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ON SECTION SETS OF NEIGHBORHOODS OF GRAPHS OF SEMICONTINUOUS FUNCTIONS

Abstract

We prove that for any lower semicontinuous function $f : [0,1] \to [0,1]$ with purely unrectifiable graph and for any $\varepsilon > 0$ there is an open set $U \supset \operatorname{graph} f$ with every vertical section set of one-dimensional Lebesgue measure at most ε.

1 Motivation and definitions

Two basic notions in geometric measure theory are those of purely and uniformly purely unrectifiable sets. A set $A \subset \mathbb{R}^2$ is purely unrectifiable if for every Lipschitz curve γ we have $\mathcal{H}^1(\operatorname{graph} \gamma \cap A) = 0$ and A is uniformly purely unrectifiable if for every $K \geq 0$ and every $\varepsilon > 0$ there an open set U with $A \subset U$ and such that for every K-Lipschitz function g in any rotated cartesian coordinates we have $\mathcal{H}^1(\operatorname{graph} g \cap U) \leq \varepsilon$. Clearly, all uniformly purely unrectifiable sets are purely unrectifiable and it is not difficult to observe that for F_σ sets these notions coincide. It is not known whether they coincide also for G_δ sets or even Borel sets (this problem was stated by Alberti, Csörnyei and Preiss, see [1], remark after Theorem 21.).

In this paper we deal with a similar but much weaker property. Our G_δ set A will be a purely unrectifiable graph of a (lower) semicontinuous function and we will look only for the existence of an open superset of its graph with small measure of its intersections with all vertical lines. Recall that $f : [0,1] \to [0,1]$ is lower semicontinuous when for every $\alpha \in [0,1]$ the set $f^{-1}([0,\alpha])$ is compact. The main result is the following:

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Theorem 1.1. *Let $f : [0,1] \to [0,1]$ be a lower semicontinuous function with purely unrectifiable graph. Then for any $\varepsilon > 0$ there in an open set $U \supset \operatorname{graph} f$ with every vertical section set of one-dimensional Lebesgue measure at most ε.*

Theorem 1.1 follows directly from Proposition 2.3. Before we proceed with the proof it will be useful to make some remarks.

1) There exists a lower semicontinuous function $f : [0,1] \to [0,1]$ with purely unrectifiable graph. To obtain such a function it is sufficient to consider κ the usual von Koch curve (which is known to be purely unrectifiable) built above the interval $[0,1]$ on the x-axis and put $f(x) = \min\{y : (x,y) \in \operatorname{graph} \kappa\}$. Note that the result in [2] shows that the function f is not continuous.

2) There exists a lower semicontinuous function $f : [0,1] \to [0,1]$ such that every open set $U \supset \operatorname{graph} f$ contains the whole interval $[0,1]$ in some of its vertical section sets.

One way to construct such a function is to find some compact set $K \subset [0,1]^3$ such that for every compact set $L \subset [0,1]^3$ there is some $x \in [0,1]$ with

$$L = K_x = \{(y,z) \in [0,1]^2 : (x,y,z) \in K\}$$

and put

$$f(x) = \min(\{1\} \cup \{y \in [0,1] : (x,y,x) \in K\}).$$

It is enough to prove that the graph of f intersects every compact set $L \subset [0,1]^2$. Choose such a set L and find $x \in [0,1]$ from the definition of K. Now, we have $(x, f(x)) \in K_x = L$.

Another way is to consider any lower semicontinuous function f that is Darboux, $f(0) = 0$ and is $f = 1$ on rational numbers in $(0,1]$. (Sketch of the proof.) Again, it is enough to prove that the graph of f intersects every compact set $L \subset [0,1]^2$. Divide $[0,1]$ in two intervals of length $\frac{1}{2}$. In at least one of these intervals there is an x such that $f(x)$ is not greater than $\max\{u : (x,u) \in L\}$ (0 is always such point). Choose the interval with this property which is most to the right. Now, do the same procedure with four intervals of length $\frac{1}{4}$, eight intervals of length $\frac{1}{8}$ and so on. The chosen intervals form a monotone sequence with one point z in its intersection. It is not difficult to observe that $(z, f(z)) \in L$.

