RESEARCH

Real Analysis Exchange
Vol. 24(1), 1998/99, pp. 229–262

Valery A. Zheludev, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv 69978, Isreal e-mail: `zhel@math.tau.ac.il`

WAVELET ANALYSIS IN SPACES OF SLOWLY GROWING SPLINES VIA INTEGRAL REPRESENTATION

Abstract

In this paper we consider polynomial splines with equidistant nodes which may grow as $\mathrm{O}|x|^s$. We present an integral representation of such splines with a distribution kernel where exponential splines are used as basic functions. By this means we characterize splines possessing the property that translations of any such spline form a basis of corresponding spline space. It is shown that any such spline is associated with a dual spline whose translations form the biorthogonal basis. We suggest a scheme of wavelet analysis in the spaces of growing splines based on integral representation of the splines. The key point of that scheme is the refinement equation for the exponential splines which contains only two terms. We construct the so called exponential wavelets. We establish conditions for a spline to be a basic wavelet which enable us to form a library of such wavelets. We give formulas for the decomposition of a spline into a weak orthogonal sum of the sparse-grid spline and an element of the corresponding wavelet space. Reconstruction formulas are presented which permit the use of arbitrary bases of spline and wavelet spaces.

1 Introduction

In this paper we consider polynomial splines $S(x)$ with equidistant nodes which may grow as $\mathrm{O}|x|^s$. We present an integral representation of such splines with the distribution kernel. This representation resembles, to some extent, the Fourier integral of slowly growing functions. Instead of the Fourier exponentials, the so called exponential splines introduced by Schoenberg [11] there are

Key Words: wavelets, splines, integral representation, distribution
Mathematical Reviews subject classification: 42C5,42A38
Received by the editors June 24, 1997

used. In particular, the latter are eigenvectors of the operator of shifts and generalized eigenvectors of the operator of differentiation.

It is worth mentioning periodized exponential splines. It was discovered in [8] that these splines form orthogonal bases of the spaces of periodic splines. In the author's papers [15], [16] this idea was used to come up with the concept of Spline Harmonic Analysis which is a version of harmonic analysis in the spaces of periodic splines. This concept allowed, in particular, a flexible computational scheme of the spline wavelet analysis to be developed. Later on a related approach was applied in [9] to an extended class of periodic functions.

Integral representation provides a suitable tool for operating with growing splines. In particular, it enables us to characterize splines whose shifts form a basis of corresponding spline space. It is shown that any such spline is associated with a dual spline whose shifts form the biorthogonal basis of the same space.

The technique of integral representation proved to be highly relevant to the construction of the scheme of wavelet analysis in the spaces of growing splines. An unusual feature in this case is that the spaces we operate in are non-Hilbert. However, due to our approach, we circumvent this obstacle. We base that scheme on the refinement equation for the exponential splines containing only two terms. Further, we introduce the so-called exponential wavelets. The latter are related to the exponential splines and provide an integral representation of elements of the wavelet spaces. We establish conditions for a spline to be a basic wavelet which results in the creation of a library of such wavelets. This library includes the well known Battle–Lemarié wavelets ([2], [10]) as well as two kinds of Chui–Wang wavelets ([6]).

We establish formulas for decomposition of a spline into a weak orthogonal sum of the sparse-grid spline and an element of the corresponding wavelet space. Reciprocal reconstruction formulas are given as well. We stress that these formulas allow decomposition from an arbitrary basis of a spline space into arbitrary bases of the sparse-grid spline space and the wavelet space. The same may be said of the reconstruction. We also present formulas for the wavelet transformation of a growing signal.

It should be pointed out that most of the formulas of spline wavelet analysis established in this paper have their prototypes in the analysis in the space L^2 ([6]), ([1]). We admit that those relations could be derived by other means. However, in our opinion, the approach to be presented is most relevant to this purpose. Moreover, it is rather universal and can be applied to an extended set of problems. Therefore, to some extent, the paper is of methodological character.

The paper consists of two parts. Part 1 is devoted to a general description of the integral representation of splines. It is auxiliary to Part 2 which is concerned with wavelet analysis. More detailed discussion of integral representation of splines can be found in [17], [18].

In the introductory Section 2 we outline the necessary properties of splines with equidistant nodes and, especially, of the B-splines.

Section 3 is basic to the whole work. At the beginning of this section we discuss some properties of the periodic distributions. Then we introduce the exponential splines. In the concluding subsection we derive the integral representation of the growing splines and present a Parseval type identity.

In Section 4 we establish conditions to be imposed on a spline to ensure that its translations form a basis of the corresponding spline space. We call such splines the TB-splines. Dual splines are constructed as well and some examples of TB-splines are given. Further we discuss the projection of a growing function onto the spline spaces.

In Part 2 we apply the techniques developed to the wavelet analysis in the spaces of growing splines. In Section 5 we establish the refinement equation for the exponential splines and some of its consequences.

In Section6 we introduce exponential wavelets and study their properties. Further, we obtain the integral representation of elements of the wavelet spaces and a Parseval type identity in the wavelet spaces.

In Section 7 we construct splines whose translations form a basis of the wavelet space. We call such splines the TB-wavelets. Dual TB-wavelets are constructed as well and some examples of TB-wavelets are given.

Decomposition of a spline and its reconstruction are discussed in Section 8 together with formulas for the wavelet transformation of growing signals.

Part I
Integral Representation of Splines

2 Some Properties of Splines with Equidistant Nodes

This section is an introductory one. We outline here properties of the polynomial splines with equidistant nodes most of which are known [6], [11].

A function ${}_pS_h$ will be referred to as a spline of order p if

1. ${}_pS_h$ is $p-2$ times continuously differentiable,

2. ${}_pS_h(x) = P_k(x)$ as $x \in (x_k, x_{k+1}), \quad x_k = hk, \quad P_k \in \Pi_{p-1}$,

where Π_{p-1} is the space of polynomials whose degree does not exceed $p-1$.

Splines with $h = 1$ are called the cardinal ones.

The remarkable feature of the splines defined above is that the space of these splines is shift invariant [3]. This means that the space of splines of order p can be looked upon as the span of shifts of a single spline, the so-called B-spline.

The B-splines ${}_pB_h$ of order p are defined as follows.

$$ {}_1B_h(x) = \begin{cases} 1/h & \text{as } x \in (0,h) \\ 0 & \text{else.} \end{cases} \qquad {}_pB_h(x) := {}_1B_h(x)^{[p]}. $$

Here $f^{[p]}$ means the p-th convolution power of the function f.

Throughout, $\sum_r$ will stand for $\sum_{r=-\infty}^{\infty}$. We will omit the index ${}_p\cdot$ as long as it is not essential for the considerations, and similarly for the index $\cdot_h$

Properties of the B-splines.

1. $\mathsf{supp}\,{}_pB_h(x) = (0, hp)$.

2. ${}_pB_h(x) > 0$ as $x \in (0, ph)$.

3. The B-spline ${}_pB_h(x)$ is symmetric about $x = hp/2$ where it attains its unique maximum.

4. $h\sum_r B_h(x - rh) \equiv 1$.

5. The product
$$ h\,B_h(xh) = B_1(x), \tag{1} $$
i.e. does not depend on h.

6. The convolution is
$$ \Big({}_pB * {}_qB\Big)(x) = \int_{-\infty}^{\infty} {}_pB_h(x-y)\,{}_qB_h(y)\,dy = {}_{p+q}B_h(x). $$

7. The derivatives are
$$ {}_pB_h^{(q)}(x) = h^{-q}\,\nabla_h^q({}_{p-q}B_h(x)) = h^{-s}\sum_{l=0}^{q}(-1)^l\binom{q}{l}\,{}_{p-q}B_h(x - hl). \tag{2} $$

In what follows we will repeatedly use the 1-periodic function

$$ {}_pu(v) := h\sum_k e^{2\pi ivk}{}_pB_h((p/2-k)h) = \sum_k e^{2\pi ivk}{}_pB_1(p/2-k). \tag{3} $$

These functions were studied extensively in [13], [11]. They are related to the Euler-Frobenius polynomials. It is important that for any real v the values ${}_pu(v)$ are strictly positive.

The Fourier Transform of the B-spline is

$$ {}_p\widehat{B}_h(v) = \left(\frac{1-e^{-ivh}}{ivh}\right)^p = e^{-\frac{ipvh}{2}}\left(\frac{\sin vh/2}{vh/2}\right)^p. \tag{4} $$

Proposition 2.1. *[12]. Any spline of order p–${}_pS_h$ with its nodes at the points $\{hk\}_{-\infty}^{\infty}$ can be represented by*

$$ {}_pS_h(x) = h\sum_k q_k\, {}_pB_h(x-hk). \tag{5} $$

Remark. If x is any fixed value such that $lh \le x \le (l+1)h$ then the series (5) contains only p nonzero terms, $l-p+1 \le k \le l$. So, given a sequence of coefficients $\{q_k\}$, the values of the spline S can be computed immediately. Moreover, Property 4 of the B-splines implies that in this case

$$ \Big|S(x)\Big| \le \max\{|q_l|\}, \quad l-p+1 \le k \le l. \tag{6} $$

3 Integral Representation

In this section we restrict the class of splines under consideration and introduce a transform in spline spaces which results in the integral representation of the splines related to the Fourier integral.

Definition 3.1. We denote by $\mathbf{G}^s$ the space of sequences $\vec{a} = \{a_k\}_{-\infty}^{\infty}$ which meet the requirement $|a_k| \le M|k|^s\,\forall k$ with a fixed integer s and any positive constant M. The space $\mathbf{G} := \bigcup_{s=-\infty}^{\infty}\mathbf{G}^s$ is said to be the space of sequences of slow growth. Correspondingly, we denote by $\mathbf{F}^s$ the space of locally integrable functions $\Big\{f : |f(x)| \le M|x|^s\,\forall x\Big\}$. The space $\mathbf{F} := \bigcup_{s=-\infty}^{\infty}\mathbf{F}^s$ is referred to as the space of functions of slow growth.

Definition 3.2. We denote by ${}_p\mathbf{V}_h^s$ the space of splines ${}_pS_h$ such that the sequences $\vec{q} = \{q_k\}_{-\infty}^{\infty}$ in the representation (5) belong to $\mathbf{G}^s$ and the space ${}_p\mathbf{V}_h$ we define as follows: ${}_p\mathbf{V}_h := \bigcup_{s=-\infty}^{\infty}{}_p\mathbf{V}_h^s$.

Remark. We stress that for any spline $S \in {}_p\mathbf{V}_h^s$ the inequality

$$\Big|S(x)\Big| \leq L|x|^s \tag{7}$$

holds, with some positive constant L. This follows immediately from (6). Therefore ${}_p\mathbf{V}_h^s \subset \mathbf{F}^s$.

3.1 Some Remarks on Periodic Distributions

Let $\vec{a} = \{a_k\}_{-\infty}^{\infty} \in \mathbf{G}$. Put

$$\mathcal{F}(\vec{a}, v) = \sum_k e^{-2\pi ikv} a_k. \tag{8}$$

This series is a 1-periodic distribution [14], p. 331.

