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Tamás Keleti,* Mathematical Institute of the Hungarian Academy of Sciences, 1364 Budapest, POB 127, Hungary (e-mail: `elek@cs.elte.hu`)

PERIODIC L_p FUNCTIONS WITH L_q DIFFERENCE FUNCTIONS†

Abstract

Let $0 < p < q < \infty$. We investigate the following question: For which subsets H of the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is it true that if $f \in L_p$ and $\Delta_h f(x) = f(x+h) - f(x) \in L_q$ for any $h \in H$, then $f \in L_q$? We prove that this is not true for pseudo-Dirichlet sets. Evidence is gathered for the conjecture that the class of counter-examples is precisely the class of N-sets.

1 Introduction

In [7] the following notion was introduced. Let $\mathcal{F}$ es $\mathcal{G}$ be classes of functions on the circle group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $\mathcal{F} \supset \mathcal{G}$. We denote by $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ the class of those subsets H of $\mathbb{T}$, for which a function $f \in \mathcal{F}$ can have difference functions $\Delta_h f(x) = f(x+h) - f(x)$ in $\mathcal{G}$ for every $h \in H$ without f belonging to $\mathcal{G}$. That is,

$$\mathfrak{H}(\mathcal{F}, \mathcal{G}) = \Big\{ H \subset \mathbb{T} : (\exists f \in \mathcal{F} \setminus \mathcal{G})\ (\forall h \in H)\ \Delta_h f \in \mathcal{G} \Big\}.$$

We denote by L_0 the class of measurable real functions on $\mathbb{T}$, and L_p denotes the class of those measurable real functions f on $\mathbb{T}$ for which $\|f\|_p = \left(\int_T |f|^p\right)^{1/p} < \infty$. It was proved in [7] (Theorem 4.10) that for any $0 \le p < q < \infty$ we have $\mathfrak{H}(L_p, L_q) \subset \mathfrak{F}_\sigma$ where $\mathfrak{F}_\sigma$ denotes the family of those subsets of $\mathbb{T}$ that can be covered by a proper F_σ subgroup of $\mathbb{T}$.

The classes $\mathfrak{H}(\mathcal{F}, \mathcal{G})$ are often related to some classes of thin sets in harmonic analysis. Now we define those classes that will arise in our results. Detailed explanation of this topic can be found in the monographs [2], [10], in the recent research papers [4] and [5] or in the recent topical survey [3].

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- A set $H \subset \mathbb{T}$ is called a *Dirichlet set* if there exists an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ we have $|\sin q_n \pi x| < \varepsilon_n$ for every $n \in \mathbb{N}$.
- A set $H \subset \mathbb{T}$ is called a *pseudo-Dirichlet set* if there exist an increasing sequence of integers (q_n) and a sequence (ε_n) converging to zero such that for any $x \in H$ there exists a $k(x)$ such that $|\sin q_n \pi x| < \varepsilon_n$ if $n \geq k(x)$.
- A set $H \subset \mathbb{T}$ is called an *N-set* if there exists a trigonometric series on $\mathbb{T}$ that is absolutely convergent on H but is not absolutely convergent everywhere.

The family of Dirichlet sets, pseudo-Dirichlet sets and N-sets are denoted by $\mathfrak{D}$, $\mathfrak{pD}$ and $\mathfrak{N}$, respectively. It is known ([5],[9]) that

$$\mathfrak{D} \subsetneq \mathfrak{pD} \subsetneq \mathfrak{N} \subsetneq \mathfrak{F}_\sigma.$$

In this paper we prove that for and $0 < p < q$ we have $\mathfrak{pD} \subset \mathfrak{H}(L_p, L_q)$. We also investigate the possible improvement of the earlier mentioned inclusion $\mathfrak{H}(L_p, L_q) \subset \mathfrak{F}_\sigma$ of [7].