Note that in the second case it is simple to observe that f could not have purely unrectifiable graph, since $\phi : y \to \max(f^{-1}([0,y])$ is strictly monotone function from $[0,1]$ to $[0,1]$ whose graph (in the y-coordinate) lies on the graph of f. We use the fact that graph of any monotone function lies on some Lipschitz curve and also that $1 = \mathcal{H}^1([0,1]) = \mathcal{H}^1(P_y(\operatorname{graph} f)) \le \mathcal{H}^1(\operatorname{graph} f)$, where P_y is orthogonal projection to the y-axis.

We will need the following notation:

We will use $B(z,r)$ for the open ball in $\mathbb{R}^2$ with center z and radius r and I will be used for the unit interval $[0,1]$. For a set $A \subset \mathbb{R}$ we will use $|A|$ for (one-dimensional) Lebesgue measure of A.

For $t \in \{0,1\}^{<\omega}$ we will denote $|t|$ the length of t, and $\prec$ will be used for classical lexicographic ordering (the same symbol will be used for lexicographic ordering on $\{0,1\}^{\omega}$).

For $t \in \{0,1\}^{<\omega}$ or $t \in \{0,1\}^{\omega}$ and $n \in \mathbb{N}$ denote $t(n)$ the n-th coordinate of t and define $t|n \in \{0,1\}^n$ as $t|n(i) = t(i)$ for $i = 1,..,n$.

For $t, u \in \{0,1\}^{<\omega}$ define $t^*u \in \{0,1\}^{|t|+|u|}$ as $t^*u(i) = t(i)$ for $i = 1,..,|t|$ and $t^*u(|t|+i) = u(i)$ for $i = 1,..,|u|$.

We will write $u \lhd t$ if there is $n \in \mathbb{N}$ such that $u = t|n$.

For $t \in \{0,1\}^{<\omega}$ we will use I_t for the dyadic interval

$$I_t = [a_t, b_t] = \Big[\sum_{i=1}^{|t|} t(i)2^{-i}, 2^{-|t|} + \sum_{i=1}^{|t|} t(i)2^{-i}\Big].$$

We will use P_x or P_y for the orthogonal projection to the x or y-axis.

For $A \subset I^2$ and $w \in I$ put $A^w = \{z \in I : (w,z) \in A\}$. For $B \subset I$ denote B° the interior relative to I of B. We will use $\mathcal{K}(I^2)$ for the system of all compact subsets of I^2.

2 Proof of the theorem

Throughout the whole section fix $\varepsilon > 0$ and a lower semicontinuous function $f : I \to I$ with the property that there is no open, relatively in I^2, set U with $\operatorname{graph} f \subset U \subset I^2$ with $|U^z| < \varepsilon$ for any $z \in I$. Put $\alpha = 1 - \varepsilon$.

Since f is lower semicontinuous, we can find for every $z \in I$ and $\delta > 0$ some $\beta(z,\delta) > 0$ with $\min_{v \in [z-\beta(z,\delta), w+\beta(z,\delta)]} f(v) \geq f(z) - \delta$. Fix some such $\beta(z,\delta)$ for every such z and δ.

For $z \in I$ and $J \subseteq I$ interval define $\mathcal{K}(s,z,J) \subset \mathcal{K}(I^2)$ as a system of all $K \in \mathcal{K}(I^2)$ with $P_y K \subseteq J$, $z \in P_x K^\circ$ and for all $w \in P_x K$ we have $|K^w| \geq s$ and $K \cap \operatorname{graph} f = \emptyset$. Then define

$$s(x, J) = \sup\{s : \mathcal{K}(s,z,J) \neq \emptyset\}.$$

Lemma 2.1. *1. there is $z \in I$ with $s(z, I) \leq \alpha$.*

2. if $\rho, \delta > 0$ and $s(z,J) < |J| - \delta$ then there is a $z_{\rho,\delta} \in I$ with $0 < |z - z_{\rho,\delta}| \leq \rho$, $s(z_{\rho,\delta}, J) < s(z,J) + \delta$ and $f(z_{\rho,\delta}) \in J$.