Definition 3.3. We denote by $\mathbf{D}^s$ the space of 1-periodic distributions given by (8) with $\vec{a} \in \mathbf{G}^s$, and $\mathbf{D} := \bigcup_{s=-\infty}^{\infty} \mathbf{D}^s$. The space of 1-periodic complex-valued s-time continuously differentiable functions we denote by $\mathbf{C}^s$.

We emphasize that $\mathbf{D}^{-s-2} \subset \mathbf{C}^s$.

Given a sequence $\vec{a} \in \mathbf{G}^s$, we define the function

$$\Phi(\vec{a}, v) := \Big(\sum_{k=-\infty}^{-1} + \sum_{k=1}^{\infty}\Big) \frac{a_k}{(-2\pi ik)^{s+2}} e^{-2\pi ikv} \ \in \mathbf{D}^{-2} \subset \mathbf{C}^0.$$

Then the distribution $\mathcal{F}(\vec{a}, v)$ can be represented by $\mathcal{F}(\vec{a}, v) = a_0 + \Phi^{(s+2)}(\vec{a}, v)$, where the derivative is used in the sense of the distribution theory [14].

The distribution $\mathcal{F}(\vec{a}, v)$ determines a functional on the space $\mathbf{C}^{s+2}$ which we denote as an integral with a central dot. To be specific, $\forall\, g(v) \in \mathbf{C}^{s+2}$

$$\begin{aligned} &\int_\alpha^{\alpha+1} \mathcal{F}(\vec{a}, v) \ \cdot \overline{g(v)}\, dv \\ :=& a_0 \int_\alpha^{\alpha+1} \overline{g(v)}\, dv + \int_\alpha^{\alpha+1} \Phi(\vec{a}, v)\, \overline{g^{(s+2)}}\, dv = \sum_k a_k\, \overline{g_k}. \end{aligned} \tag{9}$$

Here $\hat{g} = \{g_k\}$ is the sequence of the Fourier coefficients of the function g. The integral on the right hand side should be understood in the ordinary sense.

Remark. If $\vec{a} \in l_1$ then $\mathcal{F}(\vec{a}, v) \in \mathbf{C}^0$. In this case the integral (9) turns out to be an ordinary one provided g is an integrable function.

The series in (8) is the Fourier series of the distribution $\mathcal{F}_h(\vec{a},v)$. Hence the integrals $a_k = \int_0^1 \mathcal{F}(\vec{a},v) \cdot e^{2\pi ivk}\,dv$ are the Fourier coefficients of the distribution.

Let us discuss multiplication of the distribution $\mathcal{F}(\vec{a},v) \in \mathbf{D}^s$ with a function $g(v) = \mathcal{F}(\hat{g},v) \in \mathbf{C}^{s+2}$. The sequence $\hat{g}$ of the Fourier coefficients of the function $g(v)$ belongs to $\mathbf{G}^{-s-2}$. The discrete convolution of the sequences $\vec{a}$ and $\hat{g}$ is $\vec{b} := \{b_k\} = \vec{a} * \hat{g} = \left\{\sum_l a_{k-l}\, g_l\right\}$. The following assertion is readily verified.

Proposition 3.1. *The discrete convolution with a sequence from $\mathbf{G}^{-s-2}$ maps the space $\mathbf{G}^s$ into itself. The continuous convolution with a function from $\mathbf{F}^{-s-2}$ maps the space $\mathbf{F}^s$ into itself.*

The proposition implies that the series $b(v) := \sum_k e^{-2\pi ikv} b_k = \mathcal{F}(\vec{b},v)$ is the distribution from the space $\mathbf{D}^s$ as well as $\mathcal{F}(\vec{a},v)$. Now let $\varphi(v)$ be any testing function of $\mathbf{C}^{s+2}$ and $\{\varphi_k\}$ be its Fourier coefficients. Let us consider the integral

$$\begin{aligned}\int_0^1 b(v) \cdot \overline{\varphi(v)}\,dv &= \sum_k \overline{\varphi_k} \sum_l g_{k-l} a_l \\ = \sum_l a_l \sum_k \overline{\varphi_k\, g_{l-k}} &= \int_0^1 \mathcal{F}_h(\vec{a},v) \cdot \overline{g(v)\varphi(v)}\,dv.\end{aligned}$$

This relation justifies the following.

Definition 3.4. The product of a distribution g from $\mathbf{D}^s$ with a function $f = \mathcal{F}(\vec{a},\cdot)$ from $\mathbf{C}^{s+2}$ will be understood as follows.

$$g(v)\,\mathcal{F}(\vec{a},v) := \mathcal{F}(\vec{a} * \vec{g},v) \in \mathbf{D}^s. \tag{10}$$

It corresponds with the conventional definition of multiplication of a distribution with a function.

3.2 Exponential Splines

Let us return to the $B-$spline. Eq. (4) implies that

$$\begin{aligned}{}_pB_h(x-kh) &= \int_{-\infty}^{\infty} e^{2\pi i\omega(x-kh)} \Big(\frac{1-e^{-2\pi i\omega h}}{2\pi i\omega h}\Big)^p\,d\omega \\ &= \frac{1}{h}\sum_l \int_0^1 e^{2\pi i(v-l)(x/h-k)}\,\frac{(1-e^{-2\pi iv})^p}{(2\pi i(v-l))^p}\,dv.\end{aligned}$$

Integration and summation here could be transposed and we appear at the following representation

$$ {}_pB_h(x-kh) = \frac{1}{h}\int_0^1 {}_pm_h(v,x)e^{-2\pi ikv}\,dv, \quad \text{where} \tag{11}$$

$$ {}_pm_h(v,x) := \sum_l e^{2\pi i(v-l)x/h}\left(\frac{1-e^{-2\pi iv}}{2\pi i(v-l)}\right)^p. \tag{12}$$

Here the product $h\,{}_pB_h(x-kh)$ presents the k-th Fourier coefficient of the 1-periodic with respect to the variable v function ${}_pm_h(v,x)$. Hence

$$ {}_pm_h(v,x) = h\sum_k e^{2\pi ikv}\,{}_pB_h(x-kh) \,\in\, {}_p\mathbf{V}_h^0. \tag{13}$$

It is apparent from this relation that with any x, $m_h(\cdot,x)\in \mathbf{C}^\infty$. As for the variable x, with any v, ${}_pm_h(v,\cdot)$ is a spline from ${}_p\mathbf{V}_h^0$. The spline ${}_pm_h(v,x)$ is nothing but the exponential spline $\Phi_n(x;t)$ by Schoenberg, [11], p.17, with $t=e^{2\pi iv}, n=p-1$.

We will not discuss the numerous noteworthy properties of the splines m here, but will point out only those to be used in what follows. First we mention that Eq. (1) implies that, $m_h(v,x)=m_1\left(v,\frac{x}{h}\right)$.

Proposition 3.2. *The splines $m_h(v,x)$ are eigenvectors of the shift operator. To be specific*

$$ m_h(v,x+lh) = \, e^{2\pi ilv}m_h(v,x). \tag{14}$$

Now put $x=ph/2$. Then we have

$$\begin{aligned} {}_pm_h(v,ph/2) =&h\sum_k e^{2\pi ikv}\,{}_pB_h((p/2-k)h)\\ =\,&{}_pu(v) =\left(\frac{\sin\pi v}{\pi}\right)^p\sum_n \frac{(-1)^{pn}}{(v-n)^p}. \end{aligned}$$

Recall that the function ${}_pu$ was primarily defined in (3).

Proposition 3.3. *The exponential spline m is a generalized eigenvector of the operator of differentiation in the sense that*

$$ {}_pm_h(v,x)_x^{(s)} = \left(\frac{1-e^{-2\pi iv}}{h}\right)^s\,{}_{p-s}m_h(v,x).$$

PROOF. Eq. (12) implies

$$
\begin{aligned}
{}_p m_h(v,x)_x^{(s)} =& h^{-s} \sum_l e^{2\pi i(v-l)x/h} \frac{\left(1-e^{-2\pi i v}\right)^p}{(2\pi i(v-n))^{p-s}} \\
=& \left(\frac{1-e^{-2\pi i v}}{h}\right)^s \sum_l e^{2\pi i(v-n)x/h} \left(\frac{1-e^{-2\pi i v}}{2\pi i(v-n)}\right)^{p-s} \\
=& \left(\frac{1-e^{-2\pi i v}}{h}\right)^s {}_{p-s} m_h(v,x).
\end{aligned}
$$

□

Proposition 3.4. *The derivative $({}_p m_h)_v^{(s)}$ is a spline from the space ${}_p\mathbf{V}_h^s$.*

PROOF. The statement follows immediately from (13). □

The convolution of the exponential spline with the B-spline is

$$
\begin{aligned}
\Big({}_p m(v,\cdot) * {}_q B\Big)(x) =& h \sum_k e^{2\pi i k v} \Big({}_p B(\cdot - kh) * {}_q B\Big)(x) \\
=& h \sum_k e^{2\pi i k v} {}_{p+q} B_h(x-kh) = {}_{p+q} m(v,x).
\end{aligned}
$$

Due to the symmetry of the B-splines we have ${}_p B_h(x) = {}_p B_h(ph-x)$. Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty} {}_p m(v,x)\, {}_p B(x-kh)\, dx =& \Big({}_p m_h(v,\cdot) * {}_p B_h\Big)((k+p)h) \\
= {}_{2p} m(v,(k+p)h) =& e^{2\pi i k v}\, {}_{2p} m(v,ph) = e^{2\pi i k v}\, {}_{2p} u(v).
\end{aligned}
\tag{15}
$$

3.3 Integral Representation and a Parseval Type Identity

We proceed now to establishing the central results of the section.

Theorem 3.5. *Let a distribution ξ belong to $\mathbf{D}^s$ with an integer s. Then the function*

$$
\sigma(\cdot) := \int_0^1 \xi(v) \cdot {}_p m_h(v,\cdot)\, dv \tag{16}
$$

is a spline from ${}_p\mathbf{V}^s \subset \mathbf{F}^s$ and

$$
\sigma(x) = S(x) = h \sum_k q_k\, {}_p B(x-hk), \quad \vec{q} = \{q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s. \tag{17}
$$

Then the coefficients in (17) are $q_k = \int_0^1 \xi(v) \cdot e^{2\pi i v k}\, dv$. Conversely, any spline S from ${}_p\mathbf{V}^s$ can be represented as the integral (16) with some $\xi \in \mathbf{D}^s$.