2 The Main Result

Lemma 2.1. *Suppose that $0 < p < q < \infty$ and (a_n) is a sequence of positive reals such that*

$$\sum_{j=1}^{\infty} a_j < \infty \text{ and } \sum_{j=k}^{\infty} a_j \geq C/k^N \text{ for fixed } C > 0 \text{ and } N \geq 2.$$

Then there exists sequence of positive reals (c_j) such that

$$\sum_{j=1}^{\infty} a_j c_j^p < \infty, \tag{A}$$

$$\sum_{j=1}^{\infty} a_j c_j^q = \infty, \tag{B}$$

and

$$\sum_{j=2}^{\infty} a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|) \big)^q < \infty. \tag{C}$$

PROOF. First we define a sequence of integers $0 = n_0 < n_1 < n_2 < \ldots$ such that

(i) $$\sum_{j=n_k}^{\infty} a_j \geq C'/k^N \text{ for infinitely many } k \text{ for a fixed } C' > 0 \text{ and}$$

(ii) $$\sum_{j=n_k+1}^{\infty} a_j < C/k^N \qquad \text{for every } k \in \mathbb{N}.$$

Let $k \in \mathbb{N}$ and suppose that n_{k-1} is already defined. If $\sum_{j=n_{k-1}+1}^{\infty} a_j \geq C/k^N$, then let $n_k = \max\left\{ i : \sum_{j=i}^{\infty} a_j \geq C/k^N \right\}$. Otherwise let $n_k = n_{k-1} + 1$. Then clearly $n_k > n_{k-1}$ and (ii) holds. If $\sum_{j=n_{k-1}+1}^{\infty} a_j \geq C/k^N$ for infinitely many k, then (i) clearly holds with $C' = C$. Otherwise, there exists an $m \in \mathbb{N}$ such that $n_k = n_m + (k - m)$ for every $k \geq m$. Then for every $k \geq m$,

$$\sum_{j=n_k}^{\infty} a_j = \sum_{j=n_m+k-m}^{\infty} a_j \geq \frac{C}{(n_m + k - m)^N} \geq \frac{C}{((n_m + 1 - m)k)^N}.$$

Thus (i) holds for $C' = \frac{C}{(n_m+1-m)^N}$.

Let $b_k = a_{n_k+1} + a_{n_k+2} + \ldots + a_{n_{k+1}}$ $(k = 0, 1, 2, \ldots)$, and $\alpha = \sup\left\{ \beta : \sum_{k=1}^{\infty} b_k k^{\beta} < \infty \right\}$. We claim that $1 \leq \alpha \leq N$.

$1 \leq \alpha$: By (ii),

$$\sum_{k=1}^{\infty} b_k k = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} b_n = \sum_{k=1}^{\infty} \sum_{j=n_k+1}^{\infty} a_j < \sum_{k=1}^{\infty} C/k^N < \infty.$$

$\alpha \leq N$: Let $\varepsilon > 0$ arbitrary. If (i) holds for $k \geq 2$, then

$$\begin{aligned}\sum_{n=1}^{\infty} b_n n^{N+\varepsilon} &\geq \sum_{n=k-1}^{\infty} b_n n^{N+\varepsilon} \geq \sum_{n=k-1}^{\infty} b_n (k-1)^{N+\varepsilon} \\ &= (k-1)^{N+\varepsilon} \sum_{j=n_{k-1}+1}^{\infty} a_j \geq (k-1)^{N+\varepsilon} \sum_{j=n_k}^{\infty} a_j \\ &\geq (k-1)^{N+\varepsilon} \frac{C'}{k^N}.\end{aligned}$$

Since k can be arbitrarily big this implies that $\sum_{n=1}^{\infty} b_n n^{N+\varepsilon} = \infty$. Thus $\alpha \leq N$.

