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UNIFORM CONTINUITY OF A PRODUCT OF REAL FUNCTIONS

Abstract

We produce necessary and sufficient conditions for the pointwise product of two uniformly continuous real-valued functions defined on a metric space to be uniformly continuous.

1 Introduction

Let $\langle X, d\rangle$ be a metric space and let f, g be uniformly continuous real-valued functions on X. The sum of f and g is again uniformly continuous, but their (pointwise) product need not be, even if one of the functions is bounded [1]. While it is a standard exercise in advanced calculus texts to show that uniform continuity of the product holds provided f and g are both bounded, conditions that are necessary as well as sufficient seem elusive. It is the purpose of this note to present such conditions. There are other related problems of interest, e.g., determining conditions on a metric space that guarantee that the product of each pair of real-valued uniformly continuous functions remains uniformly conditions, on which some progress has been made [4, 6, 7].

As we shall see, our conditions prove sufficient without assuming anything whatsoever about the factors. On the other hand, necessity holds for a class of pairs Δ that is much broader than the class of all pairs of functions that

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are both uniformly continuous. Seemingly, one must put some restriction on the class of pairs of functions under consideration for uniform continuity of their product, in that given any strictly positive function f no matter how pathological it may be, $f \cdot \frac{1}{f}$ will be uniformly continuous.

Our distinguished class of function pairs Δ is characterized by a uniform joint oscillation condition. Let us write $\mathbb{R}^X$ for the family of all real-valued functions on X. For $\{f, g\} \subseteq \mathbb{R}^X$ and $\delta > 0$ put

$$\lambda(f, g, \delta) := \sup\{|(f(x) - f(p))(g(x) - g(p))| : \{x, p\} \subseteq X \text{ and } d(x, p) < \delta\}.$$

Notice that when $\delta_1 < \delta_2$ then $\lambda(f, g, \delta_1) \leq \lambda(f, g, \delta_2)$. We now define Δ by

$$\Delta := \{(f, g) \in \mathbb{R}^X \times \mathbb{R}^X : \lim_{n\to\infty} \lambda(f, g, \frac{1}{n}) = \inf_{n\in\mathbb{N}} \lambda(f, g, \frac{1}{n}) = 0\}.$$

Evidently, Δ contains all pairs (f, g) where both functions are uniformly continuous. It also contains all (f, g) where one function is uniformly continuous and the other function is bounded.

Example 1.1. As an example of (f, g) in Δ where neither function is uniformly continuous nor bounded, let $X = \{1, \frac{3}{2}, 2, \frac{7}{3}, 3, \frac{13}{4}, 4, \ldots\}$, and let $f \in \mathbb{R}^X$ and $g \in \mathbb{R}^X$ be defined by

$$f(x) = \begin{cases} n & \text{if } x = n \text{ and } n \text{ is even} \\ 2n & \text{if } x = n + \frac{1}{n+1} \text{ and } n \text{ is even} \\ 0 & \text{otherwise} \end{cases},$$

$$g(x) = \begin{cases} n & \text{if } x = n \text{ and } n \text{ is odd} \\ 2n & \text{if } x = n + \frac{1}{n+1} \text{ and } n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}.$$

This example shows that we can have $(f, g) \in \Delta$ while for each $n \in \mathbb{N}$,

$$\sup\ \{|f(x) - f(p)| : d(x, p) < \frac{1}{n}\} = \sup\ \{|g(x) - g(p)| : d(x, p) < \frac{1}{n}\} = \infty,$$

because large local variability in one function is corrected by small local variability in the other.

The reader might expect that we will produce a condition on a function pair that combined with uniform continuity of each function yields uniform continuity of the product. Instead, we introduce a continuity notion for a pair (f, g) that is properly stronger than uniform continuity of their product, and which for uniformly continuous factors, reduces to uniform continuity of their product. We call this condition on a function pair *emphatic uniform continuity* of the product.

Definition 1.2. Let f, g be real-valued functions on a metric space $\langle X, d\rangle$. We say the pair (f, g) has an *emphatically uniformly continuous product* provided $\forall\varepsilon > 0,\ \exists\delta > 0$ such that $\forall x \in X,\ \forall p \in X,\ d(x, p) < \delta$ implies

$$|\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| < \varepsilon.$$

We deliberately avoid the usage "strong uniform continuity" here, as this language already has an established meaning in the literature (see, e.g., [2, 3]).

