RESEARCH

Real Analysis Exchange
Vol. 36(1), 2010/2011, pp. 29–44

Saibal Ranjan Ghosh, 13/13, Sibtala Road, Bagmore, Kanchrapara, 24 Parganas (N), Pin-743145, West Bengal, India. email: cool.soibal@rediffmail.com

Hiranmay Dasgupta, Department of Mathematics, JIS College of Engineering, Phase-III, Block-A, Kalyani, Nadia, Pin-741235, West Bengal, India. email: hiranmaydg@yahoo.com

CLASSIFICATION OF POINTS OF LOWER SEMI-CONTINUITY OF A MULTIFUNCTION IN TOPOLOGICAL SPACES

Abstract

In this paper we introduce the notion of y-lower semi-continuity and point out a distinction between a point of lower semi-continuity in global sense and a point of lower semi-continuity in local sense in general topological spaces after classifying points of y-lower semi-continuity (resp. lower semi-continuity) and also study their interrelationships. In particular, we find a necessary and sufficient condition for a bijective open multifunction on a T_2 space to be lower semi-continuous. Finally, a sufficient condition for an open bijective multifunction on the real line to have at most countable points of lower semi-discontinuity is formulated.

1 Introduction.

In this paper X, Y always denote topological spaces, ϕ the empty set, $\mathbb{N}$ the set of natural numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{U}$ the usual topology. A multifunction $F : X \to Y$ is a point to set correspondence and we assume $F(x) \neq \phi$ for all $x \in X$. If $A \subset X, F(A) = \bigcup\{F(x) : x \in A\}$, and for $B \subset Y$,

Mathematical Reviews subject classification: Primary: 54C05, 54C60; Secondary: 54A99
Key words: lower semi-continuity, lower semi-discontinuity, *y-l.s.c.*, *y-l.s.d.*, s_y-point, w_y-point, honest point, s-point, w-point, c-point
Received by the editors June 7, 2008
Communicated by: Udayan B. Darji

$F^{-1}(B) = \{x : F(x) \cap B \neq \phi\}$. A multifunction $F : X \to Y$ is called strongly injective [1] (resp. surjective [2], [6]) if $x_1, x_2 \in X$ with $x_1 \neq x_2$ implies $F(x_1) \cap F(x_2) = \phi$ (resp. $F(X)=Y$); F is called bijective (or a bijection) if F is both strongly injective and surjective, and F is called open (resp. closed) if $F(U)$ is open (resp. closed) in Y for every open (resp. closed) U in X. Clearly, if F is surjective then $B \subset FF^{-1}(B)$ for each $B \subset Y$. The closure and the interior of a set A is denoted by $Cl\ A$ and $Int\,A$ respectively. The boundary [5] of a set A is defined by $Bd\,A = A - Int\,A$. A is said to be weakly separated [5] from B if and only if $A \cap Cl\,B = \phi$ for $A, B \subset X$. A multifunction $F : X \to Y$ is called lower semi-continuous [6] if $F^{-1}(V)$ is open in X for each open V in Y and F is called lower semi-continuous at $x \in X$ [4] if whenever $y \in F(x)$ and for every open V containing y, there exists an open U with $x \in U$ such that $F(z) \cap V \neq \phi$ for all $z \in U$. Clearly F is lower semi-continuous if and only if F is lower semi-continuous at each point in the domain space. From now on both of the phrases 'lower semi-continuous' and 'lower semi-continuity' (resp. 'lower semi-discontinuous' and 'lower semi-discontinuity') will often be abbreviated by *l.s.c.* (resp. *l.s.d.*). The logical constant 'exclusive or' will be denoted by 'XOR'.

This paper is based on two simple observations. Firstly, it is trivial that if $F : X \to Y$ is not *l.s.c.* then it fails to be so for at least one $x \in X$. It may happen that F fails to satisfy the requisites if not all but for at least one $y \in F(x)$ and becomes lower semi-discontinuous at $x \in X$ (i.e., not lower semi-continuous at $x \in X$). This observation leads to the definition of *l.s.c.* (resp. *l.s.d.*) for $F : X \to Y$ at $x \in X$ with respect to an $y \in F(x)$, or briefly, y-*l.s.c.* (resp. y-*l.s.d.*) at $x \in X$. Our second observation stems from a natural question (cf. [3]) that when a multifunction $F : X \to Y$ is *l.s.c.* for some $z \in X$ but not all, then whether a point of *l.s.c.* for such a multifunction shares the same characteristic with a point of *l.s.c.* for an *l.s.c.* multifunction. We have seen that there is a difference and used it to classify points of y-*l.s.c.* (resp. *l.s.c.*) by defining s_y-points and w_y-points (resp. by defining s-points, w-points and c-points) and studied their intrinsic properties : for instance, under certain condition the domain space is partitioned; the existence of a point of *l.s.d.* is ensured by the existence of a w_y-point, or in other words, for an *l.s.c.* multifunction all points of the domain space are characterized by s-points. Also the existence of the maximum open neighbourhood of y for an s_y-point is found and with the help of this largest open neighbourhood of y for a special type of s_y-point we have got a necessary and sufficient condition for a bijective open multifunction on a Hausdorff space to be *l.s.c.* Further properties of this largest open neighbourhood is also studied. Finally a sufficient condition for an open bijective multifunction on the real line to have at most countable points of lower semi-discontinuity is formulated.

