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# BOX DIMENSION OF THE GRAPH OF THE LIMIT OF A SEQUENCE OF HÖLDER FUNCTIONS

**Abstract**

In this note a theorem to determine the box dimension of the graph of the limit of a sequence of $\alpha$-Hölder functions is established. By application of such a theorem the box dimensions of the graphs of some functions that are generalizations of Weierstrass-type functions are determined.

## 1 Introduction and prerequisites

In this paper the box-dimension of the graph $G$ of a real function $f$ that is the limit of a sequence of Hölder continuous functions is determined under suitable hypotheses. Recall that the graph of a function $f$ defined in the real interval $[a, b]$ is the set:

$$G = G_f = \{(x, y) \in R^2 : x \in [a, b], y = f(x)\}.$$

In order to introduce the box-dimension of $G$ recall that in general a $2^{-n}-$ *mesh* is a closed interval of $R^k$ of the form:

$$\{x \in R^k : h_i 2^{-n} \leq x_i \leq (h_i + 1)2^{-n}, i = 1, 2, ..., k\},$$

where $h_i$ are arbitrary integers. Let $E \subseteq R^k$; the number of $2^{-n}$-meshes meeting $E$ is denoted by $N_{2^{-n}}(E)$. Consider the following indexes:

$$\underline{lim}_{n\to\infty} \frac{log N_{2^{-n}}(E)}{log 2^n}.$$

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and
$$\overline{lim}_{n\to\infty}\frac{logN_{2^{-n}}(E)}{log2^n}.$$

They are called respectively the *lower box dimension* and the *upper box dimension* of $E$; if they agree their common value is called *box dimension* of $E$ and is denoted by $\Delta(E)$ (see [7] and [11]). It is possible to consider $\delta-meshes$ in place of $2^{-n}-meshes$, that is intervals of the form

$$\{x\in R^k : h_i\delta\le x_i\le (h_i+1)\delta, i=1,2,...k\},$$

where $h_i$ are integers and (see [7]) it is obvious that:

$$\Delta(E)=lim_{\delta\to 0}\frac{logN_\delta(E)}{-log\delta},$$

if the limit on the right exists. In general for the graph $G$ of a continuous function in an interval $[a,b]$ it is:

$$\underline{dim}_B(G)\ge 1;$$

recall that a function $f$ defined in an interval $[a,b]$ is $\alpha$-Hölder continuous in $[a,b]$ $(0<\alpha\le 1)$, if there exists a constant $C>0$ such that for every $x$ and $y\in[a,b]$ it is:
$$|f(x)-f(y)|\le C|x-y|^\alpha.$$
It is easy to see that if $f$ is $\alpha$-Hölder continuous in $[a,b]$ then we have:

$$\overline{dim}_B(G)\le 2-\alpha;$$

(see [7] and [8]).
As far as I know, besides the considerations that can be found in [13] or in [6], no general treatment has been developed up to now for the determination of the box-dimension (and Hausdorff dimension) of the graph of a continuous function, but there are several partial results valid for particular classes of functions, as one can find for example in [1], [2], [9], [10], [12] and [14].
In particular T. Bousch and Y. Heurteaux in [2] and Heurteaux in [9] and [10] consider Weierstrass-type functions

$$f(x)=\sum_{n\in N} b^{-n\alpha}g(b^n x) \tag{1}$$

where $g$ is almost periodic in $R$, Lipschitz continuous and $1<b<+\infty$. They claim that if $0<\alpha<1$ then $f$ is $\alpha$–Hölder continuous and for every interval

$I$ it is $\Delta(G) = 2 - \alpha$. Other considerations about Weierstrass- type functions have been recently developed by S.P. Zhou and G. L. He in [14]. In [3], [4] and [5] I try to study general properties of the functions that can be useful in this framework. Moreover, the following definition can be introduced: an $\alpha$-Hölder continuous function $f : [a, b] \to R$ is said to be *uniformly essentially $\alpha$-Hölder continuous* in $[a, b]$ if there exists a positive number $C > 0$, such that, for every interval $I \subseteq [a, b]$, it is:

$$\omega(f, I) \geq C\ |I|^{\alpha},$$

where, $\omega(f, I)$ is the oscillation of $f$ in $I$ and $|I|$ denotes the length of $I$. If $f$ is uniformly essentially $\alpha-$Hölder continuous in $[a, b]$ or in a subinterval of $[a, b]$, then it is easy to see that the graph of $f$ has the greatest upper box dimension a graph of such a function can achieve, namely 2-$\alpha$.
However if the upper box dimension of $G$ is equal to $2 - \alpha$ it may be that in no subinterval of $[a, b]$ $f$ satisfies the previous condition, as is possible to see with examples (see Example 3.1 of [3]). In [4] I show that some functions that are generalizations of Weierstrass-type functions such as (1), where a $\delta$-Hölder continuous function $g$ with exponent $\delta$ greater than $\alpha$ and less or equal to 1 instead of a Lipschitz function appears are uniformly essentially $\alpha-$Hölder continuous in $[0, 1]$ (Theorem 2.2 of [4]). The previous theorem does not hold in the case that $\alpha = \delta$. In order to consider also this case, in this paper a theorem about the upper box dimension of the graph of the limit of a uniformly convergent sequence of functions that are uniformly essentially Hölder continuous is given and it is immediately applied to determine the box dimension of the graph of a Weierstrass generalized function like (1), where $g$ is Hölder continuous with an exponent equal to the limit value $\alpha$ $(0 < \alpha \leq 1)$.

## 2 A theorem about the box dimension of the graph of the limit of a sequence of Hölder functions

The following theorem yields a sufficient condition in order to determine the box dimension of the graph of the limit of a uniformly convergent sequence of Hölder continuous functions.

**Theorem 2.1.** *Let, for every $n \in N$, $f_n : [0, 1] \to R$ be a continuous function, let $0 < \alpha \leq 1$ and let $(\alpha_n)_{n \in N}$ be a sequence such that $0 < \alpha_n \leq 1$, $\alpha_n \to \alpha$. Let $d \in N$, $d > 1$, $x = \frac{p}{d^r}$, $r \in N$, $p = 1, 2, ..., d^r - 1$. Let $(s_n)_{n \in N}$, $s_n \in N$, be a sequence such that, for every $r \in N$, putting $m = s_n + r$ and $h = \frac{1}{2d^m}$, we have, for enough large $n \in N$:*

$$\frac{|f_n(x + h) - f_n(x)|}{h^{\alpha_n}} > \lambda. \tag{2}$$

*If there exists an increasing sequence* $(g(k))_{k\in N}$ *of natural numbers, such that for enough large* $k$ *and for every* $x \in [0,1]$ *it is:*

$$|f(x) - f_{g(k)}(x)| < \frac{1}{d^k} \tag{3}$$

*and if*

$$lim_{k\to\infty}\frac{s_{g(k)}}{k} = 0 \tag{4}$$

*then, if* $G$ *is the graph of* $f$*, it is:*

$$\underline{dim}_B(G) \geq 2 - \alpha.$$

PROOF. By (3) for enough large $k$ and for every $x \in [0,1]$ it is:

$$f(x) - \frac{1}{d^k} < f_{g(k)} < f(x) + \frac{1}{d^k}.$$

Let $\delta > 0$ and let $k \in N$ be such that $2d^{-k} \leq \delta < 2d^{1-k}$. If $G_{f_{g(k)}}$ denotes as usual the graph of the function $f_{g(k)}$ then, since $f(x) - \delta < f_{g(k)} < f(x) + \delta$, it is: $N_\delta(G) + \frac{2}{\delta} \geq N_\delta(G_{f_{g(k)}})$. It follows:

$$\underline{lim}_{\delta\to 0}\frac{logN_\delta(G)}{-log\delta} = \underline{lim}_{\delta\to 0}\frac{log[N_\delta(G) + \frac{2}{\delta}]}{-log\delta} \geq \underline{lim}_{n\to\infty}\frac{logN_\delta(G_{f_{g(k)}})}{-log\delta}. \tag{5}$$

By (2) written with $n = g(k)$ and $r = k$ it is:

$$N_\delta(G_{f_{g(k)}}) \geq \frac{\lambda}{\delta^2 2^{\alpha_{g(k)}} d^{(k+s_{g(k)})\alpha_{g(k)}}} \geq \frac{\lambda\delta^{-2+\alpha_{g(k)}}}{d^{\alpha_{g(k)}} 2^{2\alpha_{g(k)}} d^{s_{g(k)}\alpha_{g(k)}}}.$$

and therefore, for every $\delta > 0$, since $2d^{-k} \leq \delta < 2d^{1-k}$ :

$$\frac{logN_\delta(G_{f_{g(k)}})}{-log\delta} \geq 2 - \alpha_{g(k)} + \frac{log\frac{\lambda}{d^{\alpha_{g(k)}}2^{2\alpha_{g(k)}}}}{-log\delta} - \frac{\alpha_{g(k)}s_{g(k)}logd}{(k-1)logd - log2}.$$