PROOF. Let $q_k = c_k\big(\xi\big) = \int_0^1 \xi(v) \cdot e^{2\pi i vkh}\, dv$ be the Fourier coefficients of a distribution $\xi \in \mathbf{D}^s$. The sequence
$\vec{q} = \{q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s$. Then, as it follows from (9),

$$\sigma(x) = \sum_k q_k\, c_k\Big({}_p m(\cdot, x)\Big) = h \sum_k q_k\, {}_pB(x - hk) = S(x) \in {}_p\mathbf{V}^s.$$

Conversely, assume that S is a spline belonging to ${}_p\mathbf{V}^s$ and is given as in (17). Then $\mathcal{F}(\vec{q}, v); = \sum_k e^{-2\pi ikv} q_k \in \mathbf{D}^s$ and the function $\sigma(\cdot) := \int_0^1 \mathcal{F}(\vec{q}, v) \cdot {}_pm(v, \cdot)\, dv$ is a spline from ${}_p\mathbf{V}^s$. Its $B-$spline coefficients are

$$Q_k = \int_0^1 \xi(v) \cdot e^{2\pi ivk}\, dv = q_k.$$

Therefore, $S(x) \equiv \sigma(x) = \int_0^1 \mathcal{F}(\vec{q}, v) \cdot {}_pm(v, x)\, dv.$ $\square$

Example. The $B-$spline coefficients of the spline ${}_pm$ are $q_k = e^{2\pi iuk}$ and, therefore,

$$\mathcal{F}(\vec{q}, v) = h \sum_k e^{-2\pi ik(v-u)} = \sum_l \delta(u - v - l). \tag{18}$$

Here $\delta(u)$ is the Dirac delta.

We will now present an identity related to the Parseval one. It is fundamental for operating with integral-represented splines.

Theorem 3.6. *Let some splines S and T belonging to ${}_p\mathbf{V}^s$ and ${}_p\mathbf{V}^{-s-4}$, $s \geq 0$ respectively, be given in the integral form by*

$$S(x) = \int_0^1 \xi(v) \cdot {}_pm(v, x)\, dv = h \sum_k q_k\, {}_pB(x - hk),$$
$$T(x) = \int_0^1 \eta(v)\, {}_pm(v, x)\, dv = h \sum_k t_k\, {}_pB(x - hk).$$

Then the following identity holds.

$$\int_{-\infty}^{\infty} S(x)\, \overline{T(x)}\, dx = h \int_0^1 \xi(v) \cdot \overline{\eta(v)}\, {}_{2p}u(v)\, dv. \tag{19}$$

Recall that the function ${}_pu$ was defined in (3). To avoid overloading the paper, we refer to [17], [18] for the proof of this theorem.

4 Splines of Type B

4.1 TB-Splines and Their Duals

Definition 4.1. We say that a sequence of splines $\{s_k\}_{-\infty}^{\infty}$ forms a basis of the space ${}_p\mathbf{V}_h$ if any spline $S \in {}_p\mathbf{V}_h$ can be represented uniquely as the series $S = \sum_{-\infty}^{\infty} a_k\, s_k$ which converges uniformly on any compact set of the real line.

The h-shifts of the B-spline form a basis of the space ${}_p\mathbf{V}_h$. We describe now a class of splines which offers the similar property.

Definition 4.2. A spline $\varphi \in {}_p\mathbf{V}_h^{-\infty}$ is said to be a spline of *type B* (TB - spline) if its shifts $\{\varphi(\cdot - kh)\}_{-\infty}^{\infty}$ form a basis of the space ${}_p\mathbf{V}_h$.

Any TB-spline $\varphi \in {}_p\mathbf{V}_h^{-\infty}$, if it exists, can be represented as the integral

$$\varphi(x) = \int_0^1 \rho(v)\, {}_pm_h(v,x)\, dv. \tag{20}$$

with a function $\rho \in D^{-\infty} = \mathbf{C}^{\infty}$. Moreover, the inequality (7) enables us to affirm that for all x belonging to any compact set of the real line

$$|\varphi(x - kh)| \leq C\nu_k, \tag{21}$$

where the sequence $\{\nu_k\} \subset \mathbf{G}^{-\infty}$.

Theorem 4.1. *Let a spline φ from ${}_p\mathbf{V}_h^{-\infty}$ be represented as in (20). Then it is a TB-spline if and only if $|\rho(v)|$ is strictly positive for all real v. Herewith, the two expansions of a spline $S \in {}_p\mathbf{V}_h^{s}$*

$$S(x) = \sum_k Q_k\, \varphi(x - kh) = S(x) = \int_0^1 \xi(v) \cdot m(v,x)\, dv \tag{22}$$

are related by

$$\xi(v) = \rho(v) \sum_k e^{-2\pi ikv} Q_k, \qquad Q_k = \int_0^1 \frac{\xi(v)}{\rho(v)} \cdot e^{2\pi ivk}\, dv.$$

PROOF. We first point out that ρ belongs to $\mathbf{C}^{\infty}$. Due to (14), we may write

$$\varphi(x - kh) = \int_0^1 e^{-2\pi ikv} \rho(v)\, m(v,x)\, dv. \tag{23}$$

These are the Fourier coefficients of the function $\rho\, m(\cdot, x) \in \mathbf{C}^\infty$. Hence

$$\rho(v)\, m(v,x) = \sum_k e^{2\pi i k v} \varphi(x - kh). \tag{24}$$

1. Let $|\rho(v)|$ be strictly positive. Suppose a spline S from ${}_p\mathbf{V}_h^s$ is represented as in (16):

$$S(x) = \int_0^1 \xi(v) \cdot m(v,x)\, dv.$$

Then, referring to (9), we may write

$$\begin{aligned} S(x) &= \int_0^1 \frac{\xi(v)}{\rho(v)} \cdot \rho(v)\, m(v,x)\, dv = \sum_k Q_k \varphi(x - kh), \\ \xi(v) &= \rho(v) \sum_k e^{-2\pi i k v} Q_k, \qquad Q_k = \int_0^1 \frac{\xi(v)}{\rho(v)} \cdot e^{2\pi i k v}\, dv. \end{aligned}$$

Since the values Q_k are the Fourier coefficients of the distribution $\rho^{-1}\, \xi \in \mathbf{D}^s$, the sequence $\vec{Q} = \{Q_k\}_{-\infty}^{\infty} \in \mathbf{G}^s$. Then we see from (21) that the series on the right hand side converges uniformly on any compact set of the real line. This implies that φ is a TB-spline.

2. Conversely, suppose that the function φ given by (20) is a TB-spline. Then its translations form a basis of the space ${}_p\mathbf{V}_h$. Let us expand the exponential spline in terms of this basis

$$m(v,x) = \sum_k \mu_k(v) \varphi(x - kh).$$

Substituting it into (24) we get that

$$\rho(v) \sum_k \mu_k(v) \varphi(x - kh) = \sum_k e^{2\pi i k v} \varphi(x - kh) \Longrightarrow \rho(v)\, \mu_k(v) = e^{2\pi i k v}.$$

Hence it follows that $\rho(v) \neq 0$ for all real v. But ρ is a continuous 1-periodic function. Therefore $|\rho(v)|$ is strictly positive. $\square$

Generally, bases formed from translations of TB-splines are non-orthogonal in the L_2 sense. However, biorthogonal bases exist.

Definition 4.3. Let two splines φ, $\tilde{\varphi}$ from ${}_p\mathbf{V}_h$ be TB-splines. These splines are said to be *dual* to each other if the following relation holds.

$$\int_{-\infty}^{\infty} \varphi(x - kh)\, \overline{\tilde{\varphi}(x - lh)}\, dx = \delta_l^k,$$

where δ_l^k means the Kroneker delta.

We will show that any TB-spline has a dual one.

Theorem 4.2. *Let a TB-spline be represented by $\varphi(x) = \int_0^1 \rho(v)\, m(v,x)\, dv$. Then there exists a unique TB-spline $\tilde{\varphi}$ dual to φ.*

$$\tilde{\varphi}(x) = \int_0^1 \tilde{\rho}(v)\, m(v,x)\, dv, \tag{25}$$

$$\tilde{\rho}(v) = \Big(\overline{\rho(v)}\,{}_{2p}u(v)\, h\Big)^{-1}. \tag{26}$$

Proof. Let a spline $\tilde{\varphi}$ be given in the shape (25). Then, due to the identity (19) and Eq. (23), we have

$$\int_{-\infty}^{\infty} \varphi(x-kh)\, \overline{\tilde{\varphi}(x-lh)}\, dx = h \int_0^1 e^{-2\pi i(k-l)v}\, \rho(v)\, \overline{\tilde{\rho}(v)}\,{}_{2p}u(v)\, dv.$$

Provided (25) holds, the integral is

$$\int_{-\infty}^{\infty} \varphi(x-kh)\, \overline{\tilde{\varphi}(x-lh)}\, dx = \int_0^1 e^{-2\pi i(k-l)v}\, dv = \delta_l^k.$$

We stress that in this case $\tilde{\varphi} \in {}_p\mathbf{V}_h^{-\infty}$ just as φ.

Conversely, the relation

$$h \int_0^1 e^{-2\pi i(k-l)v}\, \rho(v)\, \overline{\tilde{\rho}(v)}\,{}_{2p}u(v)\, dv = \delta_l^k$$

means that all the Fourier coefficients c_k of the continuous function $\rho\, \overline{\tilde{\rho}}\,{}_{2p}u$, with $k \neq 0$ are zero, whereas $c_0 = 1/h$. Therefore (26) is true. That implies the uniqueness of the dual spline. □

Theorem 4.3. *Let two TB-splines φ^i, $\quad i = 1,2$, be represented in the integral form by $\varphi^i(x) = \int_0^1 \rho^i(v)\, m(v,x)\, dv$, and a spline $S \in {}_p\mathbf{V}_h^s$ be expanded with respect to the two bases*

$$S(x) = \sum_k Q_k^1\, \varphi^1(x-kh) = S(x) = \sum_k Q_k^2\, \varphi^2(x-kh). \tag{27}$$

Then the coordinates Q are linked via convolution with the sequence

$$b_r^{1,2} := \int_0^1 \frac{\rho^1(v)}{\rho^2(v)} e^{2\pi i v r}\, dv$$

by $Q_k^2 = \sum_l b_{k-l}^{1,2}\, Q_l^1$. In particular,

$$\varphi^1(x-lh) = \sum_k b_{k-l}^{1,2}\, \varphi^2(x-kh). \tag{28}$$

PROOF. The spline S may be written as in (22). Then

$$\xi(v) = \rho^1(v)\mathcal{F}(\vec{Q}^1, v) = \xi(v) = \rho^2(v)\mathcal{F}(\vec{Q}^2, v).$$

Hence $\mathcal{F}(\vec{Q}^2, v) = \frac{\rho^1(v)}{\rho^2(v)}\mathcal{F}(\vec{Q}^1, v)$. Since ρ^1/ρ^2 is a function from $\mathbf{D}^{-\infty}$, we arrive at (27), keeping in mind (10). Eq. (28) is a special case of (27) with $Q_k^1 = \delta_k^l$. □

4.2 Galerkin Projections

Since the spaces of functions we operate in are non-Hilbert, we should introduce, instead of the notion of orthogonal projection, its weak substitution.