3. if $J_i = [a_i, b_i]$, $i = 1,2$ are two intervals with $b_1 = a_2$ then for $J = J_1 \cup J_2$ we have $s(z,J_1) + s(z,J_2) \leq s(z,J)$.

Proof. 1. Suppose for contradiction that for every $z \in I$ there is $s(z,I) > \alpha$. This means that for any $z \in I$ there is a compact set $K_z \in \mathcal{K}(s_z, z, I)$ with $s_z > \alpha$. Since I is a compact set there is $k \in \mathbb{N}$ and $z_1, ..., z_k$ such that $I \subset \cup_{i=1}^k P_x K_{z_i}^\circ$. But then $U = I^2 \setminus \cup_{i=1}^k K_{z_i}$ is an open (relatively to I^2) superset of graph f with $|U^w| \le 1 - \min_i s_{z_i} < \varepsilon$ for every $w \in I$, which is a contradiction with the definition of f.

2. Let $J = [a,b]$. Suppose that for some $\delta > 0$ and $\rho > 0$ there is no such $z_{\rho,\delta}$. This means that for any $w \in I$ with $0 < |w - z| \le \rho$ and $f(w) \in J$ we have $s(w,J) \ge s(z,J) + \delta$.

Now, since f is lower semicontinuous, the set $f^{-1}([0,a])$ is compact. Which means that the set $V = [z-\rho, z+\rho] \setminus f^{-1}([0,a])$ is open relatively in $[z-\rho, z+\rho]$, in particular, can be written in the form $V = \cup_{n\in\mathbb{N}} K_n$ for K_n compact and $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$.

Now observe that $s(w,J) \ge s(z,J) + \delta$ for any $w \in V$. We assumed this for w with $f(w) \in J$ and for w with $f(w) > b$ we can find $\kappa > 0$ with $f(w) - \kappa > b$ and then $[w - \beta(w,\kappa), w + \beta(w,\kappa)] \times J \in \mathcal{K}(s(z,J) + \delta, w, J)$.

As in the previous case find $k_n \in \mathbb{N}$, $s_1^n, ..., s_{k_n}^n \ge s(z,J) + 3\delta/4$, $z_1^n, ..., z_{k_n}^n \in K_n$ and $K_{z_i^n} \in \mathcal{K}(s_i^n, z_i^n, J)$ with $K_n \subset \cup_{i=1}^{k_n} P_x K_{z_i^n}^\circ$ for every $n \in \mathbb{N}$. Put

$$\tilde{L}_n = \bigcup_{i=1}^{k_n} K_{z_i^n}, \quad \tilde{K}_n = K_n \setminus \bigcup_{i=1}^{n-1} K_i^\circ \quad \text{and} \quad L_n = \{(u_1,u_2) \in \tilde{L}_n : u_1 \in \tilde{K}_n\}$$

and define

$$K = \overline{\bigcup_{n\in\mathbb{N}} L_n} \setminus ((I \times [a, a + \delta/4)) \cup B((z, f(z)), \delta/8)).$$

It is easy to verify that $K \in \mathcal{K}(s(z,J) + \delta/4, z, J)$ which contradicts the definition of $s(z,J)$.