2 Classification of points of lower semi-continuity.

In this section we first introduce the following notions.

Definition 2.1. A multifunction $F : X \to Y$ is called lower semi-continuous at $x \in X$ with respect to an $y \in F(x)$ (or simply, *y-l.s.c.* at $x \in X$) if for every open V containing y, there exists an open U with $x \in U$ such that $F(z) \cap V \neq \phi$ for all $z \in U$. x is then called an *y-l.s.c.* point.

Definition 2.2. A multifunction $F : X \to Y$ is called lower semi-discontinuous at $x \in X$ with respect to an $y \in F(x)$ (or simply, *y-l.s.d.* at $x \in X$) if F is not *y-l.s.c.* at $x \in X$. x is then called an *y-l.s.d.* point.

Naturally examples for Definition 2.1 and Definition 2.2 are due. But before that we give a necessary and sufficient condition for a multifunction F to be *y-l.s.d.* at $x \in X$ in the following theorem.

Theorem 2.3. *A multifunction $F : X \to Y$ is y-l.s.d. at $x \in X$ if and only if there exists an open neighbourhood V of y such that $x \in Bd\, F^{-1}(V)$.*

PROOF. Let F be *y-l.s.d.* at $x \in X$. Then there exists an open V containing y such that for every open U with $x \in U$ we have $F(z) \cap V = \phi$ for at least one $z \in U$. If possible, for every open neighbourhood of y (and hence for V), $x \in Int\, F^{-1}(V)$. We set $U = Int\, F^{-1}(V)$. Clearly $x \in U$. Now let $z \in U \subset F^{-1}(V)$. Then $F(z) \cap V \neq \phi$, a contradiction. So, $x \in Bd\, F^{-1}(V)$. Conversely, let there be an open neighbourhood V of $y \in F(x)$ such that $x \in Bd\, F^{-1}(V)$. If possible, let F be *y-l.s.c.* at x. Then for V, there exists an open U with $x \in U$ such that $F(z) \cap V \neq \phi$ for all $z \in U$, i.e., $U \subset F^{-1}(V)$. So $x \in U = Int\, U \subset Int\, F^{-1}(V)$ which contradicts with $x \in Bd\, F^{-1}(V)$. Hence F is *y-l.s.d.* at x. ☐

Example 2.4. Let $F : [0, 1] \to [0, 1]$ be defined by $F(1) = \{1\}$, $F(\frac{1}{2}) = \{\frac{3}{4}, 1\}$, $F(x) = \{x, 1\}$ for $x \neq \frac{1}{2}, 1$. For $1 \in F(\frac{1}{2})$ we see $F^{-1}(V) = [0, 1]$ is open where V is any open set containing 1 as $F^{-1}(\{1\}) = [0, 1]$. So F is 1-*l.s.c.* at each $x \in [0, 1]$. But for $\frac{3}{4} \in F(\frac{1}{2})$, taking $V = (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$, $0 < \epsilon < \frac{1}{4}$ we see $\frac{1}{2} \in Bd\, F^{-1}(V)$ because $F^{-1}(V) = (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon) \cup \{\frac{1}{2}\}$ and $Int\, F^{-1}(V) = (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$. So F is $\frac{3}{4}$-*l.s.d.* at $x = \frac{1}{2}$.

Corollary 2.5. *Let $F : X \to Y$ be a multifunction and O be any open set in Y. If $Bd\, F^{-1}(O) \neq \phi$ then each $x \in Bd\, F^{-1}(O)$ is a point of l.s.d. with respect to each of the image points of x in O.*

Now we classify points of y-*l.s.c.* (respectively *l.s.c.*) for a multifunction $F : X \to Y$ in the following.

Definition 2.6. A point $x \in X$ is called an s_y-point of $F : X \to Y$ with respect to an $y \in F(x)$ (or simply, an s_y-point) if there exists an open neighbourhood $N(y)$ of y such that for any open sub-neighbourhood $O(y)$ of y [i.e., $O(y)$ is a neighbourhood of y such that $O(y) \subset N(y)$], $F^{-1}(O(y))$ is open. $x \in X$ is called an s-point of $F : X \to Y$ (or simply, an s-point) if for each $y \in F(x)$, x is an s_y-point.

Example 2.7. In Example 2.4, it is easy to verify that $x = \frac{3}{4}$ is an s_1-point and x $(\neq \frac{1}{2}, \frac{3}{4})$ is an s-point. But $x = \frac{3}{4}$ is not an s-point because $x = \frac{3}{4}$ is not an $s_{\frac{3}{4}}$-point.