Definition 4.4. Let f be a function of slow growth. We call a spline $S(f) \in {}_p\mathbf{V}_h$ the Galerkin projection (GP) of the function f onto the spline space ${}_p\mathbf{V}_h$ if for all integers k the following relations hold

$$\int_{-\infty}^{\infty} S(f,x)\,{}_pB_h(x-kh)\,dx = \int_{-\infty}^{\infty} f(x)\,{}_pB_h(x-kh)\,dx := \Phi_k. \tag{29}$$

Remark. In the case when the function f is square integrable on the real line, its GP is just the same as the conventional orthogonal projection.

To construct the GP of a function we need the TB-spline dual to the B-spline. Since in the integral representation of the B-spline the function $\rho(v) \equiv 1/h$, the condition (25) implies that the spline $\varphi^d(x) = \int_0^1 \frac{{}_pm(v,x)}{{}_{2p}u(v)}\,dv$ is dual to the B-spline. We stress that ${}_{2p}u(v)^{-1} \in \mathbf{C}^\infty$. Thus $\varphi^d \in {}_p\mathbf{S}_h^{-\infty} \subset \mathbf{F}^{-\infty}$. In fact, the coefficients of the B-spline representation of the spline φ^d are of exponential decay (see [11] e.g.) and the same may be said on the very spline φ^d.

With the dual spline φ^d at hand the following theorem can be established immediately.

Theorem 4.4. *Let $f(x)$ be a function of slow growth. Then there exists the unique GP $S(f)$ of the function onto ${}_p\mathbf{V}_h$. Moreover, if $f \in \mathbf{F}^s$ then the spline $S(f) \in {}_p\mathbf{V}_h^s \subset \mathbf{F}^s$. This spline is equal to $S(f,x) = \sum_k \Phi_k\,\varphi^d(x-kh)$. In the integral form $S(f,x) = \int_0^1 \frac{\mathcal{F}(\vec{\Phi},v)}{{}_{2p}u(v)} \cdot {}_pm_h(v,x)\,dv$.*

The sequence $\left\{\Phi_k\right\}$ was defined in (29).

Corollary 4.5. *The GP of a polynomial $P \in \Pi_{p-1}$ onto the spline space ${}_p\mathbf{V}_h$ is the very polynomial P.*

4.3 Two More Examples of TB-Splines

4.3.1 Fundamental TB-splines

Let us consider the TB-spline ${}_pL_h$:

$$ {}_pL_h(x) := \int_0^1 \frac{{}_pm_h(v,x)}{{}_pu(v)}\, dv. \tag{30} $$

This spline interpolates the data $\left\{\delta_k^0\right\}$. Setting $x = (k + p/2)h$, we have

$$ {}_pL_h((k+\frac{p}{2})h) = \int_0^1 \frac{{}_pm_h(v,(k+p/2)h))}{{}_pu(v)}\, dv $$

$$ = \int_0^1 e^{2\pi ivk}\, \frac{{}_pm_h(v,hp/2)}{{}_pu(v)}\, dv = \int_0^1 e^{2\pi ivk}\, \frac{{}_pu(v)}{{}_pu(v)}\, dv = \delta_k^0. $$

Such a spline is called a fundamental one.

4.3.2 Selfdual TB-Splines

Let us define the TB-spline ${}_p\varphi^o$ by ${}_p\varphi^o(x) = \int_0^1 \frac{{}_pm(v,x)}{\sqrt{{}_{2p}u(v)\,h}}\, dv$. It is readily seen from (25) that this TB-spline coincides with its dual one. So, it is pertinent to call it the selfdual TB-spline. The shifts $\left\{\varphi_h^o(x-kh)\right\}_{-\infty}^{\infty}$ form an orthonormal basis (in the sense of L_2) of the space ${}_p\mathbf{V}_h$. These TB-splines were discovered by Battle and Lemarié [2], [10].

Part II

Basics of the Spline-Wavelet Analysis

In this part we establish some relations of the spline-wavelet analysis of the functions of slow growth. Most of those formulas are related to corresponding formulas for square integrable functions and splines [6]. However the integral representation approach provides remarkably simple tools for deriving them. It offers some advantages even for the L_2 case.

First we should define wavelet spaces. Since we cannot use the conventional definition of such spaces as the orthogonal complements of the sparse-grid spaces in the fine-grid ones, we introduce wavelet spaces by proceeding as follows.

Definition 4.5. A function f from $\mathbf{F}$ satisfying the conditions

$$\int_{-\infty}^{\infty} f(x)\, {}_pB_h(x-kh)\, dx = 0$$

with all integers k, is said to be weak orthogonal to the space ${}_p\mathbf{V}_h^s$.

Here ${}_pB_h \in {}_p\mathbf{V}_h^s$ is the B-spline.

We point out that the space ${}_p\mathbf{V}_{2h}^s$ is the subspace of ${}_p\mathbf{V}_h^s$. It is apparent that if a spline S_h belongs to ${}_p\mathbf{V}_h^s$ and S_{2h} is its GP onto ${}_p\mathbf{V}_{2h}^s$ then the spline

$$W_{2h} := S_h - S_{2h} \tag{31}$$

is weak orthogonal to ${}_p\mathbf{V}_{2h}^s$. Therefore the space ${}_p\mathbf{V}_h^s$ may be represented as the direct sum

$${}_p\mathbf{V}_h^s = {}_p\mathbf{V}_{2h}^s \oplus {}_p\mathbf{W}_{2h}^s, \quad {}_p\mathbf{V}_h = {}_p\mathbf{V}_{2h} \oplus {}_p\mathbf{W}_{2h}, \tag{32}$$

where we have denoted by ${}_p\mathbf{W}_{2h}^s$ the subspace of ${}_p\mathbf{V}_h^s$ consisting of all splines weak orthogonal to ${}_p\mathbf{V}_{2h}^s$. Correspondingly, we denote ${}_p\mathbf{W}_{2h} := \bigcup_{s=-\infty}^{\infty} {}_p\mathbf{W}_{2h}^s$. We may consider sums in (32) as weak orthogonal sums.

Definition 4.6. The subspace ${}_p\mathbf{W}_{2h}^s\big({}_p\mathbf{W}_{2h}\big)$ we call the weak orthogonal complement of ${}_p\mathbf{V}_{2h}^s\big({}_p\mathbf{V}_{2h}\big)$ in ${}_p\mathbf{V}_h^s\big({}_p\mathbf{V}_h\big)$ and refer to it as to the wavelet space. The spline $W_{2h} \in {}_p\mathbf{W}_{2h}^s$ defined by (31) we call the GP of S_h onto ${}_p\mathbf{W}_{2h}^s$.

5 Refinement Equation

We start with the so called refinement equation which is fundamental for any wavelet construction as well as for subdivision schemes [7]. This equation links basic splines of the spaces ${}_p\mathbf{V}_h$ and ${}_p\mathbf{V}_{2h}$.

The term $m_\nu(v,x)$ will stand for ${}_pm_\nu(v,x)$ and $u(v)$ for ${}_{2p}u(v)$. The following theorem relates the exponential splines $m_\nu(v,x)$.

Theorem 5.1. *The following refinement equation holds.*

$$m_{2h}(v,x) = b(v)\, m_h\Big(\frac{v}{2},x\Big) + b(v+1)\, m_h\Big(\frac{v+1}{2},x\Big), \tag{33}$$

where $b(v) = 2^{-p}(1+e^{-\pi i v})^p$.

PROOF. Let us rewrite Eq. (12) in such a manner

$$m_h(v,x) = e^{2\pi ivx/h}\left[\frac{1-e^{-2\pi iv}}{2\pi i}\right]^p \sum_n \frac{e^{-2\pi inx/h}}{(v-n)^p}.$$

Then, putting $t = v/2$, we transform the sparse-grid spline as follows:

$$\begin{aligned} m_{2h}(v,x) &= e^{2\pi itx/h}\left[\frac{1-e^{-4\pi it}}{2\pi i}\right]^p \sum_n \frac{e^{-\pi inx/h}}{(2t-n)^p} \\ =&\left(\frac{1+e^{2\pi it}}{2}\right)^p e^{2\pi itx/h}\left[\frac{1-e^{-2\pi it}}{2\pi i}\right]^p \sum_n \frac{e^{-2\pi inx/h}}{(t-n)^p} \\ +&\left(\frac{1+e^{2\pi i(t+1/2)}}{2}\right)^p e^{2\pi i(t+1/2)x/h}\left[\frac{1-e^{-2\pi i(t+1/2)}}{2\pi i}\right]^p \sum_n \frac{e^{-2\pi inx/h}}{(t+1/2-n)^p} \\ =&\left(\frac{1+e^{2\pi it}}{2}\right)^p m_h(t,x) + \left(\frac{1+e^{2\pi i(t+1/2)}}{2}\right)^p m_h(t+1/2,x). \end{aligned}$$ □

The relation (33) implies a useful identity.

Corollary 5.2. *The following identity holds.*

$$u(v) = |b(v)|^2\, u\Big(\frac{v}{2}\Big) + |b(v+1)|^2\, u\Big(\frac{v+1}{2}\Big). \tag{34}$$

PROOF. Let us rewrite (33) for the splines of the order $2p$ and put $x = 2hp$. Then we have

$$\begin{aligned} u(v) =&{}_{2p}m_{2h}(v,2hp) = 4^{-p}\Big(1+e^{-\pi iv}\Big)^{2p}{}_{2p}m_h\Big(\frac{v}{2},2hp\Big) \\ &+4^{-p}\Big(1-e^{-\pi iv}\Big)^{2p}{}_{2p}m_h\Big(\frac{v+1}{2},2hp\Big). \end{aligned}$$

But since

$${}_{2p}m_h\Big(\frac{v}{2},2hp\Big) = e^{\pi ivp}{}_{2p}m_h\Big(\frac{v}{2},hp\Big) = e^{\pi ivp}u\Big(\frac{v}{2}\Big),$$

we get

$$u(v) = 4^{-p}e^{\pi ivp}\left[(1+e^{-\pi iv})^{2p}u\Big(\frac{v}{2}\Big) + (-1)^p\Big(1-e^{-\pi iv}\Big)^{2p}u\Big(\frac{v+1}{2}\Big)\right].$$

Hence (34) follows. □

Corollary 5.3. *For the sparse-grid $B-$splines the following integral representation holds.*

$$B_{2h}(x) = \frac{1}{h}\int_0^1 b(2v)\, m_h(v,x)\, dv = \frac{2^{-p}}{h}\int_0^1 \left(1+e^{-2\pi iv}\right)^p m_h(v,x)\, dv \quad (35)$$

PROOF. Due to (11) we may write

$$\begin{aligned} B_{2h}(x) =& \frac{1}{2h}\int_0^1 m_{2h}(v,x)\, dv \\ =& \frac{1}{2h}\int_0^1 \left(b(v)\, m_h\Big(\frac{v}{2},x\Big) + b(v+1)\, m_h\Big(\frac{v+1}{2},x\Big)\right) dv \\ =& \frac{1}{h}\int_0^{1/2} b(2v)\, m_h(v,x)\, dv + \frac{1}{h}\int_{1/2}^1 b(2v)\, m_h(v,x)\, dv. \end{aligned} \qquad \square$$

The following simple assertion will enable us to construct basic elements for the integral representation of elements of wavelet spaces.