2 The Main Results

To be worthy of its name, emphatic uniform continuity of fg for a pair (f, g) ought to force uniform continuity of fg, and it does. In fact, emphatic uniform continuity is equivalent to uniform continuity of fg plus membership of (f, g) to our distinguished family Δ.

Proposition 2.1. *Let $\langle X, d\rangle$ be a metric space and let $\{f, g\} \subseteq \mathbb{R}^X$. Then (f, g) has an emphatically uniformly continuous product if and only if fg is uniformly continuous and $(f, g) \in \Delta$*

PROOF. For necessity, let $\varepsilon > 0$ and choose $\delta > 0$ such $\forall x \in X, \forall p \in X$, $d(x, p) < \delta \Rightarrow |\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| < \frac{\varepsilon}{2}$. By symmetry, it is clear that also $|f(p)g(p) - \frac{1}{2}(f(x)g(p) + f(p)g(x))| < \frac{\varepsilon}{2}$. Let us put $\alpha = \frac{1}{2}(f(x)g(p)+f(p)g(x))-f(x)g(x)$ and $\beta = f(p)g(p)-\frac{1}{2}(f(x)g(p)+f(p)g(x))$. Then the inequality $|\alpha + \beta| \le |\alpha| + |\beta|$ gives

$$\sup\{|f(p)g(p) - f(x)g(x)| : d(x, p) < \delta\} \le \varepsilon,$$

while the inequality $|\beta - \alpha| \le |\beta| + |\alpha|$ gives

$$\sup\{|(f(x) - f(p))(g(x) - g(p))| : d(x, p) < \delta\} \le \varepsilon.$$

For sufficiency, let $\varepsilon > 0$, and choose $n \in \mathbb{N}$ so large that both

(1) $d(x,p) < \frac{1}{n} \Rightarrow |f(p)g(p) - f(x)g(x)| < \varepsilon$, and

(2) $\lambda(f, g, \frac{1}{n}) < \varepsilon$.

We compute for $d(x,p) < \frac{1}{n}$

$$|(f(x)g(p)+f(p)g(x))-2f(x)g(x)| = |f(x)(g(p)-g(x))+g(x)(f(p)-f(x))|$$

$$\leq |f(x)(g(p)-g(x))+g(p)(f(p)-f(x))|+|(g(x)-g(p))(f(p)-f(x))|$$

$$\leq |f(x)(g(p) - g(x)) + g(p)(f(p) - f(x))| + \lambda(f, g, \tfrac{1}{n})$$
$$= |f(p)g(p) - f(x)g(x)| + \lambda(f, g, \tfrac{1}{n}) < 2\varepsilon.$$

It now follows that $|\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| < \varepsilon$ as required. □

It is illustrative to present some examples showing how fg can be uniformly continuous yet not emphatically uniformly continuous, equivalently $(f, g) \notin \Delta$.

Example 2.2. Let f and g be two functions defined on $(0, \infty)$ by $f(x) = \frac{1}{x}$ and $g(x) = x$. To show emphatic uniform continuity fails, we show that for each $\delta > 0$, we can find $x > 0$ and $p > 0$ with $d(x, p) < \delta$ yet

$$|\frac{1}{2}(f(x)g(p) + f(p)g(x)) - f(x)g(x)| = |\frac{1}{2}(\frac{1}{x}p + \frac{1}{p}x) - 1| > 1.$$

Take $p = \delta$ and $x = \frac{p}{5}$; we compute

$$\frac{1}{x}p + \frac{1}{p}x = \frac{p^2 + x^2}{xp} > \frac{p^2}{\frac{1}{5}p^2} = 5,$$

from which the desired estimate follows.

In the last example, one function is uniformly continuous and the other is continuous and unbounded. In fact, it is not possible to have a counterexample where one function is uniformly continuous and the other is bounded. In the next example, we give a pair of continuous bounded functions whose product is uniformly continuous but not emphatically uniformly continuous.