Definition 2.8. A point $x \in X$ is called a w_y-point of $F : X \to Y$ with respect to an $y \in F(x)$ (or simply, a w_y-point) if for any open neighbourhood of y there exists an open sub-neighbourhood $O(y)$ of y such that $x \in Int\, F^{-1}(O(y))$ and $Bd\, F^{-1}(O(y)) \neq \phi$. $x \in X$ is called a w-point of $F : X \to Y$ (or simply, a w-point) if for each $y \in F(x)$, x is a w_y-point.

Example 2.9. In Example 2.4, $x = \frac{3}{4}$ is a $w_{\frac{3}{4}}$-point but clearly $x = \frac{3}{4}$ is not a w-point. Next we consider the multifunction $F : [0,1] \to [0,1]$ defined by $F(\frac{1}{2}) = \{\frac{3}{4}, 1\}, F(\frac{3}{4}) = \{\frac{3}{4}\}$, $F(1) = \{1\}$ and $F(x) = \{x, 1\}$ otherwise. It is easy to check that $x = \frac{3}{4}$ is a w-point.

Notation. We use the following notation throughout the paper.

(i) X_l^y (X_l), X_s^y (X_s), X_w^y (X_w), X_{ld}^y (X_{ld}) denote respectively the set of all points of y-*l.s.c.* (*l.s.c.*), the set of all s_y-points (s-points), the set of all w_y-points (w-points), the set of all points of y-*l.s.d.* (*l.s.d.*) of a multifunction $F : X \to Y$.

(ii) $F_l(x) = \{y : y \in F(x)$ and x is a point of y-*l.s.c.* $\}$,

$F_{ld}(x) = \{y : y \in F(x)$ and x is a point of y-*l.s.d.*$\}$,

$F_s(x) = \{y : y \in F(x)$ and x is an s_y-point $\}$,

$F_w(x) = \{y : y \in F(x)$ and x is a w_y-point $\}$.

Also by the 'image points of a point $x \in X$', '$\downarrow$-image points of a point $x \in X$' and '$\uparrow$-image points of a point $x \in X$' we mean the points of $F(x)$, $F_{ld}(x)$ and $F_l(x)$ respectively.

Definition 2.10. A point $x \in X$ is called a c-point of $F : X \to Y$ (or simply, a c-point) if x is an s_y-point for at least one $y \in F(x)$ and also x is a w_y-point for at least one $y \in F(x)$ and $F(x) = F_s(x) \cup F_w(x)$, or in other words, $F_s(x) \neq \phi$, $F_w(x) \neq \phi$ and $F(x) = F_s(x) \cup F_w(x)$. The set of all c-points of $F : X \to Y$ is denoted by X_c.

Example 2.11. In Example 2.4, $F(\frac{3}{4}) = \{\frac{3}{4}, 1\}$ and $x = \frac{3}{4}$ is a $w_{\frac{3}{4}}$-point and also an s_1-point. Hence $x = \frac{3}{4}$ is a c-point.

Theorem 2.12. *Let $F : X \to Y$ be a multifunction. If x is either an s_y-point or a w_y-point then F is y-l.s.c. at x.*

Proof. We prove only the first case. Let $x \in X$ be an s_y-point. Then there exists an open neighbourhood $N(y)$ of y such that $F^{-1}(O(y))$ is open for all open sub-neighbourhoods $O(y)$ of y. Let V be an open set containing y. Now $V_1 = N(y) \cap V$ is an open neighbourhood of y and $V_1 \subset N(y)$, so $F^{-1}(V_1) = U$ is open. Clearly $x \in U$, and if $z \in U$ then $\phi \neq F(z) \cap V_1 \subset F(z) \cap V$. So F is y-*l.s.c.* at x. □

Corollary 2.13. *If $x \in X$ is an s-point or a w-point or a c-point then F is l.s.c. at x.*

Theorem 2.14. *If x is an s_y-point of $F : X \to Y$ then there is no y-l.s.d. points of F in X, or in other words, $X_s^y \neq \phi$ implies $X_{ld}^y = \phi$.*

Proof. Let $x \in X$ be an s_y-point. Then there is an open neighbourhood $N(y)$ of y such that for each open sub-neighbourhood $M(y)$ of y, $F^{-1}(M(y))$ is open. If possible, let x_1 be a point of y-*l.s.d.* Then by Theorem 2.3, there exists an open neighbourhood $O(y)$ of y such that $x_1 \in Bd\, F^{-1}(O(y))$. Let $O'(y) = N(y) \cap O(y)$. Then $x_1 \in Bd\, F^{-1}(O'(y))$, a contradiction because $O'(y) \subset N(y)$ implies $F^{-1}(O'(y))$ is open. □

The converse of Theorem 2.14 is not true in general as shown by the following example.