Proposition 5.4. *The GP of the spline $m_h(v,\cdot)$ onto the subspace ${}_p\mathbf{V}_{2h}$ is the spline*

$$\tilde{m}_{2h}(v,x) := \frac{\overline{b(2v)}\, u(v)}{u(2v)} m_{2h}(2v,x). \quad (36)$$

PROOF. Let us consider the integral $I_k := \int_{-\infty}^{\infty} m_h(v,x)\, B_{2h}(x-k2h)\, dx$. Due to (18) we may write $m_h(v,x) = \int_0^1 \delta(v-\xi)\cdot m_h(\xi,x)\, d\xi$. Further we apply to our integral the "Parseval identity" (19). Then we have

$$I_k = \int_0^1 \delta(v-\xi)\cdot e^{2\pi i\xi 2k}\overline{b(2\xi)}\, u(\xi)\, d\xi = e^{2\pi iv2k}\, \overline{b(2v)}\, u(v).$$

On the other hand, we may write this integral as in (15).

$$\int_{-\infty}^{\infty} m_{2h}(2v,x)\, B_{2h}(x-k2h)\, dx = e^{2\pi i2vk}\, u(2v). \qquad \square$$

6 Exponential Wavelets

Now we are in a position to construct some splines in the wavelet space ${}_p\mathbf{W}_{2h} \subset {}_p\mathbf{V}_h$ which are related to the exponential splines m_{2h}. To start with, we find the difference $\tilde{w}_{2h} := m_h - \tilde{m}_{2h}$ between the exponential spline m_h and its GP onto the space ${}_p\mathbf{V}_{2h}$. So,

$$\begin{aligned}
&\tilde{w}_{2h}(v,x)\\
=&m_h(v,x) - \frac{\overline{b(2v)}\,u(v)}{u(2v)}\Big[b(2v)\,m_h(v,x) + b(2v+1)\,m_h(v+1/2,x)\Big]\\
=&\frac{m_h(v,x)\big(u(2v) - |b(2v)|^2\,u(v)\big) - m_h(v+1/2,x)\overline{b(2v)}\,b(2v+1)\,u(v)}{u(2v)}.
\end{aligned}$$

Eq. (34) leads us to the following representation.

$$\begin{aligned}
&\tilde{w}_{2h}(v,x)\\
=&\frac{m_h(v,x)\,|b(2v+1)|^2\,u(v+1/2) - m_h(v+1/2,x)\overline{b(2v)}\,b(2v+1)\,u(v)}{u(2v)}\\
=&\frac{b(2v+1)}{u(2v)}\Big(m_h(v,x)\,\overline{b(2v+1)}\,u(v+1/2) - m_h(v+1/2,x)\overline{b(2v)}\,u(v)\Big).
\end{aligned}$$

We want to write the spline $\tilde{w}_{2h}(v,x)$ in a shape similar to (36). For this purpose we denote

$$a(v) := e^{\pi i v}\overline{b(v+1)}\,u\Big(\frac{v+1}{2}\Big) = 2^{-p}e^{\pi i v}\,(1-e^{\pi i v})^p\,u\Big(\frac{v+1}{2}\Big)$$

and introduce the spline w_{2h} as follows.

$$w_{2h}(v,x) := a(v)\,m_h\Big(\frac{v}{2},x\Big) + a(v+1)\,m_h\Big(\frac{v+1}{2},x\Big). \tag{37}$$

We call the spline w_{2h} the *exponential wavelet* because its properties are related to those of the exponential splines m_{2h}.

Note that, due to periodicity, $b(2v+1+1) = b(2v)$, $u(v+1/2+1/2) = u(v)$ and define the function t by

$$t(v) := u\Big(\frac{v}{2}\Big)u\Big(\frac{v+1}{2}\Big)u(v). \tag{38}$$

This function belongs to $\mathbf{C}^\infty$ and has no zeros on the real line. Then the spline $\tilde{w}_{2h}$ may be written similarly to $\tilde{m}_{2h}$. $\tilde{w}_{2h}(v,x) := \frac{\overline{a(2v)}\,u(v)}{t(2v)}w_{2h}(2v,x)$. We mention some properties of the splines w_{2h}.

Properties of Exponential Wavelets

1. The splines w_{2h} as well as $\tilde{w}_{2h}$ belong to the wavelet space ${}_p\mathbf{W}_{2h} \subset {}_p\mathbf{V}_h$.

2. With any fixed x, $w_{2h}(\cdot, x) \in \mathbf{C}^\infty$.
3. The splines w_{2h} are eigenvectors of the shift operator

$$w_{2h}(v, x + l2h) = e^{2\pi i v l} w_{2h}(v, x). \tag{39}$$

This fact stems from the corresponding property (14) of the splines m_h.

4. The exponential wavelet is a derivative of a combination of the exponential splines. Namely,

$$\begin{aligned}{}_p w_{2h}(v, x) = &\left(\frac{-h}{2}\right)^p \frac{d^p}{dx^p}\left[{}_{2p}u\Big(\frac{v+1}{2}\Big)\, {}_{2p}m_h\Big(\frac{v}{2}, x + (1+p)h\Big)\right. \\ &\left. + \ {}_{2p}u\Big(\frac{v}{2}\Big)\, {}_{2p}m_h\Big(\frac{v+1}{2}, x + (1+p)h\Big)\right].\end{aligned} \tag{40}$$

This relation is an immediate consequence of Proposition 3.3.

We proceed now to establishing the integral representation of elements of the wavelet spaces.

Proposition 6.1. *Any spline $W \in {}_p\mathbf{V}_h^s$ which can be represented as the integral $W(x) = \int_0^1 \eta(v) \cdot w_{2h}(v, x)\, dv$ with some $1-$periodic distribution $\eta \in \mathbf{D}^s$, belongs to ${}_p\mathbf{W}_{2h}^s$.*

PROOF. Substituting $w_{2h}(v, x)$ from (37) into the integral, we obtain

$$W(x) = \int_0^1 \eta(v) \cdot \left[a(v)\, m_h\left(\frac{v}{2}, x\right) + a(v+1)\, m_h\left(\frac{v+1}{2}, x\right)\right] dv.$$

Hence

$$W(x) = 2\int_0^1 \eta(2v) \cdot\ a(2v)\, m_h(v, x)\, dv. \tag{41}$$

Invoking the "Parseval identity" (19) and the integral representation (35) of the $B-$spline, we find the integral

$$J_k := \int_{-\infty}^{\infty} W(x)\, B_{2h}(x - k2h)\, dx = \int_0^1 \chi(v)\, dv \quad \text{where}$$

$$\begin{aligned}\chi(v) :=& \eta(2v)\, e^{4\pi i k v}\, a(2v)\overline{b(2v)}\, u(v) \\ =& \eta(2v)\, e^{2\pi i(2k+1)v}\, \overline{b(2v+1) b(2v)}\, u(v + 1/2).\end{aligned}$$

It is readily verified that $\chi(v + 1/2) = -\chi(v)$. Therefore $J_k = 0$ for all k. Hence we see that $W \in {}_p\mathbf{W}_{2h}^s$. $\square$

Proposition 6.2. *Let a spline S from $\mathbf{V}_{2h}^s$ be given by*

$$S(x) = \int_0^1 \xi(v) \cdot m_{2h}(v, x),\, dv.$$

Then, being regarded as an element of the space $\mathbf{V}_h^s$, it can be represented by

$$S(x) = 2\int_0^1 \xi(2v) \cdot\, b(2v)\, m_h(v, x)\, dv. \tag{42}$$

Eq. (42) is derived similarly to (41).

We emphasize that

$$\begin{aligned} m_h(v, x) =& \tilde{m}_{2h}(v, x) + \tilde{w}_{2h}(v, x) \\ =& \frac{\overline{b(2v)}\, u(v)}{u(2v)} m_{2h}(2v, x) + \frac{\overline{a(2v)}\, u(v)}{t(2v)} w_{2h}(2v, x). \end{aligned} \tag{43}$$

This equation will be used when proving the following theorem.

Theorem 6.3. *Suppose that a spline $S_h \in {}_p\mathbf{V}_h^s$ is given in integral form.*

$$S_h(x) = \int_0^1 \xi_h(v) \cdot m_h(v, x)\, dv.$$

Then this spline is equal to the sum

$$S_h(x) = S_{2h}(x) + W_{2h}(x) \tag{44}$$

where the splines S_{2h}, W_{2h} are the GP of the spline S_h onto the subspaces ${}_p\mathbf{V}_{2h}$, ${}_p\mathbf{W}_{2h}$, correspondingly. Moreover, the following representations hold.

$$S_{2h}(x) = \int_0^1 \xi_{2h}(v) \cdot m_{2h}(v, x)\, dv, \tag{45}$$

$$\xi_{2h}(v) = \left(\overline{b(v)}\, u\Big(\frac{v}{2}\Big)\, \xi_h\Big(\frac{v}{2}\Big) + \overline{b(v+1)}\, u\Big(\frac{v+1}{2}\Big)\, \xi_h\Big(\frac{v+1}{2}\Big) \right) u^{-1}(v).$$

$$W_{2h}(x) = \int_0^1 \eta_{2h}(v) \cdot w_{2h}(v, x)\, dv, \tag{46}$$

$$\eta_{2h}(v) = \left(\overline{a(v)}\, u\Big(\frac{v}{2}\Big)\, \xi_h\Big(\frac{v}{2}\Big) + \overline{a(v+1)}\, u\Big(\frac{v+1}{2}\Big)\, \xi_h\Big(\frac{v+1}{2}\Big) \right) t^{-1}(v).$$

$$\xi_h(v) = 2\Big(\xi_{2h}(2v)\, b(2v) + \eta_{2h}(2v)\, a(2v) \Big). \tag{47}$$

PROOF. Eq. (43) implies that

$$S_h(x) = \int_0^1 \xi_h(v) \cdot \frac{\overline{b(2v)}\, u(v)}{u(2v)} m_{2h}(2v,x)\, dv + \int_0^1 \xi_h(v) \cdot \frac{\overline{a(2v)}\, u(v)}{t(2v)} w_{2h}(2v,x)\, dv.$$

Let us consider, for example, the latter integral

$$\int_0^1 \xi_h(v) \cdot \frac{\overline{a(2v)}\, u(v)}{t(2v)} w_{2h}(2v,x)\, dv$$

$$= \int_0^1 \left(\overline{a(v)}\, u\Big(\frac{v}{2}\Big)\, \xi_h\Big(\frac{v}{2}\Big) + \overline{a(v+1)}\, u\Big(\frac{v+1}{2}\Big)\, \xi_h\Big(\frac{v+1}{2}\Big) \right) \cdot \frac{w_{2h}(v,x)}{t(v)}\, dv.$$

Hence (46) follows. Eq. (45) is derived in the same manner. Since the spline W_{2h} is represented in the integral form (46), it belongs to the subspace ${}_p\mathbf{W}_{2h}$, due to Proposition 6.1. Similarly, S_{2h} belongs to ${}_p\mathbf{S}_{2h}$. Therefore the splines S_{2h} and W_{2h} are GPs of S_h onto the corresponding subspaces. Eq. (47) results immediately from (42), (41). □

Proposition 6.1 together with Theorem 6.3 leads to the following assertion.

