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RADII OF CONVERGENCE OF POWER SERIES

Abstract

The paper is closely related to an earlier paper of two authors of this paper (cf. [4]). In the paper radii of convergence of power series are investigated as values of a function defined on the Fréchet's space using the well–known formula of Cauchy and Hadamard.

Introduction

This paper is closely related to the paper [4]. In [4] radii of convergence of power series $\sum_{n=0}^{\infty} a_n x^n$ as functions of sequences $(a_n)_{n=0}^{\infty}$ are investigated. The sequences $(a_n)_{n=0}^{\infty}$ are considered as points of the Fréchet metric space $\mathbf{s}$ of all sequences of real numbers with the metric

$$d(\mathbf{y},\mathbf{z}) = \sum_{k=0}^{\infty} 2^{-k} \frac{|y_k - z_k|}{1+|y_k - z_k|}, \quad \mathbf{y} = (y_k)_0^{\infty} \in \mathbf{s}, \quad \mathbf{z} = (z_k)_0^{\infty} \in \mathbf{s}.$$

For $\mathbf{a} = (a_k)_0^{\infty} \in \mathbf{s}$ we put (in agreement with [4])

$$\sigma(\mathbf{a}) = \sigma(a_0, a_1, \dots) = \limsup_{n\to\infty} \sqrt[n]{|a_n|}, \quad r(\mathbf{a}) = r(a_0, a_1, \dots) = \frac{1}{\sigma(\mathbf{a})}.$$

(We put $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$ in the last equality.)

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In [4] the sets

$$H_t = \{\mathbf{x} = (x_k)_0^\infty \in \mathbf{s} : \sigma(\mathbf{x}) = t\},$$
$$H_t^* = \{\mathbf{x} = (x_k)_0^\infty \in \mathbf{s} : r(\mathbf{x}) = t\},$$
$$P_t = \{\mathbf{x} = (x_k)_0^\infty \in \mathbf{s} : \sigma(\mathbf{x}) < t\}$$

($t \in [0, \infty]$) were introduced. It is shown in [4] [1] that P_t is an F_σ - set (for each $t \in [0, \infty]$) in $\mathbf{s}$. Further the set $H_\infty = H_0^*$ is a residual set in $\mathbf{s}$. Hence the set $P_\infty = \{\mathbf{x} = (x_k)_0^\infty \in \mathbf{s} : \sigma(\mathbf{x}) < \infty\} = \{\mathbf{x} = (x_k)_0^\infty \in \mathbf{s} : r(\mathbf{x}) > 0\} = B$ is a set of the first Baire category.

In the first part of this paper we shall strengthen this result by more detailed study of the structure of the set B using the concept of porosity for sets in metric spaces.

Let $\sum_{n=0}^\infty a_n x^n$ be a fixed power series, $a_n \in \mathbb{R}$ $(n = 0, 1, 2, \dots), x \in \mathbb{R}$. If $(\varepsilon_n)_0^\infty$ is a sequence of 0's and 1's with an infinite number of 1's, then the series $\sum_{n=0}^\infty \varepsilon_n a_n x^n$ is said to be a subseries of the series $\sum_{n=0}^\infty a_n x^n$. Using the above notation we have obviously $\sigma(\varepsilon_0 a_0, \varepsilon_1 a_1, \dots) \leq \sigma(a_0, a_1, \dots)$. Denote by $S = S(a_0, a_1, \dots)$ the set of all $(\varepsilon_n)_0^\infty$ such that the subseries $\sum\limits_{n=0}^\infty \varepsilon_n a_n x^n$ has the same radius of convergence as the power series $\sum_{n=0}^\infty a_n x^n$, i.e.

$$r(\varepsilon_0 a_0, \varepsilon_1 a_1, \dots) = r(a_0, a_1, \dots).$$

Denote by $W = W(a_0, a_1, \dots)$ the set $U \setminus S$. Hence $W = U \setminus S$, where U denotes the set of all sequences $(\varepsilon_n)_0^\infty$ of 0's and 1's with an infinite number of 1's. To each $(\varepsilon_n)_0^\infty \in U$ there corresponds a number

$$\rho(\varepsilon_0, \varepsilon_1, \dots) = \sum_{k=0}^\infty \frac{\varepsilon_k}{2^{k+1}} \in (0, 1].$$