Example 2.15. Let $F : (X, \tau) \to (Y, \tau^*)$ be defined by $F(x) = \{1, 2\}, F(y) = \{2, 3\}, F(z) = \{3\}$ where $X = \{x, y, z\}, Y = \{1, 2, 3\}, \tau = \big\{\phi, X, \{x\}\big\}$ and $\tau^* = \big\{\phi, Y, \{1, 2\}\big\}$. Clearly both of X_{ld}^1 and X_s^1 are void.

Theorem 2.16. *If x is an s_y-point of $F : X \to Y$ then there is no w_y-points of F in X, or in other words, $X_s^y \neq \phi$ implies $X_w^y = \phi$.*

Proof. Let $x \in X$ be an s_y-point. Then there is an open neighbourhood $N(y)$ of y such that for each open sub-neighbourhood $M(y)$ of y, $F^{-1}(M(y))$ is open. If possible, let x_1 be a w_y-point. Then for each open neighbourhood of y and hence for $N(y)$ there exists an open sub-neighbourhood $O(y)$ of y such that $x_1 \in Int\, F^{-1}(O(y))$ and $Bd\, F^{-1}(O(y)) \neq \phi$. Hence $F^{-1}(O(y))$ is not open which is a contradiction. So x_1 is not a w_y-point. □

The converse of Theorem 2.16 is not true in general as shown by the following example.

Example 2.17. Let $F : [0,1] \to [0,1]$ be a multifunction defined by $F(x) = \{\frac{x}{2}\}$, $0 \leqslant x < \frac{1}{2}$, $F(\frac{1}{2}) = [\frac{1}{4}, \frac{3}{4}]$, $F(x) = \{\frac{1}{2}(x+1)\}$, $\frac{1}{2} < x \leqslant 1$. We consider the image point $\frac{1}{3}$. The only $x \in [0,1]$ which is mapped into $\frac{1}{3}$ is $\frac{1}{2}$. For the open neighbourhood $V = (\frac{1}{3} - \epsilon, \frac{1}{3} + \epsilon)$ of $\frac{1}{3}$, $0 < \epsilon < \frac{1}{12}$ we have $F^{-1}(V) = \{\frac{1}{2}\}$. So $Int\, F^{-1}(V) = \phi$ and $Bd\, F^{-1}(V) = \{\frac{1}{2}\}$. Hence $x = \frac{1}{2}$ is a point of $\frac{1}{3}$-*l.s.d.*, i.e., $x = \frac{1}{2}$ is not a point of $\frac{1}{3}$-*l.s.c.* Then by Theorem 2.12, $x = \frac{1}{2}$ is neither an $s_{\frac{1}{3}}$-point nor a $w_{\frac{1}{3}}$-point.

It is not necessarily true that $X_s \neq \phi$ implies $X_w = \phi$ as follows from Example 2.9 where $x = \frac{3}{4}$ is a w-point and $x = \frac{1}{3}$ is an s-point. However, we have

Corollary 2.18. *For $F : X \to Y$, (a) $X_s \cap X_w = \phi$, (b) $X_s \cap X_c = \phi$, (c) $X_w \cap X_c = \phi$.*

Theorem 2.19. *For $F : X \to Y$, $X_l^y = X_s^y$ XOR $X_l^y = X_w^y$, or in other words, all points of y-l.s.c. are s_y-points* XOR *w_y-points.*

Proof. By Theorem 2.12 and Theorem 2.16 we get $X_s^y \subset X_l^y$ XOR $X_w^y \subset X_l^y$. Now, let $x \in X_l^y$ and $x \notin X_w^y$. Then there exists an open neighbourhood $N(y)$ of y such that for every open sub-neighbourhood $O(y)$ of y we have either (i) $x \notin Int\, F^{-1}(O(y))$ or (ii) $x \in Int\, F^{-1}(O(y))$ and $Bd\, F^{-1}(O(y)) = \phi$. If at least one $O(y)$ satisfies (i) then x is a point of y-*l.s.d.*, a contradiction. So all $O(y)$ satisfy (ii) and hence x is an s_y-point, i.e., $x \in X_s^y$. Now, $X_w^y = \phi$ by Theorem 2.16. Consequently, $X_l^y \subset X_s^y$ XOR $X_l^y \subset X_w^y$. Hence $X_l^y = X_s^y$ XOR $X_l^y = X_w^y$. □

Corollary 2.20. *For a multifunction $F : X \to Y$, (i) $X_l = X_s \cup X_w \cup X_c$ and (ii) $X = X_s \cup X_w \cup X_c \cup X_{ld}$.*

In the following theorem the existence of a w_y-point is ensured by the existence of a point of *l.s.d.* in a certain manner.