By this inequality and (5) it follows:

$$\underline{lim}_{\delta\to 0}\frac{logN_\delta(G)}{-log\delta} \geq 2 - \alpha - \alpha lim_{k\to\infty}\frac{s_{g(k)}logd}{(k-1)logd - log2},$$

whence by (4), it is: $\underline{lim}_{\delta\to 0}\frac{logN_\delta(G)}{-log\delta} \geq 2 - \alpha$ and the theorem is completely proven. □

**Remark 2.2.** *It is possible to prove Theorem 2.1 even if (2) holds only for* $r = ku$*, where* $u > 1$ *is a fractional number. Indeed in the previous proof, given* $\delta > 0$ *it is possible to choose* $k \in N$ *such that* $2d^{-ku} \leq \delta < 2d^{(1-k)u}$*. Then we have that* $f(x) - \delta < f_{g(ku)} < f(x) + \delta$*, whence, proceeding as before, by (2), written with* $n = g(ku)$ *and* $r = ku$*, the thesis follows.*

## 3 An application

It is possible to apply immediately Theorem 2.1 in order to determine the box dimension of the graph of a Weierstrass generalized function like (1), where g is Hölder continuous with exponent equal to the limit value $\alpha$.

**Theorem 3.1.** Let $\varphi : R \to R$ be an $\alpha$-Hölder continuous function, $(0 < \alpha \leq 1)$, periodic with period 1, such that $\varphi(0) = \varphi(1) = 0$, $0 \leq \varphi(x) \leq \varphi(1/2) = 1$ for every $x \in R$. Let

$$f(x) = \Sigma_{n\in N} d^{-n\alpha}\varphi(d^n x),$$

where $d > 1$ is a natural number. Then $f$ is ($\alpha$-$\varepsilon$)-Hölder continuous for every $\varepsilon \in ]0, \alpha[$ and $\Delta(G) = 2 - \alpha$.

Proof. We first prove that $f$ is ($\alpha$-$\varepsilon$)-Hölder continuous for every $\varepsilon > 0$, $\varepsilon < \alpha$. Let $x \in R$ and let $\nu \in N$ be such that $\frac{1}{d^{\nu+1}} \leq h < \frac{1}{d^\nu}$. Then

$$f(x+h)-f(x) = \Sigma_{n=1}^{n=\nu}\frac{\varphi(d^n x + d^n h) - \varphi(d^n x)}{d^{n\alpha}} + \Sigma_{n=\nu+1}^{n=\infty}\frac{\varphi(d^n x + d^n h) - \varphi(d^n x)}{d^{n\alpha}}$$

whence:

$$|f(x+h) - f(x)| \leq I_1 + I_2,$$

where

$$I_1 \leq \Sigma_{n=1}^{n=\nu}\frac{c(d^n h)^\alpha}{d^{n\alpha}} = ch^\alpha\nu,$$

$c$ being the Hölder coefficient of $\varphi$. Since, for every $\varepsilon > 0$, $h^\varepsilon\nu < \frac{\nu}{d^{\nu\varepsilon}}$ is infinitesimal for diverging $\nu$, we have that there exists $M_\varepsilon > 0$ such that $ch^\varepsilon\nu < M_\varepsilon$ and therefore

$$I_1 \leq M_\varepsilon h^{\alpha-\varepsilon};$$

since it is also:

$$I_2 \leq 2\Sigma_{n=\nu+1}^{n=\infty}\frac{1}{d^{n\alpha}} \leq \frac{2(hd)^\alpha}{d^\alpha - 1},$$

