \setminus \mathrm{PC} \neq \emptyset$.

Let $\mathcal{A}$ denote the family of all associated sets of $\mathcal{F}$. Divide the set $\mathbb{R} \setminus \{0\}$ onto two sets C and D, each Cantor dense in $\mathbb{R}$. Since the characteristic functions $\chi_C, \chi_D \in \mathrm{PR} \subset \mathcal{F}$, the sets C, $\mathbb{R} \setminus C$, D, and $\mathbb{R} \setminus D$ belong to $\mathcal{A}$. Then $f = \chi_C - \chi_D \in \mathcal{F} \setminus \mathrm{PC}$, with 0 being a point in which f is not peripherally continuous. □

3 The Class of Sierpiński-Zygmund Functions

In this section we consider the problem whether the class of all Sierpiński-Zygmund functions can be characterized by images or by preimages. Recall that for $X \subset \mathbb{R}$ the class $\mathrm{SZ}(X)$ of *Sierpiński-Zygmund functions* is the class of all functions $f\colon X \to \mathbb{R}$ whose restrictions $f \restriction Y$ are discontinuous for all subsets Y of X of cardinality continuum. We will write SZ for $\mathrm{SZ}(\mathbb{R})$.

Theorem 3.1. *The class* SZ *can be characterized neither by images nor preimages of sets.*

PROOF. First, by way of contradiction assume that $\mathrm{SZ} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ for some $\mathcal{A}, \mathcal{B} \subset \mathbb{R}$. Note that $\mathcal{B} \not\subset \{\emptyset, \mathbb{R}\}$, since otherwise either $\mathrm{SZ} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \mathbb{R}^{\mathbb{R}}$ (if $\mathcal{B} \subset \mathcal{A}$) or $\mathrm{SZ} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}} = \emptyset$ (if $\mathcal{B} \not\subset \mathcal{A}$), a contradiction. So, let $B_0 \in \mathcal{B} \setminus \{\emptyset, \mathbb{R}\}$ and pick $x \in B_0$. If every non-empty $B \in \mathcal{B}$ has cardinality $< \mathfrak{c}$, then $\mathcal{A}$ contains every subset A of cardinality $< \mathfrak{c}$. (Since $A = f^{-1}(B_0) \in \mathcal{A}$, where $f \in \mathrm{SZ}$ is such that $f[A] = \{x\}$ and $f[\mathbb{R} \setminus A] \subset \mathbb{R} \setminus B_0$.) Then the identity is in $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, a contradiction. So, $\mathcal{B}$ contains a set B of cardinality $\mathfrak{c}$. Thus, $\mathbb{R} \in \mathcal{A}$, since $\mathbb{R} = f^{-1}(B) \in \mathcal{A}$, where $f \in \mathrm{SZ}$ is such that $f[\mathbb{R}] \subset B$. Notice also that $\emptyset \in \mathcal{A}$. Indeed, if there is $B \in \mathcal{B}$ such that $|\mathbb{R} \setminus B| = \mathfrak{c}$, then $\emptyset = f^{-1}(B) \in \mathcal{A}$, where $f \in \mathrm{SZ}$ is such that $f[\mathbb{R}] \subset \mathbb{R} \setminus B$. So, by way of contradiction assume that $|\mathbb{R} \setminus B| < \mathfrak{c}$ for every $B \in \mathcal{B}$. Then $\mathcal{A}$ contains every set $A \subset \mathbb{R}$ with $|\mathbb{R} \setminus A| < \mathfrak{c}$ since $A = f^{-1}(B_0) \in \mathcal{A}$, where $f \in \mathrm{SZ}$ is such that $f[A] \subset B_0$ and $f[\mathbb{R} \setminus A] \subset \mathbb{R} \setminus B_0$. But then the identity is in $\mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$, a contradiction. So, $\emptyset, \mathbb{R} \in \mathcal{A}$, implying that $\mathrm{SZ} = \mathcal{C}^{-1}_{\mathcal{A},\mathcal{B}}$ contains all constants, a contradiction.