Theorem 2.21. *Let $x \in X_w^y$ for a multifunction $F : X \to Y$. Then either there is an y-l.s.d. point or y is a limit point of the ↓-image points of l.s.d. points.*

PROOF. Since x is a w_y-point, for every open neighbourhood U of y there exists an open neighbourhood $O(y)$ of y with $O(y) \subset U$ such that $x \in Int\, F^{-1}(O(y))$ and $Bd\, F^{-1}(O(y)) \neq \phi$. Let $x_1 \in Bd\, F^{-1}(O(y))$. Now either $y \in F(x_1)$ or $y \notin F(x_1)$. If $y \in F(x_1)$ then by Theorem 2.3, x_1 is an y-*l.s.d.* point. If $y \notin F(x_1)$ then by Corollary 2.5, there is an *l.s.d.* point in $F^{-1}(O(y))$ and hence in $F^{-1}(U)$, i.e., y is a limit point of the ↓-image points of *l.s.d.* points. □

Remark 2.22. From Theorem 2.21 it follows that for $F : X \to Y$, $X_w \neq \phi$ or $X_c \neq \phi$ implies $X_{ld} \neq \phi$, and consequently, for F to be *l.s.c.* (i.e., *l.s.c.* at each point of the domain space) each point of the domain space must be an s-point.

A natural question now arises whether the existence of a point of *l.s.d.* ensures the existence of a w_y-point. The answer is in negation which is evident from the following example.

Example 2.23. We consider the multifunction $F : [0, 1] \to [0, 1]$ of Example 2.17. Clearly except $x = \frac{1}{2}$ each $x \in [0, 1]$ is an s-point and $x = \frac{1}{2}$ is a point of *l.s.d.* for each $y \in F(\frac{1}{2})$. So there is no w_y-point for F.

3 Maximum open neighbourhood, its applications and the cardinality of $\mathbb{R}_{ld}$.

In this section we prove the existence of a unique maximum open neighbourhood of an $y \in F(x)$ for an s_y-point x and its consequences in different aspects.

Theorem 3.1. *If $F : X \to Y$ is a multifunction and $x \in X_s^y$ then there exists a unique largest open neighbourhood $M_s(y)$ of y such that for any open sub-neighbourhood $O(y)$ of y, $F^{-1}(O(y))$ is open.*

PROOF. Since $x \in X_s^y$ there exists an open neighbourhood $N(y)$ of y such that $F^{-1}(O(y))$ is open whenever $O(y)$ is any open sub-neighbourhood of y. Let $\{N^\alpha(y)\}$ be the family of all such open neighbourhoods $N^\alpha(y)$ where α belongs to an index set I. Clearly, $\{N^\alpha(y)\}$ is partially ordered by the set inclusion relation. Then $M_s(y) = \bigcup_{\alpha \in I} N^\alpha(y)$ is clearly the largest open neighbourhood of y sought in the theorem. Let $M(y)$ be any open neighbourhood of y such that $M(y) \subset M_s(y)$. Now

$$\begin{aligned} F^{-1}(M(y)) &= F^{-1}(M(y) \cap M_s(y)) \\ &= F^{-1}(\bigcup_{\alpha \in I}(M(y) \cap N^{\alpha}(y))) \\ &= \bigcup_{\alpha \in I} F^{-1}(M(y) \cap N^{\alpha}(y)). \end{aligned}$$

As $M(y) \cap N^{\alpha}(y)$ is an open neighbourhood of y and contained in $N^{\alpha}(y)$, $F^{-1}(M(y) \cap N^{\alpha}(y))$ is open for each α according to the property of $N^{\alpha}(y)$. Consequently, $F^{-1}(M(y))$ is open. □

We observe that the largest open neighbourhood $M_s(y)$ of y for an s_y-point may or may not intersect $F_{ld}(x_1)$ and $F_w(x_2)$ for some $x_1, x_2 \in X$. For this we furnish the following examples.

Example 3.2. Let $F : [0,1] \to [0,1]$ be defined by $F(\frac{1}{2}) = [\frac{1}{2}, \frac{3}{4}]$ and $F(x) = \{x\}$ otherwise. It is easy to verify that $\frac{1}{2}$ is an $s_{\frac{1}{2}}$-point and $M_s(\frac{1}{2}) = [0,1]$. Also, for $0 < \epsilon < \frac{1}{4}$, $F^{-1}((\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)) = (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon) \cup \{\frac{1}{2}\}$. Therefore, $\frac{1}{2}$ is a point of $\frac{3}{4}$-*l.s.d.* and $\frac{3}{4}$ is a $w_{\frac{3}{4}}$-point. Hence $M_s(\frac{1}{2})$ intersects $F_{ld}(\frac{1}{2})$ and also $F_w(\frac{3}{4})$.

Example 3.3. We consider the multifunction $F : [0,1] \to [0,1]$ of Example 2.17. Clearly 1 is an s_1-point and $M_s(1) = [0,1] - [\frac{1}{4}, \frac{3}{4}]$. Also for each $y \in [\frac{1}{4}, \frac{3}{4}]$, $\frac{1}{2}$ is an y-*l.s.d.* point and this is the only *l.s.d.* point. It is evident that there is no w_y-point in [0,1]. Hence $M_s(1)$ does not intersect $F_{ld}(x_1)$ and also $F_w(x_2)$ for any $x_1, x_2 \in [0,1]$.