Choose γ such that $\alpha/q < \gamma < \min(\alpha/p, (\alpha/q)+1)$, and let $c_n = k^\gamma$ for any $n_k < n \le n_{k+1}$ $(k = 0, 1, \ldots)$. Then $\sum_{j=1}^\infty a_j c_j^p = \sum_{k=0}^\infty b_k (k^\gamma)^p < \infty$, since $\gamma p < \alpha$. We also have $\sum_{j=1}^\infty a_j c_j^q = \sum_{k=0}^\infty b_k (k^\gamma)^q = \infty$, since $\gamma q > \alpha$. If $\gamma \le 1$, then $|c_{j+1} - c_j| \le 1$ for any $j \in \mathbb{N}$; so we have

$$\sum_{j=2}^\infty a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|)\big)^q \le \sum_{j=2}^\infty a_j < \infty.$$

If $\gamma > 1$, then, applying the mean value theorem, we have

$$\max(|k^\gamma - (k-1)^\gamma|, |(k+1)^\gamma - k^\gamma|) \le \gamma(k+1)^{\gamma-1} \le \gamma(2k)^{\gamma-1}$$

for any $k \in \mathbb{N}$. Thus

$$\begin{aligned}
&\sum_{j=n_1+1}^\infty a_j \big(\max(|c_j - c_{j-1}|, |c_{j+1} - c_j|)\big)^q \\
\le &\sum_{k=1}^\infty b_k \big(\max(|k^\gamma - (k-1)^\gamma|, |(k+1)^\gamma - k^\gamma|)\big)^q \\
\le &\sum_{k=1}^\infty b_k \left(\gamma(2k)^{\gamma-1}\right)^q = (2^{\gamma-1}\gamma)^q \sum_{k=1}^\infty b_k k^{q(\gamma-1)} < \infty,
\end{aligned}$$

since $q(\gamma - 1) < \alpha$. □

Notation 2.2. If $A, B \subset \mathbb{T}$, then we put $A + B = \{a + b : a \in A, b \in B\}$. The sets $A - B$ and $-A$ are defined similarly. If $k \in \mathbb{N}$, the k-fold sum $A + \ldots + A$ is denoted by kA. The Lebesgue outer measure of H is denoted by $|H|$. By the measure of a set we mean its outer Lebesgue measure. Sometimes we identify $\mathbb{T}$ with $[0, 1)$. If $x \in \mathbb{T}$, then by $|x|$ we mean $\min(x, 1 - x)$.

Lemma 2.3. *Suppose that A and H are closed subsets of $\mathbb{T}$, $A = -A$, $0 \in A$, H has positive measure and there exist constants $C > 0$ and $N \ge 2$ such that*

$$\big|H + kA\big| \le 1 - C/k^N \quad (\forall k \in \mathbb{N}). \tag{1}$$

Then $A \in \mathfrak{H}(L_p, L_q)$ for any $0 < p < q < \infty$.

PROOF. For given $0 < p < q < \infty$ we construct a function $g \in L_p \setminus L_q$ such that $\Delta_h g \in L_q$ for any $h \in A$. The construction is a modification of the construction of Balcerzak, Buczolich and Laczkovich ([1], proof of the (i)⇒(ii) part of Theorem 1.1). Using the same notation, we let $H_j = H + jA$, $H_\infty = \cup_{j \in \mathbb{N}} H_j$. If A is a finite subset of $\mathbb{Q} \cap \mathbb{T}$, then an arbitrary periodic function $g \in L_p \setminus L_q$ with period $1/m$ (where m is a common denominator of the elements of A) satisfies the conditions. Otherwise, since $A=-A$, H_∞ has infinitely many periods; thus $|H|>0$ implies $|H_\infty|=1$.

Let $a_j=|H_j\setminus H_{j-1}|$ ($j\in\mathbb{N}$). Then, using $|H_\infty|=1$ and (1), we get

$$\sum_{n=k}^{\infty} a_n = \left|H_\infty\setminus H_{k-1}\right| = \left|\mathbb{T}\setminus H_{k-1}\right| \ge \left|\mathbb{T}\setminus H_k\right| \ge C/k^N.$$

Then, according to Lemma 2.1, there exists a sequence of positive reals (c_j) such that (A), (B) and (C) hold.