$f$ is ($\alpha$-$\varepsilon$)-Hölder continuous. Since $\varepsilon > 0$ is arbitrary, we have that $\overline{dim}_B(G) \leq 2-\alpha$ and the proof ends here for $\alpha = 1$. In order to prove the converse relation for $0 < \alpha < 1$, it is enough to prove that there exists a sequence of functions uniformly convergent to $f$ such that the conditions of Theorem 2.1 are satisfied. To this end observe that it is possible to choose $\varepsilon \in ]0, 1[$ in such a way that it is $\alpha\varepsilon = \frac{1}{u}$, with a rational $u > 1$ and consider for every $k \in N$, such that $ku \in N$, the function:

$$f_k(x) = \sum_{n=1}^{n=ku-1}\frac{1}{d^{n\alpha}}\varphi(d^n x) + \sum_{n\geq ku}(\frac{\gamma_k}{d^\alpha})^n\varphi(d^n x), \tag{6}$$

where $\gamma_k = 1+a_k$, $(a_k)_{k\in N}$ being a decreasing infinitesimal sequence of positive numbers less than 1 to be fixed in the sequel. For every $k \in N$ $f_k$ is $\alpha_k-$Hölder continuous, with $\alpha_k = \alpha - log_d\gamma_k$ (see [4]). Obviously $\alpha_k < \alpha$ for every $k \in N$ and $\alpha_k \to \alpha$, increasingly. Moreover:

$$|f(x) - f_k(x)| \le \Sigma_{n\ge ku}\frac{(1+a_k)^n - 1}{d^{n\alpha}} =$$

$$= \Sigma_{n\ge ku}\frac{a_k}{d^{n\alpha}}[(1+a_k)^{n-1} + (1+a_k)^{n-2} + ... + 1] < a_k\Sigma_{n\ge ku}n(\frac{1+a_k}{d^\alpha})^n;$$

since $a_k$ is infinitesimal, there exists $\nu \in N$ such that, for every $k \ge \nu$ it is $1 + a_k \le 1 + a_\nu < d^\alpha$ and therefore:

$$|f(x) - f_k(x)| \le a_k\Sigma_{n\ge ku}n(\frac{1+a_\nu}{d^\alpha})^n.$$

Since the series on the right converges we have that $(f_k)_{k\in N}$ converges uniformly to $f$. Moreover observe that:

$$|f(x) - f_k(x)| < a_k(\frac{1}{d^{\alpha\varepsilon}})^{ku}\sum_{n\ge ku} n(\frac{1+a_\nu}{d^{(1-\varepsilon)\alpha}})^n;$$

then, if $\nu$ is such that $1 + a_\nu < d^{(1-\varepsilon)\alpha}$, then also the series on the right of this inequality is convergent and infinitesimal when $k$ diverges; therefore it is possible to assume that for enough large $k$ it is $\sum_{n\ge ku} n(\frac{1+a_\nu}{d^{(1-\varepsilon)\alpha}})^n < 1$. Then we obtain for every $x \in [0, 1]$ :

$$|f(x) - f_k(x)| < \frac{1}{d^k}. \tag{7}$$

Since $\varphi$ is periodic with period 1, if $x = \frac{p}{d^r}$ with $p = 0, 1, 2, ..., d^r - 1$, then $\varphi(d^n x) = 0$ for $n \ge r$. Let $h = \frac{1}{2d^m}$, where $m = r + s_k$; then if $d$ for example is an odd number (the proof runs in as similar way if $d$ is even), we have, putting $c_k^n = 1$ if $n < ku$ and $c_k^n = (1 + a_k)^n$ if $n \ge ku$ :

$$f_k(x) = \Sigma_{n<r}\frac{c_n^k}{d^{n\alpha}}\varphi(d^n x),$$

$$f_k(x+h) = \sum_{n>m}\frac{c_k^n}{d^{n\alpha}} + \sum_{r\le n\le m}\frac{c_k^n}{d^{n\alpha}}\varphi(d^n\frac{1}{2d^m}) + \sum_{n<r}\frac{c_k^n}{d^{n\alpha}}\varphi(d^n x + d^n\frac{1}{2d^m}),$$

since $d^n x + d^n \frac{1}{2d^m} = \frac{2v+1}{2}$ with $v \in N$ for $n > m$, and therefore $\varphi(d^n x + d^n \frac{1}{2d^m}) = 1$ for such values of n, while, for $r \le n \le m$ we have that $d^n \frac{1}{2d^m}$

differs from $d^n x + d^n \frac{1}{2d^m}$ by an integer and therefore $\varphi$ assumes the same value in these points. Thus:

$$f_k(x+h) - f_k(x) =$$
$$\sum_{n<r} \frac{c_k^n}{d^{n\alpha}} [\varphi(d^n x + d^n \frac{1}{2d^m}) - \varphi(d^n x)] + \sum_{n>m} \frac{c_k^n}{d^{n\alpha}} + \sum_{r \le n \le m} \frac{c_k^n}{d^{n\alpha}} \varphi(\frac{d^n}{2d^m})$$

that we write in the following form:

$$f_k(x+h) - f_k(x) = I_1 + I_2 + I_3. \tag{8}$$

Now it is easily seen that, if $r = ku$ then:

$$|I_1| \le ch^\alpha \sum_{n<ku} c_k^n \le \ ch^\alpha (ku-1), \tag{9}$$

where $c$ is the Hölder coefficient of $\varphi$;

$$\frac{I_2}{h^\alpha} \ge \frac{2^\alpha}{d^\alpha - 1}. \tag{10}$$

Finally, we have that:

$$I_3 = \sum_{ku \le n \le m} \frac{c_k^n}{d^{n\alpha}} \varphi(\frac{d^n}{2d^m}) \ge h^\alpha \gamma_k^{ku+s_k}. \tag{11}$$

We have, by (8):

$$\frac{|f_k(x+h) - f_k(x)|}{h^\alpha} \ge \frac{I_2 + I_3 - |I_1|}{h^\alpha},$$

so that, by (9), (10) and (11), for $k \in N$ and $r = ku$ it is:

$$\frac{|f_k(x+h) - f_k(x)|}{h^\alpha} \ge \frac{2^\alpha}{d^\alpha - 1} + \gamma_k^{ku+s_k} - c(ku-1);$$

therefore if $\gamma_k^{ku+s_k} \ge c(ku-1)$, that is if

$$s_k \ge \frac{logc + log(ku-1)}{log(1+a_k)} - ku$$

it is:

$$\frac{|f_k(x+h) - f_k(x)|}{h^\alpha} > \lambda, \tag{12}$$

where $\lambda = \frac{1}{d^{\alpha}-1} > 0$.

Let $a_k = \frac{log(ku-1)}{ku}$ for every $k \in N$; and let

$$s_k = 1 - ku + [\frac{logc + log(ku-1)}{log(1+a_k)}],$$

where $[v]$ denotes the integer part of the real number $v$. Then:

$$lim_{k\to\infty}\frac{s_k}{k} = 0. \tag{13}$$

Consider now (7): it is for every $x \in [0,1]$:

$$|f(x) - f_k(x)| < \frac{1}{d^k}.$$

Moreover, if $x = \frac{1}{d^{ku}}$, $h = \frac{1}{2d^{ku+s_k}}$ and $u > 1$ is a rational number, it is, by (12):

$$|f_k(x+h) - f_k(x)| > \frac{\lambda}{2^{\alpha}d^{(ku+s_k)\alpha}}.$$

Then, by (13), Theorem 2.1 and Remark 2.2 apply giving $\underline{dim}_B(G) \geq 2-\alpha$. Since the converse inequality holds the theorem is proven. $\square$

**Remark 3.2.** *Observe that in particular by Theorem 3.1, if* $0 < \alpha \leq 1$ *then the graph of the function:*

$$f(x) = \sum_{n\in N}\frac{|sin(d^n\pi x)|^{\alpha}}{d^{n\alpha}}$$

*where* $d \in N$, $d > 1$, *has the greatest box dimension it can achieve, that is* $2-\alpha$.

## 4 Extensions of previous results

In this Section some larger classes of functions will be considered and the box dimension of the graphs of such functions will be determined. Namely consider the function:

$$f(x) = \sum_{n\in N}\frac{1}{b_n^{\delta}}\varphi(b_n x) \tag{14}$$

where:

a) $(b_n)_{n\in N}$ is a sequence of real numbers such that $b_1 = b > 1$, $b_{n+1} \geq bb_n$ for every $n \in N$; $\frac{b_n}{b_m} \in N$ for every $n \geq m$; $lim_{n\to\infty}\frac{logb_{n+1}}{logb_n} = 1$.

b) $\varphi : R \to R$ is an $\alpha$-Hölder continuous function, $(0 < \alpha \leq 1)$, periodic with period 1, such that $\varphi(0) = \varphi(1) = 0$, $0 \leq \varphi(x) \leq \varphi(\frac{1}{2}) = 1$ for every $x \in R$.