This observation leads to the following simple but useful theorem which builds up almost all of our results in the sequel.

Theorem 3.4. *Let x be an s_y-point of $F : X \to Y$. If $M_s(y)$ intersects $F_{ld}(x')$ for some $x' \in X$ then $\{x'\}$ can not be weakly separated from $F^{-1}(O(y))$ where $O(y)$ is any open neighbourhood of y contained in $M_s(y)$.*

PROOF. To prove the theorem we need the following lemma which we state without proof.

Lemma 3.5. *If $A, B \subset X$ such that $A \cup B$ is open but B is not open then $Bd\, B \subset Cl\, A$.*

Continuing the proof, let $x' \in X_{ld}$ such that $F_{ld}(x') \cap M_s(y) \neq \phi$. Let $y' \in F_{ld}(x') \cap M_s(y)$. By Theorem 2.3, there exists an open neighbourhood $M(y')$ of y' such that $x' \in Bd\,F^{-1}(M(y'))$ and also $x' \in Bd\,F^{-1}(M'(y'))$ whenever $M'(y')$ is an open neighbourhood of y' for which $M'(y') \subset M(y')$. We put $N(y') = M_s(y) \cap M(y')$. Clearly $N(y')$ is an open neighbourhood of y' and can not contain y. For, if it contains y then $N(y')$ would be an open neighbourhood of y and $N(y') \subset M_s(y)$. Hence $F^{-1}(N(y'))$ should be an open set, i.e., $Bd\,F^{-1}(N(y')) = \phi$, a contradiction because $x' \in Bd\,F^{-1}(N(y'))$ as $N(y') \subset M(y')$. Now we consider any open neighbourhood $O(y)$ of y such that $O(y) \subset M_s(y)$. Then x' must belong to $Cl\,F^{-1}(O(y))$ because of the following reason.

Consider the set $O(y) \cup N(y') = O'(y)$. Hence $F^{-1}(O'(y))$ is open because $O'(y)$ is an open neighbourhood of y and $O'(y) \subset M_s(y)$. Again $F^{-1}(O'(y)) = F^{-1}(O(y)) \cup F^{-1}(N(y'))$. But $x' \in Bd\,F^{-1}(N(y'))$. Hence by Lemma 3.5, $x' \in Cl\,F^{-1}(O(y))$. □

Theorem 3.6. *Let $F : X \to Y$ be a strongly injective open multifunction from a Hausdorff space X into Y. If x is an s_y-point and $M_s(y)$, the largest open neighbourhood of y intersects $F_{ld}(x')$ for some $x' \in X$ then x' must be equal to x.*

PROOF. Let $N(x)$ be any open neighbourhood of x in X. Since F is open, $F(N(x))$ will be open in Y. Consider the set $F(N(x)) \cap M_s(y) = W(y)$. Then $W(y)$ is an open neighbourhood of y for which $W(y) \subset M_s(y)$. Obviously $F^{-1}(W(y))$ is open and $F^{-1}(W(y)) \subset F^{-1}F(N(x)) = N(x)$ as F is strongly injective. Again $x' \in Cl\,F^{-1}(W(y))$ as proved in Theorem 3.4. Consequently, $x' \in Cl\,N(x)$. Since X is a Hausdorff space, x and x' must be identical as was to be proved. □

We use the following terminology for an s-point or a c-point in the rest of the paper.

Definition 3.7. A multifunction $F : X \to Y$ is called honest at $x \in X$ if x is an s-point or a c-point. x is then called honest with respect to F (or simply, honest). Clearly, an honest point x is an s_y-point for at least one $y \in F(x)$. We will often say that x is an honest s_y-point to mean it.

Corollary 3.8. *Let x be an honest s_y-point with respect to a multifunction $F : X \to Y$ and $x' \in X_{ld}$. If X is Hausdorff and F is open and strongly injective then $M_s(y) \cap F_{ld}(x') = \phi$.*

In the following theorem we exhibit the influence of an honest s_y-point on the corresponding largest open neighbourhood for a multifunction defined in a Hausdorff space with some restrictions.

Theorem 3.9. *Let x be an honest s_y-point with respect to $F : X \to Y$. If X is Hausdorff and F is open and strongly injective then $M_s(y)$ can not contain any $y_1 \in F_w(x_1)$ for any w_{y_1}-point x_1.*

PROOF. If possible, let $M_s(y)$ contain an $y_1 \in F_w(x_1)$ for a w_{y_1}-point x_1. Then by Theorem 2.21 either y_1 is a $\downarrow$-image of an *l.s.d.* point or there is a $\downarrow$-image of an *l.s.d.* point in $M_s(y)$, which contradicts with Corollary 3.8. □

It might have been known that an open bijective multifunction on a Hausdorff space need not be lower semi-continuous but we are unable to find any reference. So we give the following example.