Let $g(x)=c_j$. If $x\in H_j\setminus H_{j-1}$ ($j\in\mathbb{N}$), and $g(x)=0$ for $x\in\mathbb{T}\setminus H_\infty$, then, using (A) and (B), we get

$$\int_{\mathbb{T}} |g|^p = \sum_{j=1}^{\infty} a_j c_j^p < \infty \text{ and } \int_{\mathbb{T}} |g|^q = \sum_{j=1}^{\infty} a_j c_j^q = \infty.$$

Therefore $g\in L_p\setminus L_q$.

On the other hand, if $h\in A$, $x\in H_{j_x}\setminus H_{j_x-1}$ and $y=x+h\in H_{j_y}\setminus H_{j_y-1}$, then $|j_y-j_x|\le 1$. Thus $|f(x+h)-f(x)|\le \max(|c_{j_x}-c_{j_x-1}|,|c_{j_x+1}-c_{j_x}|)$. Hence, using (C), for any $h\in A$, we have

$$\int_{\mathbb{T}} |\Delta_h g|^q \le \sum_{j=1}^{\infty} a_j \big(\max(|c_j-c_{j-1}|,|c_{j+1}-c_j|)\big)^q < \infty.$$

Therefore $\Delta_h g\in L_q$. □

The following lemma was proved by Géza Kós ([8]).

Lemma 2.4. *For any pseudo-Dirichlet set $H\subset\mathbb{T}$ there exists a Dirichlet set $\Lambda\subset\mathbb{T}$ such that the group generated by Λ contains H.*

Proof. Take sequences $q_1<q_2<\dots$ and $\varepsilon_n\to 0$ and a function $k: H\to\mathbb{N}$ according to the pseudo-Dirichlet property of H. Taking a suitable subsequence we can assume that $\varepsilon_n=\frac{1}{n}$. Then, denoting $|\sin\pi x|$ by $\|x\|$, we have $\|q_n x\|<\frac{1}{n}$ for any $x\in H$ and $n>k(x)$.

First we show that the sequence $q_1,q_2,\dots$ can be replaced by a sequence $r_1,r_2,\dots$ such that (i) for $n>k(x)$ we still have $\|r_n x\|<\frac{1}{n}$, (ii) each r_n divides r_{n+1}, and (iii) $r_{n+1}\ge 2(n+1)r_n$ ($n\in\mathbb{N}$). We define the sequence (r_n) by induction. Let $r_1=q_1$ and $r_{n+1}=2(n+1)r_n q_{2(n+1)^2 r_n}$. Then clearly r_n divides r_{n+1} and $r_{n+1}\ge 2(n+1)r_n$ ($n\in\mathbb{N}$). For $n+1>k(x)$ we have

$$\begin{aligned}\|r_{n+1}x\| &= \|2(n+1)r_n q_{2(n+1)^2 r_n} x\| \\ &\le 2(n+1)r_n\|q_{2(n+1)^2 r_n} x\| < 2(n+1)r_n\frac{1}{2(n+1)^2 r_n} = \frac{1}{n+1}.\end{aligned}$$

Now we can define Λ. Let $\Lambda = \left\{x \in \mathbb{T}: \ \forall n \ \|r_n x\| < \frac{1}{n}\right\}$. It is clear that Λ is a Dirichlet set. We need to show that any element of H can be written as a finite sum of elements in Λ.