**Theorem 4.1.** *The function (14) under the hypotheses a) and b) is, for every $\delta \in ]0, \alpha[$, $\delta$-Hölder continuous and $\underline{dim}_B(G) = \overline{dim}_B(G) = 2 - \delta$.*

PROOF. Let $x \in R$, $h \in ]0, 1[$ and let $\nu \in N$ be such that $\frac{1}{b_{\nu+1}} \leq h < \frac{1}{b_\nu}$. Let $c$ be the Hölder coefficient of $\varphi$. Then:

$$|f(x+h) - f(x)| \leq \Sigma_{n \leq \nu} \frac{c(b_n h)^\alpha}{b_n^\delta} + \Sigma_{n > \nu} \frac{2}{b_n^\delta}$$

$$\leq ch^\delta \frac{b^{\alpha-\delta}}{b^{\alpha-\delta} - 1} + 2h^\delta \frac{b^\delta}{b^\delta - 1}$$

whence $f$ is $\delta$-Hölder continuous and $\overline{dim}_B(G) \leq 2 - \delta$.

Now it is possible to prove that it is also $\underline{dim}_B(G) \geq 2 - \delta$. Indeed let $x = \frac{p}{b_\nu}, p = 0, 1, ..., [b_\nu]$, and let $h = \frac{1}{2b_m}$, with $m = \nu + s, s \in N$. Then:

$$f(x+h) - f(x) = \Sigma_{n<\nu} \frac{1}{b_n^\delta} [\varphi(b_n x + b_n h) - \varphi(b_n x)] +$$

$$\Sigma_{\nu \leq n < m} \frac{1}{b_n^\delta} [\varphi(b_n x + b_n h) - \varphi(b_n x)] + \Sigma_{n \geq m} \frac{1}{b_n^\delta} [\varphi(b_n x + b_n h) - \varphi(b_n x)],$$

that we write in the form:

$$f(x+h) - f(x) = I_1 + I_2 + I_3. \tag{15}$$

It is, by hypotheses a) and b):

$$|I_1| \leq \frac{ch^\delta}{2^{\alpha-\delta} b^{(m-\nu)(\alpha-\delta)} (b^{\alpha-\delta} - 1)}, \tag{16}$$

$$I_2 = \Sigma_{\nu \leq n < m} \frac{\varphi(b_n h)}{b_n^\delta} \geq 0, \tag{17}$$

$$I_3 = \Sigma_{n \geq m} \frac{1}{b_n^\delta} \varphi(b_n h) \geq \frac{1}{b_m^\delta} = 2^\delta h^\delta. \tag{18}$$

By (15), (16), (17) and (18) it follows:

$$|f(x+h) - f(x)| \geq I_3 - |I_1| \geq h^\delta [2^\delta - \frac{c}{2^{\alpha-\delta} b^{(m-\nu)(\alpha-\delta)} (b^{\alpha-\delta} - 1)}] \geq h^\delta$$

if:

$$\frac{c}{2^{\alpha-\delta} b^{(m-\nu)(\alpha-\delta)} (b^{\alpha-\delta} - 1)} < 2^\delta - 1$$

that is if:

$$m - \nu > \frac{1}{\alpha - \delta} log_b \frac{c}{2^{\alpha-\delta}(2^\delta - 1)(b^{\alpha-\delta} - 1)}.$$

Thus there exists $s \in N$, given by:

$$s = 1 + [\frac{1}{\alpha - \delta} log_b \frac{c}{2^{\alpha-\delta}(2^\delta - 1)(b^{\alpha-\delta} - 1)}], \tag{19}$$

such that:

$$|f(x+h) - f(x)| \geq h^\delta \tag{20}$$

if $x = \frac{p}{b_\nu}, p = 0, 1, ..., [b_\nu]$, and $h = \frac{1}{2b_m}$, with $m = \nu + s$.

By (20) it follows that $\overline{dim}_B(G) \geq 2 - \delta$.