Example 3.10. We consider $F : [0, 1] \to [0, 1]$ of Example 2.17. It is easy to verify that F is an open bijection on a Hausdorff space. But $F^{-1}((\frac{3}{8}, \frac{5}{8})) = \{\frac{1}{2}\}$ is not open. So F is not *l.s.c.*

Now we study the role of an honest s_y-point and give a necessary and sufficient condition for an open bijection on a Hausdorff space to be lower semi-continuous.

Theorem 3.11. *Let $F : X \to Y$ be an open bijection and X be Hausdorff. Then F is l.s.c. if and only if there exists an honest s_y-point such that $M_s(y) = Y$.*

PROOF. *Sufficiency.* Corollary 3.8 ensures that no point of X is a point of X_{ld}. Also by Theorem 3.9, no $x \in X$ is in X_w or in X_c. Hence x must be in X_s as $X = X_s \cup X_w \cup X_c \cup X_{ld}$ and hence by Remark 2.22, F is *l.s.c.*

Necessity. Suppose F is *l.s.c.* Let $x \in X$ and $y \in F(x)$. Clearly, Y is an open neighbourhood of y. Since F is *l.s.c.*, x is an s-point by Remark 2.22, and hence, x is an honest s_y-point. Let $O(y)$ be any open neighbourhood of y. If possible, let $F^{-1}(O(y))$ be not open. Then $Bd\, F^{-1}(O(y)) \neq \phi$ and so by Corollary 2.5, there is an *l.s.d.* point in X, a contradiction. Hence $F^{-1}(O(y))$ is open and consequently, $M_s(y) = Y$. □

We now state a theorem (without proof) which follows directly from Corollary 2.5.

Theorem 3.12. *For a multifunction $F : X \to Y$, if V is an open set in Y such that it does not contain any $\downarrow$-image points of l.s.d. points then $F^{-1}(O)$ is open when O is any open subset of V.*

Next we prove some results on the intrinsic nature of the largest open neighbourhood $M_s(y)$ of y for an s_y-point and the point y itself.

Theorem 3.13. *Let $F : X \to Y$ be strongly injective, open and X a Hausdorff space. If $x \in X_s^y$ then y can not be a limit point of the $\downarrow$-image points of those l.s.d. points which are different from x.*

Proof. Let $x \in X_s^y$ and if possible, let y be a limit point of the $\downarrow$-image points of those *l.s.d.* points which are different from x. Let $M_s(y)$ be the largest open neighbourhood of y. Then there is an $y_1(\neq y)$ in $M_s(y)$ such that an y_1-*l.s.d.* point, say x_1 exists which is different from x, but this is a contradiction because $x_1 = x$ by Theorem 3.6. □

Theorem 3.14. *Let $F : X \to Y$ be strongly injective, $x \in X_s^y$ and $M_s(y)$ be the largest open neighbourhood of $y \in F(x)$. Then*

(a) $M_s(y)$ will contain $y_1 \in F_{ld}(x)$ (if exists) provided y_1 is not a limit point of the $\downarrow$-image points of those l.s.d. points which are different from x,

(b) $M_s(y)$ will contain $y_1 \in F_s(x_1), x_1 \neq x$ (if exists) provided F is open, X is Hausdorff and y_1 is not a limit point of the $\downarrow$-image points of x_1,

(c) $M_s(y)$ will not contain any $y_1 \in F_w(x_1)$ for $x_1 \in X$ (if exists) provided F is open, closed and X is Hausdorff.

Proof. (a). Let $x \in X_s^y$ and let $M_s(y)$ be the corresponding largest open neighbourhood of y. Then $x \in F^{-1}(M_s(y))$. Let $y_1 \in F_{ld}(x)$ and y_1 is not a limit point of the $\downarrow$-image points of those *l.s.d.* points which are different from x. If possible, let $y_1 \notin M_s(y)$. It is clear that an open neighbourhood $O(y_1)$ of y_1 exists such that $O(y_1)$ does not contain any $\downarrow$-image point of those *l.s.d.* points which are different from x. Now consider the set $N(y) = M_s(y) \cup O(y_1)$. Obviously, $N(y) \supset M_s(y)$. Now, let $N'(y)$ be any open sub-neighbourhood of y such that $N'(y) \subset N(y)$. Then

$$\begin{aligned} F^{-1}(N'(y)) &= F^{-1}(N'(y) \cap N(y)) \\ &= F^{-1}(N'(y) \cap M_s(y)) \cup F^{-1}(N'(y) \cap O(y_1)). \end{aligned}$$