Let $x \in H$ and $m > k(x)$. Clearly x can be written in the form of $x = \frac{a}{r_m} + y$, where $a \in \mathbb{Z}$ and $\|y\| \le \frac{\pi}{2r_m}$. We have $y \in \Lambda$, since if $n \ge m$, then

$$\|r_n y\| = \left\| r_n \left(x - \frac{a}{r_m} \right) \right\| \le \|r_n x\| + \left\| a \frac{r_n}{r_m} \right\| = \|r_n x\| < \frac{1}{n},$$

and if $n < m$, then

$$\|r_n y\| \le r_n \|y\| \le r_{m-1} \frac{\pi}{2r_m} \le r_{m-1} \frac{\pi}{4m r_{m-1}} = \frac{\pi}{4m} < \frac{1}{n}.$$

On the other hand $\frac{1}{r_m} \in \Lambda$, as well, since if $n \ge m$, then $\left\| r_n \frac{1}{r_m} \right\| = 0$, and if $n < m$, then $0 < r_n \frac{1}{r_m} \le \frac{r_{m-1}}{r_m} \le \frac{1}{2m} < \frac{1}{n}$. Thus x is indeed in the subgroup generated by Λ. ☐

Theorem 2.5. *For any* $0 < p < q < \infty$,

$$\mathfrak{H}(L_p, L_q) \supset \mathfrak{pD}.$$

That is, for any pseudo-Dirichlet set H_0 *there exists a function* $g \in L_p \setminus L_q$ *such that* $\Delta_h g \in L_q$ *for any* $h \in H_0$.

PROOF. By Lemma 2.4, there exists a Dirichlet set $\Lambda \subset \mathbb{T}$ such that the group generated by Λ contains H_0. Then clearly it is enough to prove that $\Lambda \in \mathfrak{H}(L_p, L_q)$.

Take a sequence $q_1 < q_2 < \dots$ and a sequence $\varepsilon_n \to 0$ according to the Dirichlet property of Λ. Taking a suitable subsequence of (q_n, ϵ_n) we can assume that $\varepsilon_n < C/2n^3$, where $C < 1/(\sum_{n=1}^{\infty} 2/n^2)$ $(= 3/\pi^2)$ is fixed.

Let

$$A = \left\{\alpha \in \mathbb{T} : (\forall n \in \mathbb{N}) \ |\sin q_n \pi \alpha| \le \varepsilon_n \right\}.$$

Then clearly $0 \in A$, $A = -A$, A is closed and $\Lambda \subset A$. Thus, according to Lemma 2.3, it is enough to find a closed set $H \subset \mathbb{T}$ with positive measure having property (1). Denoting $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$, we let

$$B_n = \bigcup_{j \in \mathbb{Z}_{q_n}} S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} \right) \quad (n \in \mathbb{N}),$$

where $S(x, r)$ denotes the open neighborhood of x with radius r. Let

$$B = \bigcup_{n=1}^{\infty} B_n \text{ and } H = \mathbb{T} \setminus B.$$

Then H is clearly closed. In addition,

$$|B| \le \sum_{n=1}^{\infty} |B_n| \le \sum_{n=1}^{\infty} q_n 2\frac{C}{n^2 q_n} = C\sum_{n=1}^{\infty} 2/n^2 < 1,$$

thus $|H| > 0$. Therefore we only need to prove (1).

We claim that if $|q_n\beta| < \varepsilon$, then

$$B_n + \beta \supset \bigcup_{j\in\mathbb{Z}_{q_n}} S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} - \frac{\varepsilon}{q_n}\right). \tag{2}$$

Indeed, since (on $\mathbb{T}$) $\beta = p_{n,\beta}/q_n + (q_n\beta)/q_n$ (for a proper $p_{n,\beta} \in \mathbb{Z}_{q_n}$),

$$S\left(\frac{j}{q_n}, \frac{C}{n^2 q_n} - \frac{\varepsilon}{q_n}\right) \subset S\left(\frac{j - p_{n,\beta}}{q_n}, \frac{C}{n^2 q_n}\right) + \beta.$$