Then ρ is a one-to-one mapping of U onto $(0, 1]$ (cf. [1], p. 17-18). For $T \subseteq U$ we put $\rho(T) = \{\rho(\varepsilon_0, \varepsilon_1, \dots) : (\varepsilon_n)_0^\infty \in T\}$. The set $\rho(T)$ is a tool for measuring the size of the set T. It is proved in [4] that $\lambda(\rho(S)) = 1$ (λ denotes Lebesgue measure). Since $\rho(W) = (0, 1] \setminus \rho(S)$, we get $\lambda(\rho(W)) = 0$. Hence it seems to be natural to investigate the Hausdorff dimension of the set $\rho(W)$. This will be done in the second part of the paper. This section is the result of collaboration arising between the first two authors at the Real Analysis Conference in Liptovský Ján Slovakia, 1996.

[1] See Lemma 2.3 in [4]. In the proof of this lemma the restriction to $t \in (0, \infty)$ is superfluous.

Notation and Definitions

In what follows we denote by $\dim M$ ($M \subseteq \mathbb{R}$) the Hausdorff dimension of the set M (cf. [2]). We now give definitions and notation from the theory of porosity of sets (cf. [5]-[7]). Let (Y, ρ) be a metric space. If $y \in Y$ and $r > 0$, then denote by $K(y, r)$ the ball with center y and radius r, i.e.

$$K(y, r) = \{x \in Y : \rho(x, y) < r\}.$$

Let $M \subseteq Y$. Put

$$\gamma(y, r, M) = \sup\{t > 0 : \underset{z \in Y}{\rightarrow} \exists \quad [K(z, t) \subseteq K(y, r)] \wedge [K(z, t) \cap M = \emptyset]\}.$$

Define the numbers:

$$\overline{p}(y, M) = \limsup_{r \to 0_+} \frac{\gamma(y, r, M)}{r}, \quad \underline{p}(y, M) = \liminf_{r \to 0_+} \frac{\gamma(y, r, M)}{r}$$

and if $\overline{p}(y, M) = \underline{p}(y, M)$, then we set

$$p(y, M) = \overline{p}(y, M) = \underline{p}(y, M) = \lim_{r \to 0_+} \frac{\gamma(y, r, M)}{r}.$$

Obviously the numbers $\overline{p}(y, M)$, $\underline{p}(y, M)$, $p(y, M)$ belong to the interval $[0, 1]$.

A set $M \subseteq Y$ is said to be porous (c-porous) at $y \in Y$ provided that $\overline{p}(y, M) > 0$ ($\overline{p}(y, M) \geq c > 0$). A set $M \subseteq Y$ is said to be σ-porous (σ-c-porous) at $y \in Y$ if $M = \bigcup_{n=1}^{\infty} M_n$ and each of the sets $M_n (n = 1, 2, \dots)$ is porous (c-porous) at y.

Let $Y_0 \subseteq Y$. A set $M \subseteq Y$ is said to be porous, c-porous, σ-porous and σ-c-porous in Y_0 if it is porous, c-porous, σ-porous and σ-c-porous at each point $y \in Y_0$, respectively.

If M is c-porous and σ-c-porous at y, then it is porous and σ-porous at y, respectively.

Every set $M \subseteq Y$ which is porous in Y is non-dense in Y. Therefore every set $M \subseteq Y$ which is σ-porous in Y, is a set of the first category in Y. The converse is not true even in $\mathbb{R}$ (cf. [5]).

According to the definition of $\overline{p}(y, M)$, $\underline{p}(y, M)$ we immediately get the following.

Theorem. *If $M_1 \subseteq M_2 \subseteq Y$, then for each $y \in Y$ we have $\overline{p}(y, M_1) \geq \overline{p}(y, M_2)$, $\underline{p}(y, M_1) \geq \underline{p}(y, M_2)$.*

A set $M \subseteq Y$ is said to be very porous at $y \in Y$ if $\underline{p}(y, M) > 0$ and very strongly porous at $y \in Y$ if $p(y, M) = 1$ (cf. [7], p. 327). A set M is said to be very (strongly) porous in $Y_0 \subseteq Y$ if it is very (strongly) porous at each $y \in Y_0$.

Obviously, if M is very porous at y, it is porous at y, as well. Analogously, if M is very strongly porous at y, it is 1-porous at y.