3. For every sufficiently small $\delta > 0$ find $K_\delta^i \in \mathcal{K}(s(z,J_i) - \delta, z, J_i)$, $i = 1,2$. Put $K_\delta = (K_\delta^1 \cup K_\delta^2) \cap ((P_x K_\delta^1 \cap P_x K_\delta^2) \times I)$. Then $K_\delta \in \mathcal{K}(s(z,J) - 2\delta, z, J)$ and it is sufficient to let $\delta \to 0$. □

Lemma 2.2. *For every $t \in \{0,1\}^{<\omega}$ there is a point $z_t \in I$ such that:*

1. *if $s(z_t, I_t) < |I_t|$ then $f(z_t) \in I_t$.*
2. *$\sum_{|t|=n} s(z_t, I_t) \le \alpha + \varepsilon \sum_{k=1}^n 2^{-(k+1)}$*
3. *if $|t| = |t'|$ then $t \prec t'$ if and only if $z_t < z_{t'}$.*
4. *$|z_{t|(|t|-1)} - z_t| \le 1/5 \min_{|t''|=|t'|=|t|-1, t''\neq t'} |z_{t'} - z_{t''}|$.*

5. *if* $t' \lhd t$ *then* $z_t \in (z_{t'} - \beta(z_{t'}, 2^{-|t|}), z_{t'} + \beta(z_{t'}, 2^{-|t|}))$

PROOF. We will proceed by induction by $|t|$. For $|t| = 0$. By property 1 in Lemma 2.1 there exists a point z with $s(z, I) \leq \alpha$. Put $z_\emptyset = z$. The fulfiment of all properties 1–5 is trivial.

Induction step: Suppose that we have z_t constructed for every $|t| \leq n-1$. We have $\{0,1\}^{n-1} = T_1 \cup T_2$, where

$$T_1 = \{t \in \{0,1\}^{n-1} : s(z_t, I_t) < |I_t| - \varepsilon 2^{-2n}\}$$

and

$$T_2 = \{t \in \{0,1\}^{n-1} : s(z_t, I_t) \geq |I_t| - \varepsilon 2^{-2n}\}.$$

Fix some $t \in \{0,1\}^{n-1}$, we will construct $t^*\{0\}$ and $t^*\{1\}$ by the following procedure:

Case 1. $t \in T_1$.

Put $d = 1/5 \min_{|t''|=|t'|=n-1, t'' \neq t'} |z_{t'} - z_{t''}|$. Using property 2 in Lemma 2.1 countable many times for $z = z_t$, $\delta = \varepsilon 2^{-(2n+1)}$ and $\rho = \rho_j$ for a suitable sequence $\rho_j \to 0$ there is a sequence $w_i \to z_t$ in I satisfying $|w_i - z_t| \leq d$, $s(w_i, I_t) < s(z_t, I_t) + \varepsilon 2^{-(2n+1)}$, $f(w_i) \in I_t$ and $w_i \in (z_t - \beta(z_t, 2^{-|t|}), z_t + \beta(z_t, 2^{-|t|}))$ for all $i \in \mathbb{N}$. Since $0 \leq s(w_i, I_{t^*\{0\}}) \leq 2^{-n}$ there is a subsequence $\{w_{i_l}\}_{l=1}^\infty$ and $s \in [0, 2^{-n}]$ such that $s(w_{i_l}, I_{t^*\{0\}}) \to s$. So we can choose l_0 and l_1 with $|s(w_{i_{l_0}}, I_{t^*\{0\}}) - s(w_{i_{l_1}}, I_{t^*\{0\}})| \leq \varepsilon 2^{-(2n+1)}$ and $w_{i_{l_0}} < w_{i_{l_1}}$. Put $z_{t^*\{0\}} = w_{i_{l_0}}$ and $z_{t^*\{1\}} = w_{i_{l_1}}$. By property 3 in Lemma 2.1 we have

$$\begin{aligned} s(z_{t^*\{0\}}, I_{t^*\{0\}}) + s(z_{t^*\{1\}}, I_{t^*\{1\}}) &= s(w_{i_{l_0}}, I_{t^*\{0\}}) + s(w_{i_{l_1}}, I_{t^*\{1\}}) \\ &\leq s(w_{i_{l_0}}, I_{t^*\{0\}}) + s(w_{i_{l_1}}, I_t) - s(w_{i_{l_1}}, I_{t^*\{0\}}) \\ &\leq s(z_t, I_t) + \varepsilon 2^{-(2n+1)} + \varepsilon 2^{-(2n+1)} \\ &= s(z_t, I_t) + \varepsilon 2^{-(2n)}. \end{aligned}$$

Case 2. $t \in T_2$.