Corollary 6.4. *A spline $W \in {}_p\mathbf{V}_h^s$ belongs to ${}_p\mathbf{W}_{2h}^s$ if and only if it can be represented as the integral $W(x) = \int_0^1 \eta(v) \cdot w_{2h}(v,x)\, dv$with some $1-$periodic distribution $\eta \in \mathbf{D}^s$.*

We will now establish the "Parseval identity" in the wavelet space.

Theorem 6.5. *Let splines W and Z belong to ${}_p\mathbf{W}_{2h}^s$, $s \geq 0$, and ${}_p\mathbf{W}_{2h}^{-s-4}$, respectively, and the following representations hold.*

$$W(x) = \int_0^1 \eta(v) \cdot w_{2h}(v,x)\, dv \text{ and } Z(x) = \int_0^1 \zeta(v)\, w_{2h}(v,x)\, dv.$$

Then

$$I := \int_{-\infty}^{\infty} W(x)\, \overline{Z(x)}\, dx = 2h \int_0^1 \eta(v) \cdot \overline{\zeta(v)}\, t(v)\, dv. \tag{48}$$

PROOF. Eq. (41) enables us to represent the splines involved by

$$W(x) = 2\int_0^1 \eta(2v) \cdot a(2v)\, m_h(v,x)\, dv,$$

$$Z(x) = 2\int_0^1 \zeta(2v)\, a(2v)\, m_h(v,x)\, dv.$$

Then invoking Theorem 3.6 to integrate the product, we get

$$
\begin{aligned}
I =&4h\int_0^1 \eta(2v)\cdot\overline{\zeta(2v)}|a(2v)|^2\,u(v)\,dv\\
=&4h\int_0^1 \eta(2v)\cdot\overline{\zeta(2v)}\Big[|b(2v+1)|^2\,u(v+1/2)\Big]\Big[u(v)\,u(v+1/2)\Big]\,dv\\
=&2h\int_0^1 \eta(v)\cdot\overline{\zeta(v)}\Big[u(\frac{v}{2})\,u(\frac{v+1}{2})\Big]\Big[|b(v+1/2)|^2\,u(\frac{v+1}{2})+|b(v)|^2\,u(\frac{v}{2})\Big]\,dv.
\end{aligned}
$$

Due to (34) $|b(v+1/2)|^2\,u(\frac{v+1}{2})+|b(v)|^2\,u(\frac{v}{2})=u(v)$. To arrive at (48) suffice it to recall the definition (38). □

7 TB-Wavelets

7.1 Definition and Basic Properties

In this subsection we construct and study basic elements of the space ${}_p\mathbf{W}_{2h}$ related to the TB-splines.

Definition 7.1. We call a spline $\psi\in{}_p\mathbf{W}_{2h}^{-\infty}\subset{}_p\mathbf{V}_h^{-\infty}$ a wavelet of type B (TB-wavelet) if its shifts $\big\{\psi(\cdot-k2h)\big\}_{-\infty}^{\infty}$ form a basis of the space ${}_p\mathbf{W}_{2h}$.

The notion of basis is understood here in the sense of Definition 4.1. We stress that any spline $\psi\in{}_p\mathbf{W}_{2h}^{-\infty}$ can be represented as the integral

$$\psi(x)=\int_0^1 \tau(v)\,w_{2h}(v,x)\,dv, \tag{49}$$

with some function $\tau(v)\in D^{-\infty}=\mathbf{C}^{\infty}$.

Theorem 7.1. *Let a spline $\psi\in{}_p\mathbf{W}_{2h}^{-\infty}$ be represented as in (49). Then it is a TB-wavelet if and only if $|\tau(v)|$ is strictly positive for all real v. Moreover if a spline $W\in{}_p\mathbf{W}_{2h}^{s}$ is represented in the following ways*

$$W(x)=\sum_k P_k\,\psi(x-2kh)=W(x)=\int_0^1 \eta(v)\cdot\,w_{2h}(v,x)\,dv.$$

Then $\eta(v)=\tau(v)\sum_k e^{-2\pi ikv}P_k$, $\quad P_k=\int_0^1 \frac{\eta(v)}{\tau(v)}\,e^{2\pi ivk}\,dv$.

Proof. Note first that τ belongs to $\mathbf{C}^{\infty}$. Due to (39) we may write

$$\psi(x-2kh)=\int_0^1 e^{-2\pi i2kv}\tau(v)\,w_{2h}(v,x)\,dv. \tag{50}$$

Then, to prove the theorem, we repeat the considerations of Theorem 4.1. □

We consider two TB-wavelets as *dual* to each other in the sense of Definition 4.3. The proof of the following theorems is quite similar to that of the corresponding theorems for TB-splines. The difference is that Eqs. (48) and (50) should be involved instead of Eqs. (19) and (23).

Theorem 7.2. *Let a TB-wavelet ψ be represented by*

$$\psi(x) = \int_0^1 \tau(v)\, w_{2h}(v,x)\, dv.$$

Then there exists a unique TB-wavelet $\tilde{\psi}$ dual to ψ.

$$\tilde{\psi}(x) = \int_0^1 \tilde{\tau}(v)\, w_{2h}(v,x)\, dv; \qquad \tilde{\tau}(v) = \left(2h\, \overline{\tau(v)}\, t(v)\right)^{-1}. \tag{51}$$

Theorem 7.3. *Let two TB-wavelets $\psi^i, \quad i = 1, 2,$ be represented by*

$$\psi^i(x) = \int_0^1 \tau^i(v)\, w_{2h}(v,x)\, dv,$$

and let a spline $W \in {}_p\mathbf{W}_{2h}$ be expanded with respect to the two bases

$$W(x) = \sum_k P_k^1\, \psi^1(x - 2kh) = W(x) = \sum_k P_k^2\, \psi^2(x - 2kh).$$

Then the coordinates are related by

$$P_k^2 = \sum_l a_{k-l}^{1,2}\, P_l^1, \text{ where } a_r^{1,2} = \int_0^1 \frac{\tau^1(v)}{\tau^2(v)}\, e^{2\pi i v r}\, dv.$$

In particular,

$$\psi^1(x - 2lh) = \sum_k a_{k-l}^{1,2}\, \psi^2(x - 2kh). \tag{52}$$

Any TB-wavelet is the derivative of a TB-spline. To be specific,

Proposition 7.4. *Let a TB-wavelet ${}_p\psi_{2h} \in {}_p\mathbf{W}_{2h}$ be represented by the integral ${}_p\psi_{2h}(x) = \int_0^1 \tau(v)\, {}_pw_{2h}(v,x)\, dv$, and the TB-spline ${}_{2p}\varphi_h \in {}_{2p}\mathbf{V}_h$ be defined by*

$${}_{2p}\varphi_h(x) = \left(\frac{-h}{2}\right)^p \int_0^1 \tau(2v)\, {}_{2p}u\Big(v + \frac{1}{2}\Big)\, {}_{2p}m_h(v,x)\, dv.$$

Then ${}_p\psi_{2h}(x) = \frac{d^p}{dx^p}\, {}_{2p}\varphi_h(x + (1+p)h)$.

PROOF. This fact stems from Eq. (40). Namely,

$$\begin{aligned}{}_p\psi_{2h}(x) =&\Big(\frac{-h}{2}\Big)^p \frac{d^p}{dx^p}\int_0^1 \tau(v)\left({}_{2p}u\Big(\frac{v+1}{2}\Big)\,{}_{2p}m_h\Big(\frac{v}{2}, x+(1+p)h\Big)\right.\\ &\left. + \,{}_{2p}u\Big(\frac{v}{2}\Big)\,{}_{2p}m_h\Big(\frac{v+1}{2}), x+(1+p)h\Big)\right) dv\\ =&2\Big(\frac{-h}{2}\Big)^p \frac{d^p}{dx^p}\int_0^1 \tau(2v)\,{}_{2p}u\Big(v+\frac{1}{2}\Big)\,{}_{2p}m_h(v, x+(1-p)h)\,dv. \qquad \square\end{aligned}$$

In particular, this proposition implies that a number of moments of any TB-wavelet vanish.

Corollary 7.5. *Let P_{p-1} be a polynomial of the degree $p-1$ and ${}_p\psi \in {}_p\mathbf{W}_{2h}$ be a TB-wavelet. Then the integral is $\int_{-\infty}^{\infty} P_{p-1}(x)\,{}_p\psi(x)\,dx = 0$.*

We consider now some examples of the TB-wavelets.

7.2 B-Wavelets

Let us put $\tau(v) \equiv (2h)^{-1}$ in (49) and define the following TB-wavelet ψ^b.

$$\psi^b(x) := \frac{1}{2h}\int_0^1 w_{2h}(v,x)\,dv. \tag{53}$$

Proposition 7.4 enables us to write $\psi^b(x) = \frac{1}{h}\frac{d^p}{dx^p}\,{}_{2p}\varphi_h(x+(1+p)h)$, where ${}_{2p}\varphi_h(x) = \Big(\frac{-h}{2}\Big)^p \int_0^1 u\Big(v+\frac{1}{2}\Big)\,{}_{2p}m_h(v,x)\,dv$ is a TB-spline. Recall that

$$u\Big(v+\frac{1}{2}\Big) = \sum_{s=-\infty}^{\infty} e^{-2\pi i(v+\frac{1}{2})s}\,{}_{2p}B_1\big(p+s\big) = \sum_{s=-\infty}^{\infty} (-1)^s\, e^{-2\pi i v s}\,{}_{2p}B_1\big(p+s\big).$$

Using this identity, we obtain

$${}_{2p}\varphi_h(x) = \Big(\frac{-h}{2}\Big)^p \sum_{s=-\infty}^{\infty} (-1)^s\,{}_{2p}B_1\big(p+s\big)\int_0^1 e^{-2\pi i v s}\,{}_{2p}m_h(v,x)\,dv.$$

Applying (11), the relation becomes

$${}_{2p}\varphi_h(x) = h\Big(\frac{-h}{2}\Big)^p \sum_{s=-\infty}^{\infty} (-1)^s\,{}_{2p}B_1\big(p+s\big)\,{}_{2p}B_h\big(x-sh\big). \tag{54}$$

The expression for the wavelet ψ^b can be derived from (54) by means of (2).

$$\psi^b(x) = \sum_{s=-\infty}^{\infty} \frac{(-1)^{s+p}}{2} \,{}_{2p}B_1\big(p+s\big) \sum_{l=0}^{p} (-1)^l \binom{p}{l} \,{}_{p}B_h\big(x-(l+s-p-1)h\big)$$
$$= \sum_{k=-2p}^{p-2} H_k \,{}_{p}B_h\big(x-kh-h\big), \tag{55}$$

where $H_k = (2)^{-p}(-1)^k \sum_{l=0}^{p} \binom{p}{l} \,{}_{2p}B_1\big(2p+1+k-l\big)$. The finite sum in the right hand side of (55) occurred because of $\mathsf{supp}\,{}_{2p}B_h = (0, 2ph)$. Therefore $\mathsf{supp}\,\psi^b = \Big((-2p+1)h, (2p-1)h\Big)$. Due to the symmetry of B-splines, the wavelet ψ^b is symmetric when p is even and antisymmetric when p is odd. It is readily seen that the wavelet ψ^b is nothing but the B -wavelet by Chui and Wang [6], (up to a shift). It is proven in [6] that ψ^b is a unique TB-wavelet of minimal support.