Further, a set $M \subseteq Y$ is said to be uniformly very porous in $Y_0 \subseteq Y$ provided that there is a $c > 0$ such that for each $y \in Y_0$ we have $\underline{p}(y, M) \geq c$ (cf. [7], p. 327). In agreement with the previous terminology and in analogy with the notion of σ-porosity, we introduce the following notions.

Definition. a; A set $M \subseteq Y$ is said to be uniformly σ-very porous in $Y_0 \subseteq Y$ provided that $M = \bigcup_{n=1}^{\infty} M_n$ and there is a $c > 0$ such that for each $y \in Y_0$ and each $n = 1, 2, \ldots$ we have $\underline{p}(y, M_n) \geq c$.

b; A set $M \subseteq Y$ is said to be uniformly σ-very strongly porous in Y_0 provided that $M = \bigcup_{n=1}^{\infty} M_n$ and for each $y \in Y_0$ and each $n = 1, 2, \ldots$ we have $p(y, M_n) = 1$.

1 Porosity Character of the Set B

We shall try to improve a result from [4] (Theorem 1.3 (iv)) where it was shown that the set

$$B = \{\mathbf{x} \in \mathbf{s} : \sigma(\mathbf{x}) < \infty\} = \{\mathbf{x} \in \mathbf{s} : r(\mathbf{x}) > 0\}$$

is a set of the first Baire category in $\mathbf{s}$. Note that this set is obviously dense in $\mathbf{s}$. We shall investigate the porosity of this set in $\mathbf{s}$. This investigation is very easy at points of the set $\mathbf{s} \setminus B$. The following simple auxiliary result enables us to prove a result in this direction (see the following Theorem 1.1).

Lemma 1.1. *Let (Y, ρ) be a metric space. Let $M \subseteq Y$, M be an F_σ-set in Y. Then M is uniformly σ-very strongly porous in $Y \setminus M$.*

Proof. By the assumption $M = \bigcup_{n=1}^{\infty} M_n$, where M_n (n=1,2,...) are closed in Y. Let $y \in Y \setminus M$, $n \in \mathbb{N}$. Then $y \notin M_n$, thus by the closedness of M_n there exists a $\delta_0 > 0$ such that $K(y, \delta_0) \cap M_n = \emptyset$. But then for each δ, $0 < \delta \leq \delta_0$ we get $K(y, \delta) \cap M_n = \emptyset$ and so $p(y, M_n) = 1$ (for each $n \in \mathbb{N}$). □

Theorem 1.1. *The set B is uniformly σ-very strongly porous in $\mathbf{s} \setminus B$.*

Proof. The set $B = P_\infty$ is an F_σ-set in $\mathbf{s}$ (cf. Lemma 2.3 in [4]). Hence the theorem follows from Lemma 1.1. □

The following theorem describes the kind of porosity of the set B globally in whole space $\mathbf{s}$.

Theorem 1.2. *The set B is uniformly σ-very porous in the space* $\mathbf{s}$.

PROOF. Evidently $B = \bigcup_{n=1}^{\infty} B_n$, where $B_n = \{\mathbf{x} \in \mathbf{s} : \underset{k \geq 1}{\rightarrow} \forall \sqrt[k]{|x_k|} \leq n\}$. We shall prove precisely that at each $\mathbf{y} \in \mathbf{s}$ we have $\underline{p}(\mathbf{y}, B_n) \geq \frac{1}{4}$ $(n = 1, 2, \dots)$. Let $\mathbf{y} = (y_k)_0^{\infty} \in \mathbf{s}$, $r > 0$, $n \in \mathbb{N}$. Construct the ball $K(\mathbf{y}, r)$. Suppose already that $0 < r < \frac{1}{2}$. Then there is an $m \in \mathbb{N}$ such that $2^{-m-1} \leq r < 2^{-m}$. Construct the sequence $\mathbf{z} = (z_j)_{j=0}^{\infty}$ as follows.

$$z_j = y_j \text{ if } j \neq m+2 \text{ and}$$
$$z_{m+2} = (n + \eta + 1)^{m+2} \quad (\eta > 0 \text{ will be chosen later})$$

Then we get
$$d(\mathbf{z}, \mathbf{y}) < 2^{-m-2}. \tag{1}$$

We show that there is a $\delta > 0$ such that

a; $K(\mathbf{z}, \delta) \subseteq K(\mathbf{y}, 2^{-m-1})$,

b; $K(\mathbf{z}, \delta) \cap B_n = \emptyset$.