Example 2.3. Let X be the metric subspace of the line introduced in Example 1.1. Note that each real-valued function on X is continuous as X has no limit points. Define $f \in \mathbb{R}^X$ by

$$f(x) = \begin{cases} 1 & \text{if } x = n \text{ for some } n \in \mathbb{N} \\ -1 & \text{if } x = n + \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \end{cases},$$

and let $g(x) = -f(x)$ for all x. As in the previous example, the product fg is constant. For variety, we show that the uniform joint oscillation condition fails. For each $n \in \mathbb{N}$ put $x_n = n$ and $p_n = n + \frac{1}{n+1}$. We compute

$$(f(x_n) - f(p_n))(g(p_n) - g(x_n)) = 4$$

while $\lim_{n\to\infty} d(x_n, p_n) = 0$. This shows that $(f, g) \notin \Delta$.

We next state our main result which is an immediate consequence of Proposition 2.1.

Theorem 2.4. *Let $\langle X, d\rangle$ be a metric space and let $(f, g) \in \Delta$. Then (f, g) has an emphatically uniformly continuous product if and only if fg is uniformly continuous.*

Our main result produces these corollaries.

Corollary 2.5. *Let $\langle X, d\rangle$ be a metric space and let f and g be uniformly continuous real functions on X. Then (f, g) has an emphatically uniformly continuous product if and only if fg is uniformly continuous.*

Corollary 2.6. *Let $\langle X, d\rangle$ be a metric space and let $f \in \mathbb{R}^X$ be uniformly continuous. Then f^2 is uniformly continuous if and only if for each $\varepsilon > 0, \exists\delta > 0$ such that whenever $d(x, p) < \delta$, we have $|f(x)(f(p) - f(x))| < \varepsilon$.*

Proof. This follows from $\frac{1}{2}(2f(x)f(p)) - (f(x))^2 = f(x)(f(p) - f(x))$. □

Corollary 2.7. *Let $\langle X, d\rangle$ be a metric space and let $f : X \to \mathbb{R}$ be uniformly continuous and let $g : X \to \mathbb{R}$ be bounded. Then (f, g) has an emphatically uniformly continuous product if and only if fg is uniformly continuous.*

Uniform continuity of real functions can be considered of course for real functions defined on a topological space equipped with a diagonal uniformity (see, e.g., [8]). The class Δ and emphatic uniform continuity of a product are defined in the obvious ways in this setting, and all of the results listed above extend without difficulty. We also note that $\mathbb{R}$ can be replaced by the complex field $\mathbb{C}$ and all of our terminology and arguments go through verbatim.

We close this note by reconciling the class Δ with oscillation as it is traditionally understood [1, 5, 8]. Let $B_d(x, \alpha)$ denote the open ball with center $x \in X$ and radius $\alpha > 0$. For each $n \in \mathbb{N}$, $f : X \to \mathbb{R}$, and $x \in X$, put

$\omega_n(f,x) :=$ diam $f(B_d(x, \frac{1}{n}))$, where of course the diameter of the image is taken with respect to the usual metric for $\mathbb{R}$. Then the *oscillation* of f at x is defined by the familiar formula

$$\omega(f,x) := \lim_{n\to\infty} \omega_n(f,x) = \inf_{n\in\mathbb{N}} \omega_n(f,x).$$

Continuity of f at $x \in X$ is equivalent to $\omega(f,x) = 0$, whereas global uniform continuity is equivalent to the uniform convergence of $\langle \omega_n(f,\cdot)\rangle$ to the zero function. For arbitrary f, $x \mapsto \omega(f,x)$ is an upper semicontinuous function with values in $[0,\infty]$ (consider $f(x) = \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$).

We will be looking at products of the form $\omega_n(f,x)\omega_n(g,x)$. Since one or both of the factors may be ∞, we adopt the convention $\infty \cdot 0 = 0$ along with the usual conventions for extended real arithmetic. Note that at least one of the functions must be continuous at x provided $\lim_{n\to\infty}\omega_n(f,x)\omega_n(g,x) = 0$.

A key tool in the proof of our final result is the following elementary inequality.