Remark. Note that (35) implies the following well known refinement equation for the B-splines. ${}_{p}B_{2h}(x) = \sum_{k=0}^{p} G_k \,{}_{p}B_h(x-kh)$ and $G_k = 2^{-p}\binom{p}{k}$.

Proposition 7.6. *Let $\psi^b \in {}_{p}\mathbf{W}_{2h}$ be the B-wavelet. Then for any spline $S \in {}_{p}\mathbf{V}_{2h}$ the integrals are*

$$\int_{-\infty}^{\infty} S(x)\psi^b(x-2lh)\,dx = 0 \quad \forall l. \tag{56}$$

PROOF. To verify (56) one should write the spline S in the B-spline basis and integrate the series obtained term by term. This is admissible because of the compact support of the B- wavelet. But, since ψ^b belongs to ${}_{p}\mathbf{W}_{2h}$,

$$\int_{-\infty}^{\infty} \psi^b(x-2lh)B_{2h}(x-2kh)\,dx = 0 \quad \forall k, l. \qquad \square$$

7.3 Interpolatory TB-Wavelets

Let us consider the following TB-wavelet $\psi^i \in {}_{p}\mathbf{W}_{2h}$:

$$\psi^i(x) = \int_0^1 \tau(v)\,{}_{p}w_{2h}(v,x)\,dv \quad \text{with } \tau(v) = \left(u\Big(\frac{v}{2}\Big)\,u\Big(\frac{v+1}{2}\Big)\right)^{-1}.$$

Proposition 7.4 implies that this wavelet is the derivative

$$\psi^i(x) = \Big(\frac{-h}{2}\Big)^p \frac{d^p}{dx^p}\,{}_{2p}\varphi^i_h(x+(p+1)h), \tag{57}$$

where the spline

$$
\begin{aligned}
{}_{2p}\varphi_h^i(x) &:= \int_0^1 \tau(2v)\, {}_{2p}u\Big(v+\frac{1}{2}\Big)\, {}_{2p}m_h(v,x)\, dv \\
&= \int_0^1 \frac{{}_{2p}m_h(v,x)}{{}_{2p}u(v)}\, dv = {}_{2p}L_h(x)
\end{aligned}
$$

is the fundamental TB-spline from the space ${}_{2p}\mathbf{V}_h$ (see (30)). TB-wavelets ψ^i were presented by Chui and Wang [4] under the name *interpolatory wavelets.*

Proposition 7.7. *Let a spline W be an element of the wavelet space ${}_p\mathbf{W}_{2h}$, and P_{p-1} be any polynomial of degree $p-1$. Then W is the p^{th} order derivative of a spline from the space ${}_{2p}\mathbf{V}_h$ interpolating the polynomial P_{p-1} in the points $\{(2k-p)h\}$. Conversely, let $\tilde{W} = {}_{2p}\tilde{S}^{(p)}$ be the p^{th} order derivative of a spline ${}_{2p}\tilde{S} \in {}_{2p}\mathbf{V}_h$ interpolating a $p-1$-degree polynomial in the points $\{(2k-p)h\}$. Then $\tilde{W}$ belongs to the wavelet space ${}_p\mathbf{W}_{2h}$.*

PROOF. Let us expand the spline W in terms of the interpolatory basic wavelets. $W(x) = \sum_k Q_k\, \psi^i(x-2kh)$ and define the spline ${}_{2p}S \in {}_{2p}\mathbf{V}_h$ by

$$
{}_{2p}S(x) := P_{p-1}(x) + \Big(\frac{-h}{2}\Big)^p \sum_k Q_k\, {}_{2p}L_h(x-(2k-1-p)h).
$$

It is readily seen that ${}_{2p}S((2k-p)h) = P_{p-1}((2k-p)h)$. At the same time (57) implies that $W = {}_{2p}S^{(p)}$. The reciprocal assertion is apparent. □

Note that the proposition includes Theorem 6.2 from [6] as a special case when $P_{p-1}(x) \equiv 0$.

7.4 Selfdual TB-Wavelets

Theorem 7.2 immediately leads to the construction of the TB-wavelet

$$
\psi^o(x) = \int_0^1 \tau(v)\, {}_p w_{2h}(v,x)\, dv \in {}_p\mathbf{W}_{2h}
$$

which is dual to itself. To do , we should choose $\tau(v) = \Big(2ht(v)\Big)^{-\frac{1}{2}}$. The shifts $\big\{\psi^o(\cdot - 2kh)\big\}_{-\infty}^{\infty}$ form an orthonormal (in the sense of L^2) basis of the space ${}_p\mathbf{W}_{2h}$. These TB-wavelets are known as the Battle- Lemarié wavelets [2], [10].

7.5 Galerkin Projections and TB-Wavelets Dual to B-Wavelets

Now we are able to give a direct definition of the GP of a function onto a wavelet space.

Definition 7.2. Let f be a function of slow growth. We call a spline $W(f,\cdot) \in {}_p\mathbf{W}_{2h}$ a Galerkin projection (GP) of the function f onto the wavelet space ${}_p\mathbf{W}_{2h}$ if for all integers k the integrals with B-wavelets ψ^b are

$$\int_{-\infty}^{\infty} W(f,x)\,\psi^b(x-2kh)\,dx = \int_{-\infty}^{\infty} f(x)\,\psi^b(x-2kh)\,dx := \Psi_k.$$

To reconcile this with the previous Definition 4.6 of the GP of a spline onto ${}_p\mathbf{W}_{2h}$, we prove the following assertion.

Proposition 7.8. *Let a spline $S(f,\cdot)$ be the GP of a function f onto the spline space ${}_p\mathbf{V}_h$ and $W(f,\cdot)$ be the GP of the spline $S(f,\cdot)$ onto the wavelet space ${}_p\mathbf{W}_{2h}$ in the sense of Definition 4.6. Then $W(f,\cdot)$ is the GP of the function f onto the space ${}_p\mathbf{W}_{2h}$.*

PROOF. Let ψ^b be the B-wavelet and B_h – the B-spline. We point out first that

$$\int_{-\infty}^{\infty} W(f,x)\,\psi^b(x-2kh)\,dx = \int_{-\infty}^{\infty} S(f,x)\,\psi^b(x-2kh)\,dx.$$

This stems from (44), (53). Then, applying (55), we may write the integral as

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\psi^b(x-2kh)\,dx &= \sum_{s=-2p}^{p-2} H_s \int_{-\infty}^{\infty} f(x)\,B_h\big(x-(s+2k)h\big)\,dx \\ &= \sum_{s=-2p}^{p-2} H_s \int_{-\infty}^{\infty} S(f,x)\,B_h\big(x-(s+2k)h\big)\,dx \\ &= \int_{-\infty}^{\infty} S(f,x)\,\psi^b(x-2kh)\,dx = \int_{-\infty}^{\infty} W(f,x)\,\psi^b(x-k2h)\,dx. \qquad \square\end{aligned}$$

In accordance with (51), the TB-wavelet $\psi^d \in {}_p\mathbf{W}_{2h}$ dual to the B-wavelet ψ^b is defined by $\psi^d(x) = \int_0^1 \frac{w_{2h}(v,x)}{t(v)}\,dv$. Once again let $W(f,\cdot)$ be the GP of f onto ${}_p\mathbf{W}_{2h}$. We expand it in terms of the dual basis $W(f,x) = \sum_k p_k\,\psi^d(x-k2h)$. Then the coordinates are

$$p_k = \int_{-\infty}^{\infty} f(x)\,\psi^b(x-k2h)\,dx = \int_{(-2p+1)h}^{(2p-1)h} f(x)\,\psi(x)^b\,dx = \Psi_k. \qquad (58)$$

The following assertion results from (52).

Proposition 7.9. *Let $\check{\psi} \in {}_p\mathbf{W}_{2h}$ be a TB-wavelet and a function f belongs to $\mathbf{F}^s$. Then a spline $W \in {}_p\mathbf{W}^s_{2h}$ is the GP of the function onto the wavelet space ${}_p\mathbf{W}_{2h}$ if and only if*

$$\int_{-\infty}^{\infty} W(x)\,\check{\psi}(x-k2h)\,dx = \int_{-\infty}^{\infty} f(x)\,\check{\psi}(x-k2h)\,dx.$$

Theorem 7.10. *Let f be a function of slow growth. Then there exists a unique GP $W(f,\cdot)$ of the function onto ${}_p\mathbf{W}_{2h}$. Moreover, if f belongs to $\mathbf{F}^s$ then the spline $W(f,\cdot)$ belongs to ${}_p\mathbf{V}^s_h \subset \mathbf{F}^s$.*

Proof. The existence and growth property of the GP follow from Proposition 7.8, while the uniqueness follows from Eq. (58). □

8 Wavelet Transformations

8.1 Decomposition and Reconstruction of Splines

By decomposition of a spline $S_h \in {}_p\mathbf{V}_h$ we mean its representation as the sum $S_h = S_{2h} + W_{2h}$ where S_{2h}, W_{2h} are GP of the spline S_h onto ${}_p\mathbf{V}_{2h}$ and ${}_p\mathbf{W}_{2h}$ correspondingly. Reconstruction is the synthesis of a spline $S_h \in {}_p\mathbf{V}_h$ from its GPs, W_{2h} and S_{2h}.

From a technical point of view, the procedures reduce to transformations of coordinates of splines involved from one TB-spline(wavelet) basis to another. We present formulas for arbitrary bases. These formulas stem from relations established for exponential splines and wavelets.