Consider the function $h(t) = t^{\frac{1}{m+2}}$, $t > 0$. By the mean value theorem we have
$$h(v_1) - h(v_2) = (v_1 - v_2) v_0^{\frac{1}{m+2} - 1} \frac{1}{m+2}, \quad v_1, v_2 > 1$$
(and v_0 is a number between v_1, v_2; hence $v_0 > 1$). Then we get
$$|h(v_1) - h(v_2)| \leq |v_1 - v_2|. \tag{2}$$

Put
$$\delta = 2^{-m-2} \frac{\eta}{1+\eta}. \tag{1'}$$

If $\mathbf{x} \in K(\mathbf{z}, \delta)$, then by (1) and (1') using the triangle inequality we get
$$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) < 2^{-m-2} + 2^{-m-2} = 2^{-m-1}.$$

Hence $K(\mathbf{z}, \delta) \subseteq K(\mathbf{y}, 2^{-m-1})$ (a; holds).

Further if $\mathbf{t} = (t_j)_{j=0}^{\infty} \in K(\mathbf{z}, \delta)$, then by definition of the metric d we get
$$2^{-m-2} \frac{|z_{m+2} - t_{m+2}|}{1 + |z_{m+2} - t_{m+2}|} < 2^{-m-2} \frac{\eta}{1+\eta}. \tag{3}$$

Consider that the function $g(t) = \frac{t}{1+t}$, $t > 0$, is increasing. Therefore (3) yields
$$|z_{m+2} - t_{m+2}| < \eta. \tag{3'}$$

Choosing $v_1 = z_{m+2}$, $v_2 = t_{m+2}$ in (2) we get by (3')

$$|z_{m+2}^{\frac{1}{m+2}} - t_{m+2}^{\frac{1}{m+2}}| \le |z_{m+2} - t_{m+2}| < \eta. \tag{4}$$

But then by (4) we have $t_{m+2}^{\frac{1}{m+2}} > z_{m+2}^{\frac{1}{m+2}} - \eta = n + \eta + 1 - \eta > n$. Thus $B_n \cap K(\mathbf{z}, \delta) = \emptyset$ (b; holds). So we get

$$\frac{\gamma(\mathbf{y}, r, B_n)}{r} \ge \frac{\delta}{r} \ge 2^{-m-2}\frac{\eta}{1+\eta}2^m = \frac{1}{4}\frac{\eta}{1+\eta}.$$

Hence $\underline{p}(\mathbf{y}, B_n) \ge \frac{1}{4}\frac{\eta}{1+\eta}$. This holds for each $\eta > 0$. If $\eta \to \infty$, we obtain $\underline{p}(\mathbf{y}, B_n) \ge \frac{1}{4}$. □

2 Hausdorff Dimension of the Set $\rho(W)$

Let

$$\sum_{n=0}^{\infty} a_n x^n \tag{5}$$

be a fixed power series, $a_n \in \mathbb{R}$ $(n = 0, 1, 2, \dots)$, $x \in \mathbb{R}$. As proved in [4] we have $\lambda(\rho(W)) = 0$, where $W = U \setminus S$, $S = S(a_0, a_1, \dots)$ being the set of all sequences $(\varepsilon_n)_0^\infty \in U$ such that $r(\varepsilon_0 a_0, \varepsilon_1 a_1, \dots) = r(a_0, a_1, \dots)$.

It is easy to see that $\dim \rho(W)$ will depend on the sequence $(a_n)_{n=0}^\infty$. We illustrate the dependence of $\dim \rho(W)$ on $(a_n)_0^\infty$ by two examples.

Example 2.1. Define $a_n = 1$ $(n = 0, 1, \dots)$, i.e. we deal with the power series $\sum\limits_{n=0}^{\infty} x^n$. Then obviously $S = S(1, 1, \dots) = U$. Hence $W = \emptyset$ and so $\dim \rho(W) = 0$.

Example 2.2. Define $\mathbf{a} = (a_n)_0^\infty$ by $a_n = 2^n$ for $n \ne k^2$ $(k = 0, 1, 2, \dots)$ and $a_{k^2} = 3^{k^2}$ for $k = 1, 2, \dots$. Then $\sigma(\mathbf{a}) = 3$, $r(\mathbf{a}) = \frac{1}{3}$.