Choose $z_{t^*\{0\}} < z_{t^*\{1\}}$ as arbitrary two points of continuity sufficiently close to z_t to satisfy conditions 4 and 5.

Property 1 in case 1 follows directly from the construction and in case 2 it is sufficient to observe that if w is a point of continuity of f, then $s(w, J) = |J|$

for every J. Properties 3–5 are clear. To verify the validity of property 2 write

$$\begin{aligned}\sum_{|t|=n} s(z_t, I_t) &= \sum_{t\in T_1} s(z_{t^*\{0\}}, I_{t^*\{0\}}) + s(z_{t^*\{1\}}, I_{t^*\{1\}})\\ &+ \sum_{t\in T_2} s(z_{t^*\{0\}}, I_{t^*\{0\}}) + s(z_{t^*\{1\}}, I_{t^*\{1\}})\\ &\le \sum_{t\in T_1} s(z_t, I_t) + \sum_{t\in T_1} \varepsilon 2^{-(2n+1)} + \sum_{t\in T_2}(|I_t| - s(z_t, I_t))\\ &\le \sum_{|t|=n-1} s(z_t, I_t) + 2^n \varepsilon 2^{-(2n+1)} \le \alpha + \varepsilon \sum_{k=1}^{n} 2^{-(k+1)}.\end{aligned}$$

□

Proposition 2.3. *The graph of the function f is not a purely unrectifiable set.*

Proof. Let $z_t, t \in \{0,1\}^{<\omega}$ be points from Lemma 2.2. For $u \in \{0,1\}^{\omega}$ denote $z_u = \lim_{n\to\infty} z_{u|n}$. This limit exists due to property 4 and by the same property together with property 3 we have $z_u < z_{u'}$ whenever $u \prec u'$. Denote h_u as the only number that lies in $\cap_n I_{u|n}$. For $n \in \mathbb{N}$ put

$$T^n = \{t \in \{0,1\}^n : s(z_t, I_t) < |I_t|\} \quad \text{and} \quad H_n = \bigcup_{t\in T^n} I_t$$

and define

$$U = \{u \in \{0,1\}^{\omega} : \ I_{u|n} \in T^n \text{ for every } n \in \mathbb{N}\},$$

$C = \{z_u : u \in U\}$ and $H = \{h_u : u \in U\}$. Note that $H_{n+1} \subset H_n$ for every $n \in \mathbb{N}$ and $H = \cap_{n\in\mathbb{N}} H_n$. So, since by property 2 we have $|H_n| \ge \frac{\varepsilon}{2}$ for each $n \in \mathbb{N}$, we have $|H| = \lim_{n\to\infty} |H_n| \ge \frac{\varepsilon}{2}$. Moreover, since

$$h_u = \lim_{n\to\infty} a_{u|(n)} - 2^{-n} \le f(z_u) \le \lim_{n\to\infty} f(z_{u|(n)}) = h_u,$$

where the first inequality is by property 5 and the second one by lower semicontinuity of f together with property 1 we obtain $f(z_u) = h_u$ for every $u \in U$. Due to this fact and property 4 we obtain that f is monotone on C.

Now, since

$$|H| = |f(C)| = \mathcal{H}^1(P_y \operatorname{graph} f|C) \le \mathcal{H}^1(\operatorname{graph} f)$$

and since graph of every monotone function lies on the graph of a Lipschitz curve, we are done.

□

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