In order to prove that it is also $\underline{dim}_B(G) \geq 2 - \delta$, consider a cover of $G$ made of $\sigma$-meshes. By a) it is, for $s$ given by (19), $lim_{\nu\to\infty} \frac{log b_{\nu+s}}{log b_\nu} = 1$. Therefore if we fix $\varepsilon > 0$, for enough large $\nu$ it is: $b_{\nu+s} < b_{\nu-1}^{1+\varepsilon}$. Let $\sigma$ be so small that $\frac{2}{b_\nu} < \sigma \leq \frac{2}{b_{\nu-1}}$ where $\nu$ verifies previous inequalities and consider (20) written with $x = \frac{p}{b_\nu}, p = 0, 1, ..., [b_\nu]$, and $h = \frac{1}{2b_m}$, where $m = \nu + s$. It is:

$$N_\sigma(G) \geq \frac{\sigma^{\delta - 2 + \varepsilon\delta}}{2^{3\delta}},$$

and therefore:

$$\underline{lim}_{\sigma\to 0} \frac{log N_\sigma(G)}{-log\sigma} \geq 2 - \delta - \delta\varepsilon;$$

by this inequality the thesis follows, since $\varepsilon$ is arbitrary. □

**Remark 4.2.** *Theorem 4.1 proves a result given by Zhou and He in [14], in the particular case that $\alpha = 1$: the treatment of these authors is different from the present one because they do not assume that $\frac{b_n}{b_m} \in N$ for every $m \in N$ and for every $n \geq m$ as is done here, but their result holds only for enough large $b$, while in the present approach nothing is required about $b$.*

**Theorem 4.3.** The function $f(x) = \Sigma_{n\in N} \frac{1}{b_n^\alpha} \varphi(b_n x)$ where $\varphi$ satisfies condition b) and $(b_n)_{n\in N}$ is a sequence of real numbers such that $b_1 = b > 1, b_{n+1} \geq b b_n$ for every $n \in N$, is $(\alpha$-$\varepsilon)$-Hölder continuous for every $\varepsilon \in ]0, 1[$ and therefore the upper box dimension of its graph is not greater than $2 - \alpha$. In particular if $\alpha = 1$ then $\overline{dim}_B(G) = \underline{dim}_B(G) = 1$.

PROOF. The proof is similar to that of the analogous property in Theorem 3.1. □

Theorem 3.1 cannot be widely generalized for $0 < \alpha < 1$. By the present approach we can only deduce that the following result holds:

**Theorem 4.4.** Consider the function $f(x) = \Sigma_{n \in N} \frac{1}{b_n^{\alpha}} \varphi(b_n x)$ where $\varphi$ satisfies condition b). Assume that $(b_n)_{n \in N}$ is a sequence such that: $b_n = Ab^n$, with suitable $A > 0$ and $b \in N$ for every $n \in N$. Then it must necessarily be $\Delta(G) = 2 - \alpha$.

Proof. See the proof of Theorem 3.1. $\square$

## References


[1] A. S. Besicovitch, H. D. Ursell, *Sets of fractional dimensions (V): on dimensional numbers of some continuous curves*, J. Lond. Math. Soc. **12** (1937), 18–25.

[2] T. Bousch, Y. Heurteaux, *Caloric measure on domains bounded by Weierstrass-type graphs*, Ann. Acad. Sci. Fenn. Math. **25** (2000), 501–522.

[3] L. Biacino, *Hausdorff dimension of the diagram of $\alpha$-Hölder continuous functions*, Ric. Mat. (2005), 229–243.

[4] L. Biacino, *A note on the box dimension of the graph of a Hölder function*, submitted for publication.

[5] L. Biacino, *Density and tangential properties of the graph of a Hölder continuous function*, Boll. Unione Mat. Ital. **9(3)** (2010), 493–503.

[6] M. Dai, L. Tian, *Fractal properties of refined box dimension on functional graph*, Chaos Solitons Fractals **23** (2005), 1371–1379.

[7] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, 1985.

[8] K. J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, John Wiley and Sons Ltd., New-York, 1990.

[9] Y. Heurteaux, *Weierstrass functions with random phases*, Trans. Amer. Math. Soc. **355(8)** (2003), 3065–3077.

[10] Y. Heurteaux, *Weierstrass functions in Zygmund's class*, Proc. Amer. Math. Soc. **133(9)** (2005), 2711–2720.

[11] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[12] R. D. Mauldin, S. C. Williams, *On the Hausdorff dimension of some graphs*, Trans. Amer. Math. Soc. **298(2)** (1986), 793–803.

[13] F. Przytycki, M. Urbanski, *On the Hausdorff dimension of some fractal sets*, Studia Math. **93** (1989), 155–186.

[14] S. P. Zhou, G. L. He, *On a class of Besicovitch functions to have exact box dimension: a necessary and sufficient condition*, Math. Nachr. **278(6)** (2005), 730–734.