Next, by way of contradiction assume that $\mathrm{SZ} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ for some families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\mathbb{R})$. Clearly we can assume that $\emptyset \notin \mathcal{A}$ and that $\mathcal{A} \neq \emptyset$. First note that $\mathcal{A} \subset [\mathbb{R}]^{\mathfrak{c}}$. Indeed, suppose that $A \in \mathcal{A} \cap [\mathbb{R}]^{<\mathfrak{c}}$. Since there exist SZ functions that are constant on A, it follows that $\mathcal{B}$ contains a singleton and consequently, $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ contains a constant function, a contradiction.

So, take $A_0 \in \mathcal{A}$ of cardinality $\mathfrak{c}$ and let $f \in \mathrm{SZ}$ be one-to-one. Then $B = f[A_0] \in \mathcal{B}$ has cardinality $\mathfrak{c}$. Note that $[B]^{\mathfrak{c}} \subset \mathcal{B}$. Indeed, if $C \in [B]^{\mathfrak{c}}$, let $X = f^{-1}(C)$ and let $g \in \mathrm{SZ}$ be such that $g\restriction X = f\restriction X$ and $g[\mathbb{R} \setminus X] \subset C$. Then $C = g[A_0] \in \mathcal{B}$. Now, pick one-to-one $g \in \mathrm{SZ}$ with $g[\mathbb{R}] \subset B$ and let $h\colon \mathbb{R} \to \mathbb{R}$ be such that $h\restriction(\mathbb{R} \setminus B) = g\restriction(\mathbb{R} \setminus B)$ and $h(x) = x$ for every $x \in B$. Then clearly $h \notin \mathrm{SZ}$. However, $h \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ since $h[A] \in \mathcal{B}$ for every $A \in \mathcal{A}$. Indeed, because $|A| = \mathfrak{c}$, we clearly have $h[A] \in [B]^{\mathfrak{c}} \subset \mathcal{B}$. This finishes the proof. $\square$

Now, recall the following theorem.

Theorem 3.2. (Balcerzak, Ciesielski, Natkaniec [1])

(a) *If $\mathbb{R}$ is not a union of less than continuum many of its meager subsets (thus under CH and MA), then there exists an $f \in \mathrm{SZ} \cap \mathrm{D}$.*

(b) *There is a model of ZFC in which every Darboux function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous on some set of cardinality $\mathfrak{c}$. In particular, in this model we have $\mathrm{SZ} \cap \mathrm{D} = \emptyset$.*

Note also that if $\mathrm{SZ} \cap \mathrm{D} = \emptyset$, which is consistent with ZFC, then

(1) $\mathrm{SZ} \cap \mathrm{D}$ can be characterized by images and by preimages;

(2) $\mathrm{D} \setminus \mathrm{SZ}$ can be characterized by images, but cannot be characterized by preimages;

(3) $\mathrm{SZ} \setminus \mathrm{D}$ can be characterized by neither images no preimages.

On the other hand, the statement (3) can be proved in ZFC, by an easy modification of the proof of Theorem 3.1. Moreover, since $\mathrm{Ext} \subset \mathrm{D} \setminus \mathrm{SZ}$

- the argument from Theorem 2.2 shows in ZFC that $\mathrm{D}\setminus\mathrm{SZ}$ cannot be characterized by preimages and
- the argument from Theorem 2.1 shows that $\mathrm{D}\setminus\mathrm{SZ}$ cannot be characterized by preimages as long as $\mathrm{D}\setminus\mathrm{SZ}\neq\mathrm{D}$; in particular, the statement

  the class $\mathrm{D}\setminus\mathrm{SZ}$ can be characterized by images

  is equivalent to the equation $\mathrm{D}\setminus\mathrm{SZ}=\mathrm{D}$ and so, it cannot be proved in ZFC.

Problem 1. *Can the class* $\mathrm{SZ}\cap\mathrm{D}$ *be characterized by images or preimages if* $\mathrm{SZ}\cap\mathrm{D}\neq\emptyset$*?*

Under CH this problem is probably not very difficult. The interesting part is, whether in ZFC alone the assumption $\mathrm{SZ}\cap\mathrm{D}\neq\emptyset$ decides whether the class $\mathrm{SZ}\cap\mathrm{D}$ can be characterized by images or preimages.

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