Lemma 2.8. *Let $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ be six real numbers, not necessarily distinct, such that*

$$(\diamondsuit) \qquad \mu := |(\alpha_1 - \alpha_2)(\beta_1 - \beta_3)| > 0.$$

Then max$\{|(\alpha_1-\alpha_2)(\beta_1-\beta_2)|, |(\alpha_1-\alpha_3)(\beta_1-\beta_3)|, |(\alpha_2-\alpha_3)(\beta_2-\beta_3)|\} > \frac{\mu}{3}$.

PROOF. Suppose that $|(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)| \leq \frac{\mu}{3}$ and $|(\alpha_1 - \alpha_3)(\beta_1 - \beta_3)| \leq \frac{\mu}{3}$. It follows from $\diamondsuit$ that $|\beta_1 - \beta_2| \leq \frac{1}{3}|\beta_1 - \beta_3|$ and $|\alpha_1 - \alpha_3| \leq \frac{1}{3}|\alpha_1 - \alpha_2|$. We compute

$$|\beta_2 - \beta_3| \geq ||\beta_3 - \beta_1| - |\beta_1 - \beta_2|| \geq \frac{2}{3}|\beta_3 - \beta_1|,$$

and

$$|\alpha_2 - \alpha_3| \geq ||\alpha_2 - \alpha_1| - |\alpha_1 - \alpha_3|| \geq \frac{2}{3}|\alpha_1 - \alpha_2|.$$

It follows that

$$|(\alpha_2 - \alpha_3)(\beta_2 - \beta_3)| \geq \frac{4}{9}\mu > \frac{1}{3}\mu,$$

and the proof is complete. □

Theorem 2.9. *Let $\langle X, d\rangle$ be a metric space and let $\{f, g\} \subseteq \mathbb{R}^X$. Then $(f, g) \in \Delta$ if and only if $lim_{n\to\infty} sup_{x\in X}\ \omega_n(f, x)\omega_n(g, x) = 0$.*

PROOF. Whenever $d(p, x) < \frac{1}{n}$, it is clear that

$$|(f(x) - f(p))(g(x) - g(p))| \leq \omega_n(f, x)\omega_n(g, x)$$

so that a function pair satisfying the condition of the theorem must belong to Δ.

The converse is more complicated. Suppose $\lim_{n\to\infty}\sup_{x\in X}\ \omega_n(f, x)\omega_n(g, x) > \rho > 0$. Then for each $n \in \mathbb{N}$, we have $\sup_{x\in X}\ \omega_{2n}(f, x)\omega_{2n}(g, x) > \rho$. From this, we intend to show that for each n, $\lambda(f, g, \frac{1}{n}) > \frac{\rho}{12}$.

Fix $n \in \mathbb{N}$; $\exists p \in X$ such that $\omega_{2n}(f, p)\omega_{2n}(g, p) > \rho$. By the definition of oscillation, we can find $\{w, x, y, z\} \subseteq B_d(p, \frac{1}{2n})$ (not necessarily distinct) such that $|(f(w) - f(x))(g(y) - g(z))| > \rho$. By the triangle inequality,

$$(|f(w) - f(p)| + |f(p) - f(x)|)(|g(y) - g(p)| + |g(p) - g(z)|) > \rho,$$

so when we distribute out the product, one of the four terms we obtain - without loss of generality the first term $|(f(w) - f(p))(g(y) - g(p))|$ - must exceed $\frac{\rho}{4}$. The last lemma guarantees that $\max\{|(f(w) - f(p))(g(w) - g(p))|, |(f(w) - f(y))(g(w) - g(y))|, |(f(y) - f(p))(g(y) - g(p))|\} > \frac{\rho}{12}$.

Since all three points lie in a common ball of radius $\frac{1}{2n}$, we conclude

$$\max\{d(w, p), d(w, y), d(y, p)\} < \frac{1}{n},$$

so that $\lambda(f, g, \frac{1}{n}) > \frac{\rho}{12}$. Since $n \in \mathbb{N}$ is arbitrary, we conclude $(f, g) \notin \Delta$. $\square$

By our final result, whenever $(f, g) \in \Delta$, then at each $x \in X$, either f or g is continuous at x. However, as shown by Example 2.2, global continuity of both f and g does not guarantee membership of (f, g) to Δ.

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