Theorem 8.1. *Let ${}^1\varphi_h$, ${}^2\varphi_{2h}$ be TB-splines from the spaces ${}_p\mathbf{V}_h$, ${}_p\mathbf{V}_{2h}$ respectively, and ${}^3\psi_{2h}$ be a TB-wavelet from the space ${}_p\mathbf{W}_{2h}$.*

$${}^1\varphi_h(x) = \int_0^1 {}^1\rho(v)\, m_h(v,x)\,dv,$$

$${}^2\varphi_{2h}(x) = \int_0^1 {}^2\rho(v)\, m_{2h}(v,x)\,dv, \quad {}^3\psi_{2h}(x) = \int_0^1 {}^2\tau(v)\, w_{2h}(v,x)\,dv.$$

Suppose that a spline $S_h \in {}_p\mathbf{V}^s_h$ and its GPs S_{2h}, W_{2h} onto the subspaces ${}_p\mathbf{V}_{2h}$, ${}_p\mathbf{W}_{2h}$ respectively, are represented in terms the TB-splines (wavelets)

$$S_h(x) = \sum_k {}^1q_k\, {}^1\varphi_h(x-kh)$$

$$S_{2h}(x) = \sum_k {}^2q_k\, {}^2\varphi_{2h}(x-k2h), \quad W_{2h}(x) = \sum_k {}^3p_k\, {}^3\psi_{2h}(x-k2h).$$

Then the coordinates are related by

$$^2q_k = \sum_l {}^1q_{2k-l} r_l, \quad r_l = \int_0^1 \frac{e^{2\pi i l v}}{{}^2\rho(2v)\, u(2v)} \overline{b(2v)}\, u(v)\, {}^1\rho(v)\, dv. \tag{59}$$

$$^3p_k = \sum_l {}^1q_{2k-l} s_l, \quad s_l = \int_0^1 \frac{e^{2\pi i l v}}{{}^3\tau(2v)\, t(2v)} \overline{a(2v)}\, u(v)\, {}^1\rho(v)\, dv, \tag{60}$$

$$^1q_k = 2\Big(\sum_l R_{k-2l}\, {}^2q_l + \sum_l S_{k-2l}\, {}^3p_l\Big), \tag{61}$$

$$R_l = \int_0^1 \frac{e^{2\pi i l v}\, {}^2\rho(2v)\, b(2v)}{{}^1\rho(v)}\, dv, \quad S_l = \int_0^1 \frac{e^{2\pi i l v}\, {}^3\tau(2v)\, a(2v)}{{}^1\rho(v)}\, dv.$$

PROOF. Let us write the splines in the integral form

$$S_h(x) = \int_0^1 {}^1\xi_h(v) \cdot m_h(v,x)\, dv,$$

$$S_{2h}(x) = \int_0^1 {}^2\xi_{2h}(v) \cdot m_{2h}(v,x)\, dv, \quad W_{2h}(x) = \int_0^1 {}^3\eta_{2h}(v) \cdot w_{2h}(v,x)\, dv,$$

where

$$^1\xi_h(v) = {}^1\rho(v) \mathcal{F}({}^1\vec{q}, v),$$
$$^2\xi_{2h}(v) = {}^2\rho(v) \mathcal{F}({}^2\vec{q}, v), \quad {}^3\eta_{2h}(v) = {}^3\tau(v) \mathcal{F}({}^3\vec{p}, v).$$

Then (45) implies

$$\mathcal{F}({}^2\vec{q}, v) = \frac{\overline{b(v)}\, u(v/2)\, {}^1\xi_h(v/2) + \overline{b(v+1)}\, u((v+1)/2)\, {}^1\xi_h((v+1)/2)}{{}^2\rho(v)\, u(v)},$$

from which we come to

$$\begin{aligned} {}^2q_k &= \int_0^1 \frac{e^{2\pi i k v}}{{}^2\rho(v)\, u(v)} \Bigg(\overline{b(v)}\, u\Big(\frac{v}{2}\Big) \cdot {}^1\xi_h\Big(\frac{v}{2}\Big) \\ &\qquad + \overline{b(v+1)}\, u\Big(\frac{v+1}{2}\Big) \cdot {}^1\xi_h\Big(\frac{v+1}{2}\Big)\Bigg)\, dv \\ &= \int_0^1 \frac{e^{2\pi i 2 k v}}{{}^2\rho(2v)\, u(2v)}\, \overline{b(2v)}\, u(v) \cdot {}^1\xi_h(v)\, dv. \end{aligned}$$

Hence (59) follows. Eq. (60) is derived similarly from (46).

Now we turn to Eq. (61). Just as in (47) we obtain

$$ {}^1\xi_h(v) = 2\Big({}^2\xi_{2h}(2v)\, b(2v) \;+\; {}^3\eta_{2h}(2v)\, a(2v)\Big). $$

Hence it follows that

$$ {}^1q_k = 2\int_0^1 \frac{e^{2\pi ikv}\, b(2v)}{{}^1\rho(v)} \cdot {}^2\xi_{2h}(2v)\, dv + 2\int_0^1 \frac{e^{2\pi ikv}\, a(2v)}{{}^1\rho(v)} \cdot {}^3\tau_{2h}(2v)\, dv $$

The latter relation implies (61). □

We stress that formulas established allow a spline given in any TB-spline basis to be decomposed into splines in any TB-spline(wavelet) bases.

8.2 Wavelet Transformations of Signals

Under the wavelet transformation of a signal $f \in \mathbf{F}$, we mean the computation of the integrals

$$ \begin{aligned} {}^\gamma\Phi(f,k,\nu) &= \int_{-\infty}^{\infty} f(x)\, {}^\gamma\varphi_{h2^\nu}(x - kh2^\nu)\, dx \\ {}^\gamma\Psi(f,k,\nu) &= \int_{-\infty}^{\infty} f(x)\, {}^\gamma\psi_{h2^\nu}(x - kh2^\nu)\, dx \end{aligned} $$

with various $\mathcal{TB}$-splines(wavelets) which are denoted by the parameter $\cdot^\gamma$, on various scales which are indicated by the parameter $\cdot^\nu$ and concerned with various translations associated with the parameter k.

Theorem 8.2. *Let ${}^1\varphi_h(x)$, ${}^1\varphi_{2h}(x)$, ${}^3\psi_{2h}(x)$ be the TB-splines(wavelets) defined in Theorem 8.1. Suppose that a signal f belongs to $\mathbf{F}$. Then*

$$ {}^2\Phi(f,k,\nu+1) = \sum_l {}^1\Phi(f,2k-l,\nu) R_{-l}, $$

$$ {}^3\Psi(f,k,\nu+1) = \sum_l {}^1\Phi(f,2k-l,\nu) S_{-l}, $$

$$ {}^1\Phi(f,k,\nu) = 2\Big(\sum_l r_{2l-k}\, {}^2\Phi(f,l,\nu+1) + \sum_l s_{2l-k}\, {}^3\Psi(f,l,\nu+1)\Big), $$

where the coefficients R_k, S_k, r_k, s_k are determined in Theorem 8.1.

Proof. The assertion is an immediate consequence of Theorem 8.1. To verify this, it suffices to recall that in the case when a spline $S \in {}_p\mathbf{V}^s_{h2^\nu}$ is the GP of a signal $f \in \mathbf{F}$, its coordinates with respect to the TB-basis $\{{}^\gamma\varphi^d_{h2^\nu}(\cdot - kh2^\nu)\}$ dual to a TB-spline basis $\{{}^\gamma\varphi_{h2^\nu}(\cdot - kh2^\nu)\}$ are

$$ {}^\gamma q_k = \int_{-\infty}^{\infty} f(x)\, {}^\gamma\varphi_{h2^\nu}(x - kh2^\nu)\, dx. $$

a similar remark is true relative to the TB-wavelet coordinates ${}^\gamma p_k$. To accomplish the proof one should apply (25) and (51). $\square$

It is readily observable that in this case decomposition and reconstruction sequences are interchanged with those in the previous subsection.

Concluding Remark

We note that the approach developed for one-dimensional polynomial splines can be applied to numerous classes of spline functions. We mention L-splines, box splines, discrete splines and Hermite splines.

The set of problems solvable by means of the techniques established is rather wide. These techniques are especially relevant for solving problems concerned with the operators of convolution and differentiation because of the intimate relationship of the exponential splines to these operators. In particular, we intend to apply these techniques to solving convolution integral equations by means of spline wavelet analysis.

Acknowledgment. The author is indebted to Professors Nira Dyn and Amos Ron for useful discussion of the work.

References

[1] A. Aldroubi, and M. Unser, *Families of multiresolution and wavelet spaces with optimal properties*, Numer. Funct. Anal. Optimiz. **14**(1993), 417–446.

[2] G. Battle, *A block spin construction of ondelettes. Part I. Lemarié functions*, Comm. Math. Phys. **110**(1987), 601–615.

[3] C. de Boor, R. DeVore, and A. Ron, Approximation from shift-invariant subspaces of $L_2(\mathbf{R}^s)$, *Trans. Amer. Math. Soc.*, **341** (1994), 787–806.

[4] C. K. Chui and J. Z. Wang, *On compactly supported spline wavelets and a duality principle*, Trans. Amer. Math. Soc.**330**(1992), 903–915.

[5] C. K. Chui and J. Z. Wang, *Computational and algorithmic aspects of cardinal spline-wavelets*, Appr. Th. Appl., **9** (1993), 53–75.

[6] C. K. Chui *An introduction to wavelets* Acad.Press, San Diego CA, 1992.

[7] N. Dyn, *Subdivision schemes in computer-aided geometric design*, in Advances in Numerical Analysis II, Wavelets, Subdivision Algorithms and Radial Functions, W.A. Light (ed.), Oxford University Press,(1992) 36–104.

[8] M. Kamada, K. Toraici and Riochi Mori, *Periodic splines orthonormal bases,* J. Approx. Th. **55**, (1988), 27–34.

[9] Y. W. Koh, S. L. Lee and H. H. Tan, *Periodic orthogonal splines and wavelets*, Appl. and Comp. Harm. Anal., 1995, 201-218.

[10] P. G. Lemarié, *Ondelettes à localization exponentielle,* J. de Math. Pure et Appl., **67** (1988) 227–236.

[11] I. J. Schoenberg , *Cardinal spline interpolation*, CBMS, **12**, SIAM, Philadelphia, 1973.

[12] I. J. Schoenberg, *Contribution to the problem of approximation of equidistant data by analytic functions*, Quart.Appl. Math., **4** (1946), 45-99, 112–141.

[13] Yu. N. Subbotin, *On the relation between finite differences and the corresponding derivatives,* Proc. Steklov Inst. Math., **78** (1965), 24–42.

[14] A. H. Zemanian, *Distribution theory and transform analysis*, McGraw-Hill, New York, 1965.

[15] V. A. Zheludev, *Spline Harmonic Analysis and Wavelet Bases,* Proc. Symp.Appl.Math., **48** (ed. W.Gautcshi), AMS 1994, 415–419.

[16] V. A. Zheludev, *Periodic splines and wavelets,* in Contemporary Mathematics, **190**, Mathematical Analysis, Wavelets & Signal Processing, (M.E.H.Ismail, M.Z.Nashed, A.I.Zayed, A.F.Ghaleb Eds.), Amer. Math. Soc., Providence, RI, pp. 339-354, 1995.

[17] V. A. Zheludev, *Integral representation of slowly growing equidistant splines and spline wavelets*, Tel Aviv University, The School of Mathematical Sciences, Technical Report 5-96, (1996).

[18] V. A. Zheludev *Integral representation of slowly growing equidistant splines*, Approximation Theory and Applications, to appear.