Denote by T_0 the set of all sequences $(\varepsilon_n)_0^\infty \in U$ such that $\varepsilon_{k^2} = 0$ $(k = 0, 1, \dots)$ and $\varepsilon_n = 0$ or 1 for $n \ne k^2$ $(k = 1, 2, \dots)$. If $(\varepsilon_n)_0^\infty \in T_0$, then for the subseries $\sum_{n=0}^\infty \varepsilon_n a_n x^n$ of $\sum_{n=0}^\infty a_n x^n$ we get $r(\varepsilon_0 a_0, \varepsilon_1 a_1, \dots) = \frac{1}{2} > r(\mathbf{a}) = \frac{1}{3}$. Therefore

$$\rho(W) \supseteq \rho(T_0). \tag{6}$$

Remember that if $M \subseteq \mathbb{N} = \{1, 2, \dots\}$, then we let

$$M(n) = \sum_{a \le n, a \in A} 1,$$

$$\underline{d}(M) = \liminf_{n\to\infty} \frac{M(n)}{n} \text{ and}$$

$$\overline{d}(M) = \limsup_{n\to\infty} \frac{M(n)}{n},$$

and if $\underline{d}(M) = \overline{d}(M) = \lim_{n\to\infty} \frac{M(n)}{n}$ then we put $d(M) = \lim_{n\to\infty} \frac{M(n)}{n}$. The numbers $\underline{d}(M)$, $\overline{d}(M)$ and $d(M)$ are called the lover, upper asymptotic density and asymptotic density of M (cf. [1], p. 71).

For the determination of $\dim \rho(T_0)$ we shall use the following result of [3] (Theorem 2.7) which in case $q_k = 2$ $(k = 1, 2, \dots)$ gives this theorem.

Theorem. *Let $A \subseteq \mathbb{N}$ and ε_k^0 be a given number for each $k \in A$ ($\varepsilon_k^0 = 0$ or 1). Denote by $Z(A; (\varepsilon_k^0), k \in A)$ the set of all $t = \sum_{j=0}^{\infty} \frac{\varepsilon_j}{2^{j+1}} \in (0, 1]$ such that $\varepsilon_j = \varepsilon_j^0$ if $j \in A$ and ε_j is arbitrary (equal to 0 or 1) for each $j \in \mathbb{N} \setminus A$. Then we have*

$$\dim Z(A; (\varepsilon_k^0), k \in A) = \liminf_{n\to\infty} \frac{\log \prod_{j \le n, j \in \mathbb{N}\setminus A} 2}{n \log 2} = \underline{d}(\mathbb{N}\setminus A). \tag{6'}$$

This result will be used very often in what follows.

Putting $A = \{1^2, 2^2, \dots\}$, $\varepsilon_{k^2} = 0$ $(k = 1, 2, \dots)$ in (6') we have $\rho(T_0) = Z(A; (\varepsilon_k^0), k \in A)$. Further $d(A) = 0$, thus $d(\mathbb{N}\setminus A) = 1$. So we get $\dim \rho(T_0) = d(\mathbb{N}\setminus A) = 1$ and by (6) we obtain $\dim \rho(W) = 1$.

We shall now prove the following general result which shows that the numbers $\dim \rho(W(a_0, a_1, \dots))$ fill up the whole interval $[0, 1]$ when $(a_n)_0^\infty$ runs over all sequences of real numbers.

Theorem 2.1. *For each $\alpha \in [0, 1]$ there exists a power series $\sum_{n=0}^{\infty} a_n x^n$ such that $\dim \rho(W(a_0, a_1, \dots)) = \alpha$.*

Proof. On account of examples 2.1, 2.2 we can assume that $\alpha \in (0, 1)$. Put for brevity $t = 1 - \alpha$ and construct the set

$$A = \left\{ \left[\frac{1}{t}\right], \left[\frac{2}{t}\right], \dots, \left[\frac{n}{t}\right], \dots \right\}$$

($[u]$ denotes the integer part of u). Construct the series $\sum_{n=0}^{\infty} b_n x^n$, where $b_k = 1$ if $k \notin A$ and $b_k = v^k$ if $k \in A$, where v is a fixed number, $0 < v < 1$. Obviously the radius of convergence of this power series is equal to 1.