For any $\alpha \in A$ we have $|q_k\alpha| \le |\sin(\pi q_k\alpha)| \le \varepsilon_k < C/2k^3$. Hence, if $\alpha_1, \ldots, \alpha_k \in A$, then $|q_k(\alpha_1 + \ldots + \alpha_k)| \le kC/2k^3 = C/2k^2$. Therefore, by (2), for any $\beta \in kA$,

$$B + \beta \supset B_k + \beta \supset \bigcup_{j\in\mathbb{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{k^2 q_k} - \frac{(C/2k^2)}{q_k}\right) = \bigcup_{j\in\mathbb{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right).$$

Thus, for any $\beta \in kA$, $H + \beta \subset \mathbb{T} \setminus \bigcup_{j\in\mathbb{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right)$. Therefore $H + kA \subset \mathbb{T} \setminus \bigcup_{j\in\mathbb{Z}_{q_k}} S\left(\frac{j}{q_k}, \frac{C}{2k^2 q_k}\right)$; so

$$|H + kA| \le 1 - q_k 2\frac{C}{2k^2 q_k} = 1 - C/k^2. \quad \square$$

Combining the previous theorem with the result of [7] mentioned in the Introduction, we get the following.

Corollary 2.6. *For any* $0 < p < q < \infty$, $\mathfrak{p}\mathfrak{D} \subset \mathfrak{H}(L_p, L_q) \subset \mathfrak{F}_\sigma$.

3 Evidence for a Conjecture

Notation 3.1. A Borel set $F \subset \mathbb{T}$ is called a *weak Dirichlet set* (see e.g. in [4] p. 48), if for every probability measure μ supported by F,

$$\limsup_{|n|\to\infty} |\hat{\mu}(n)| = 1, \text{ where } \hat{\mu}(n) = \int_{\mathbb{T}} e^{2\pi i n t} d\mu(t).$$

Theorem 3.2. *If $H \subset \mathbb{T}$ is not an N-set, $f : \mathbb{T} \to \mathbb{R}$ is a measurable function, and $\Delta_h f \in L_\infty$ for any $h \in H$, then $f \in L_p$ for any $p > 0$.*

PROOF. Let $K_m = \{h : |\Delta_h f| \leq m \text{ a. e.}\}$ $(m \in \mathbb{N})$. Then clearly $H \subset \cup_{m\in\mathbb{N}} K_m$; so $H \notin \mathfrak{N}$ implies $\cup_{m\in\mathbb{N}} K_m \notin \mathfrak{N}$. It is easy to prove (see e.g. [7], proof of Proposition 4.2) that K_m is closed. Thus (K_m) is an increasing sequence of compact sets.

It is known (see e.g. in [5] p. 190) that for each increasing sequence (K_m) of compact weak Dirichlet sets, $\cup_{m\in\mathbb{N}} K_m \in \mathfrak{N}$. Therefore, in our case, there exists an $m \in \mathbb{N}$ such that K_m is not a weak Dirichlet set. Dividing f by m, we can assume that $m = 1$. Then, denoting K_1 by K, we have

$$K = \{h : |\Delta_h f| \leq 1 \text{ a. e.}\}$$

and there exists a probability measure μ supported by K such that

$$\limsup_{|n|\to\infty} |\hat{\mu}(n)| < 1.$$

Thus there exists an $\eta > 0$ such that $\text{Re } \hat{\mu}(n) \leq 1 - \eta$ for every $n \in \mathbb{Z}$ with at most finitely many exceptions. If, for any $n \neq 0$, $\text{Re } \hat{\mu}(n) = 1$, then $e^{2\pi int} = 1$ μ-a.e.; so for any $k \in \mathbb{Z}$ also $e^{2\pi inkt} = 1$ μ-a.e., which is impossible, since $\limsup_{|n|\to\infty} |\hat{\mu}(n)| < 1$. Therefore we can assume, with a suitable $\eta > 0$, that

$$\text{Re } \hat{\mu}(n) \leq 1 - \eta \qquad (\forall n \in \mathbb{Z} \setminus \{0\}).$$