Let V_0 be the set of all sequences $(\varepsilon_j)_0^\infty \in U$ such that $\varepsilon_k = 0$ for all $k \in A$ with exception of a finite number of k's from A and ε_n is arbitrary (equal to 0 or 1) for other k's. Construct the subseries $\sum_{k=0}^\infty \varepsilon_k b_k x^k$ of $\sum_{k=0}^\infty b_k x^k$. Then the radius of convergence of $\sum_{k=0}^\infty \varepsilon_k b_k x^k$ is equal to $v^{-1} > 1$ and conversely, if the radius of convergence of a subseries $\sum_{n=0}^\infty \varepsilon'_n b_n x^n$ of $\sum_{n=0}^\infty b_n x^n$ is greater than 1, then $\varepsilon'_n = 0$ for all k's from A with the exception of only a finite number. Therefore we have

$$V_0 = W. \tag{7}$$

Let V_0^* be the set of all such $(\varepsilon_n)_0^\infty \in V_0$ that $\varepsilon_n = 0$ for each $n \in A$. Then it is easy to see that

$$\dim \rho(V_0^*) = \dim \rho(V_0). \tag{7'}$$

Using the notation of Theorem 2.7 of [3] we get $\rho(V_0^*) = Z(A; (\varepsilon_k^0), k \in A)$, where $\varepsilon_k^0 = 0$ for each $k \in A$. According to this theorem we get

$$\dim \rho(V_0^*) = \liminf_{n\to\infty} \frac{\log \prod_{j \le n, j \in \mathbb{N}\setminus A} 2}{n \log 2} = \underline{d}(\mathbb{N}\setminus A). \tag{8}$$

But $d(A) = t$. Hence $d(\mathbb{N}\setminus A) = 1 - d(A) = 1 - t = \alpha$. From (7), (7') and (8) we have $\dim \rho(W) = \alpha$. $\square$

First of all we note that W is the set of (ε_n) for which

$$\limsup_{n\to\infty} \sqrt[n]{\varepsilon_n a_n} < \sigma(\mathbf{a}). \tag{9}$$

Obviously (9) holds if and only if there are $k, m \in \mathbb{N}$ such that

$$\sqrt[n]{\varepsilon_n a_n} \le \sigma(\mathbf{a}) - \frac{1}{k}$$

for all $n \ge m$.

Put

$$H(k) = \left\{ n \in \mathbb{N} : \sqrt[n]{|a_n|} > \sigma(\mathbf{a}) - \frac{1}{k} \right\} \quad (k = 1, 2, \dots) \tag{10}$$

$$H(k, m) = H(k) \cap \{m, m+1, \dots\} \quad (k = 1, 2, \dots) \tag{10'}$$

From (10), (10') we see that

$$H(1) \supseteq H(2) \supseteq \cdots \supseteq H(k) \supseteq \dots, \tag{11}$$
$$H(k,1) \supseteq H(k,2) \supseteq \cdots \supseteq H(k,m) \supseteq \dots \text{ for every } k = 1, 2, \dots .$$

Further denote by $W(k,m)$ the class of all $(\varepsilon_j)_{j=0}^{\infty} \in U$ such that $\varepsilon_n = 0$ for each $n \in H(k,m)$. Put

$$W(k) = \bigcup_{m=1}^{\infty} W(k,m) \quad (k = 1, 2, \dots). \tag{12}$$

Note that

$$W(k,1) \subseteq W(k,2) \subseteq \dots \subseteq W(k,m) \subseteq \dots, \tag{13}$$
$$W(1) \subseteq W(2) \subseteq \dots \subseteq W(k) \subseteq \dots. \tag{14}$$

It is easy to see that

$$W = W(a_0, a_1, \dots) = \bigcup_{k=1}^{\infty} W(k). \tag{15}$$

If $B \subseteq \mathbb{N}$, then we put for brevity $C\ B = \mathbb{N}\setminus B$.