It is proved in [9] that if there exists a probability measure μ supported on K such that $\text{Re } \hat{\mu}(n) \leq 1 - \eta$ for every $n \in \mathbb{Z} \setminus \{0\}$, then K is "essentially ejective", which means that for every $x \in (0, 1]$,

$$\zeta_K(x) \geq \eta x(1 - x), \text{ where } \zeta_K(x) = \inf_{|A|=x} \sup_{h\in K} |(A + h) \setminus A|.$$

Therefore in our case $\zeta_K(x) \geq \eta x(1 - x)$. Thus

$$\sup_{h\in K} |(A + h) \setminus A| \geq \eta\, |A|\, (1 - |A|) \tag{3}$$

for any $A \subset \mathbb{T}$ with $|A| > 0$.

Now we define a sequence $A_{n_0}, A_{n_0+1}, \ldots$ of subsets of $\mathbb{T}$ by induction. Since f is measurable there exists an $n_0 \in \mathbb{N}$ such that

$$A_{n_0} = \{x \in \mathbb{T} : |f(x)| < n_0\}$$

has positive measure. Assume that A_n is already defined $(n \geq n_0)$. By (3), there exists a $h_n \in K$ such that

$$|(A_n + h_n) \setminus A_n| \geq \frac{\eta}{2}\, |A_n|\, (1 - |A_n|). \tag{4}$$

Then let $A_{n+1} = A_n \cup (A_n + h_n)$. Let $C_n = \{x \in \mathbb{T} : |f(x)| \geq n\}$ and $c_n = |C_n| \quad (n = 0, 1, \ldots)$. By the definition of K, it is easy to see by induction that $|f(x)| < n$ for a. e. $x \in A_n$, which means that $c_n \leq |\mathbb{T} \setminus A_n|$ $(n \geq n_0)$. Using the notation $b_n = |\mathbb{T} \setminus A_n|$, we use (4) to get

$$b_n - b_{n+1} \geq \frac{\eta}{2}(1 - b_n)b_n \geq \frac{\eta}{2}(1 - b_{n_0})b_n \quad (n \geq n_0).$$

Thus

$$b_{n+1} \leq b_n \left(1 - \frac{\eta}{2}(1 - b_{n_0})\right) \qquad (n \geq n_0).$$

Therefore, denoting $1 - \frac{\eta}{2}(1 - b_{n_0})$ by λ, we have

$$b_n \leq b_{n_0} \lambda^{n - n_0} \qquad (n \geq n_0).$$

Since $\eta > 0$ and $1 - b_{n_0} = |A_{n_0}| > 0$ we have $\lambda < 1$. Let $p > 0$. Then

$$\begin{aligned}
\int_{\mathbb{T}} |f|^p &= \sum_{n=1}^{\infty} \int_{C_{n-1} \setminus C_n} |f|^p \\
&\leq \sum_{n=1}^{\infty} (c_{n-1} - c_n) n^p \\
&= \sum_{m=0}^{\infty} c_m \left((m+1)^p - m^p\right) \\
&\leq O(1) + \sum_{m=n_0}^{\infty} b_m \left((m+1)^p - m^p\right) \\
&\leq O(1) + \sum_{m=n_0}^{\infty} b_{n_0} \lambda^{m-n_0} (m+1)^p \\
&< \infty.
\end{aligned}$$

□

Remark 3.3. This proof is based on the "ejectivity" property of a compact non-N-set (see [9]). On the other hand, the proof of Theorem 2.5 uses the "non-ejectivity" of a pseudo-Dirichlet set: the argument of the proof of Theorem 3.2 shows that the condition (1) of Lemma 2.3 cannot be satisfied if H is ejective. Since a set is non-ejective if and only if its closure is an N-sets (see [9] p. 162), this motivates the following conjecture.

Conjecture 3.4. *For any* $0 < p < q < \infty$, $\mathfrak{H}(L_p, L_q) = \mathfrak{N}$.

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