Theorem 2.2. *We have*

$$\dim \rho(W) = \underset{k=1,2,\dots}{\rightarrow} \sup \underline{d}(C\ H(k)) = \lim_{k\to\infty} \underline{d}(C\ H(k)).$$

PROOF. First of all we determine the Hausdorff dimension of the set $\rho(W(k,m))$ where k and m are fixed positive integers. For this we use (6') with $A = H(k,m)$, $\varepsilon_j^0 = 0$ for each $j \in A$. Then we get

$$\dim \rho(W(k,m)) = \liminf_{n\to\infty} \frac{\log \prod\limits_{j\le n, j\in\mathbb{N}\setminus A} 2}{n \log 2} = \underline{d}(C\ A).$$

By de Morgan's rule we have (see (10')) $C\ A = C\ H(k,m) = C\ H(k) \cup C\ \{m, m+1, \dots\}$. Since the second "summand" on the right-hand-side is a finite set, we get $\underline{d}(C\ A) = \underline{d}(C\ H(k))$ and so

$$\dim \rho(W(k,m)) = \underline{d}(C\ H(k)) \text{ for every } m = 1, 2, \dots. \tag{16}$$

If $M = \bigcup_{k=1}^{\infty} M_k$, $M \subseteq \mathbb{R}$, then $\dim M = \underset{k=1,2,\dots}{\rightarrow} \sup \dim M_k$. Using (16) and (12) we get

$$\dim \rho(W) \le \underset{k=1,2,\dots}{\rightarrow} \sup \dim \rho(W(k)) \le \underset{k=1,2,\dots}{\rightarrow} \sup \underline{d}(C\ H(k)). \tag{17}$$

Conversely, by (12), (15), (16) we have

$$\dim \rho(W(k)) \ge \dim \rho(W(k,m)) = \underline{d}(C\ H(k)) \tag{18}$$

for every $k = 1, 2, \dots$. Further by (15) we have

$$\dim \rho(W) \geq \dim \rho(W(k)) \quad (k = 1, 2, \dots). \tag{19}$$

Hence, by (18) and (19),

$$\dim \rho(W) \geq \underset{k=1,2,\dots}{\rightarrow} \sup \underline{d}(C\ H(k)). \tag{20}$$

The inequalities (17) and (20) yield

$$\dim \rho(W) = \underset{k=1,2,\dots}{\rightarrow} \sup \underline{d}(C\ H(k)). \tag{21}$$

By (11), the sequence $(\underline{d}(C\ H(k))_{k=1}^{\infty}$ is nondecreasing. Therefore

$$\underset{k=1,2,\dots}{\rightarrow} \sup \underline{d}(C\ H(k)) = \lim_{k \to \infty} \underline{d}(C\ H(k)).$$

From this and (21) the theorem follows. □

Example 2.3. Put $D = \bigcup_{k=0}^{\infty} D_k$, where

$$D_k = \left\{2^{k+1} + 1, 2^{k+1} + 2, \dots, 2^{k+1} + 2^{k-1}\right\} \quad (k = 1, 2, \dots).$$

It can be easily calculated that $\underline{d}(D) = \frac{1}{4}$, $\overline{d}(D) = \frac{2}{5}$. Define $\mathbf{a} = (a_n)_{n=0}^{\infty}$ by

$$\begin{aligned} &a_0 = a_1 = a_2 = 1 \\ &a_n = \left(1,5 + \frac{1}{n}\right)^n \quad \text{for } n > 2,\ n \in D \text{ and} \\ &a_n = \left(2,4 - \frac{1}{n}\right)^n \ \text{for } n > 2,\ n \in C\ D. \end{aligned}$$

Obviously $\sigma(\mathbf{a}) = 2,4$. Observe that $C\ H(k) = \{n \in \mathbb{N} : \sqrt[n]{|a_n|} \leq 2,4 - 1/k\}$ and $\sqrt[n]{|a_n|} = 2,4 - 1/n$ for $n \in C\ D$, $n > 2$ and $a_n = 1,5 + 1/n$ for $n \in D$, $n > 2$. It is clear that for $k \geq 2$ the set $C\ H(k)$ contains all but a finite number of elements of D and it contains at most a finite number of elements of $C\ D$. Thus $\underline{d}(C\ H(k)) = \underline{d}(D) = \frac{1}{4}$ and by Theorem 2.2 we have $\dim \rho(W) = \frac{1}{4}$.

Theorem 2.3. *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \sigma(\mathbf{a}) > 0$*, then* W *is empty, so* $dim\rho(W) = 0$.

PROOF. For any $(\varepsilon_n) \in U$, let (n_i) be the sequence for which $\varepsilon_{n_i} = 1$; then

$$\limsup_{n\to\infty} \sqrt[n]{|\varepsilon_n a_n|} \geq \limsup_{i\to\infty} \sqrt[n_i]{|\varepsilon_{n_i} a_{n_i}|} = \limsup_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = \sigma(\mathbf{a}).$$ □

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