Now $N'(y) \cap M_s(y)$ is an open neighbourhood of y and $N'(y) \cap M_s(y) \subset M_s(y)$. Hence $F^{-1}(N'(y) \cap M_s(y))$ is open. Again, $N'(y) \cap O(y_1) \subset O(y_1)$. Evidently $Bd\, F^{-1}(N'(y) \cap O(y_1)) \subset \{x\}$, because if there is any other point $x_1 \neq x$ and $x_1 \in Bd\, F^{-1}(N'(y) \cap O(y_1))$, then that will imply the existence of an $y_1' \in F(x_1)$ in $N'(y) \cap O(y_1) \subset O(y_1)$ such that $y_1' \neq y_1$ because F is strongly injective and so x_1 is an y_1'-*l.s.d.* point and hence $y_1' \in F_{ld}(x_1) \cap O(y_1)$, a contradiction. But $x \in Int\, F^{-1}(N'(y) \cap M_s(y))$. Hence $Bd\, F^{-1}(N'(y)) = \phi$. So $N(y) \subset M_s(y)$ which is impossible proving (a). □

(b). Let $x \in X_s^y$ and $M_s(y)$ be the corresponding largest open neighbourhood of y. Let $x_1\ (\neq x) \in X_s^{y_1}$. Since F is strongly injective $y_1 \neq y$. Suppose that y_1 is not a limit point of the $\downarrow$-image points of x_1. Hence there exists an open neighbourhood $O(y_1)$ of y_1 such that $O(y_1)$ does not contain $\downarrow$-image points of *l.s.d.* points by Theorem 3.13 and $F^{-1}(O(y_1))$ is open by Theorem 3.12. If possible, let $y_1 \notin M_s(y)$. Now, we consider the set $M(y) = M_s(y) \cup O(y_1)$. Let $M'(y)$ be any open sub-neighbourhood of y, i.e., $M'(y) \subset M(y)$. Then

$$\begin{aligned} F^{-1}(M'(y)) &= F^{-1}(M'(y) \cap M(y)) \\ &= F^{-1}(M'(y) \cap M_s(y)) \cup F^{-1}(M'(y) \cap O(y_1)). \end{aligned}$$

Since $M'(y) \cap M_s(y) \subset M_s(y)$ and is an open neighbourhood of y, hence $F^{-1}(M'(y) \cap M_s(y))$ is open. Again, since $M'(y) \cap O(y_1) \subset O(y_1)$ and is an open set, $F^{-1}(M'(y) \cap O(y_1))$ must also be open by Theorem 3.12. So $M(y) \subset M_s(y)$ which is impossible. Hence $y_1 \in M_s(y)$. □

(c). Let $x \in X_s^y$ and $M_s(y)$ be the largest open neighbourhood of y. If possible, let $M_s(y)$ contain an $y_1 \in F_w(x_1)$ for an $x_1 \in X$. Let $N(y_1)$ be any open neighbourhood of y_1 for which $N(y_1) \subset M_s(y)$. Then there exists an open neighbourhood $N'(y_1)$ of y_1 for which $N'(y_1) \subset N(y_1)$, $x_1 \in Int\, F^{-1}(N'(y_1))$ and $Bd\, F^{-1}(N'(y_1)) \neq \phi$. Let $x_2 \in Bd\, F^{-1}(N'(y_1))$ and $y_2 \in N'(y_1) \cap F(x_2)$. Then by Corollary 2.5, x_2 is an y_2-*l.s.d.* point. Clearly, $y_2 \in M_s(y)$ and then by Theorem 3.6, $x_2 = x$. Since $x_1 \neq x_2$ and F is strongly injective, we have $y_1 \neq y_2$. Hence y_1 is a limit point of $F(x)$. Since X is Hausdorff and F is closed it follows that $F(x)$ is closed and so, $y_1 \in F(x)$ which implies $F(x) \cap F(x_1) \neq \phi$, a contradiction as F is strongly injective. □

It is easy to notice (as shown in the following example) that for a point which is both an s_y-point and also an s_{y_1}-point ($y \neq y_1$), the largest open neighbourhood of y and that of y_1 may not be equal.

Example 3.15. Let $F : [0,1] \to [0,1]$ be defined by $F(\frac{1}{2}) = \{0, \frac{1}{4}, \frac{1}{2}\}$, $F(1) = \{0, \frac{1}{4}, 1\}$ and $F(x) = \{x, 0\}$ otherwise. It is easy to verify that $\frac{1}{2}$ is an s_0-point as well as an $s_{\frac{1}{2}}$-point and $M_s(0) = [0,1]$ whereas $M_s(\frac{1}{2}) = [0,1] - \{\frac{1}{4}\}$.

However we have the following theorem.

Theorem 3.16. *Let $F : X \to Y$ be strongly injective, open and X a Hausdorff space. If $x \in X_s^y$, $x \in X_s^{y_1}$ and $y \neq y_1$, then the largest open neighbourhood $M_s(y)$ of y will also be the largest open neighbourhood $M_s(y_1)$ of y_1.*

Proof. Let us consider the set $N(y) = M_s(y) \cup M_s(y_1)$. Clearly, $N(y)$ is an open neighbourhood of y. Let $N'(y)$ be any open neighbourhood of y for which $N'(y) \subset N(y)$. Then

$$\begin{aligned} F^{-1}(N'(y)) &= F^{-1}(N'(y) \cap N(y)) \\ &= F^{-1}(N'(y) \cap M_s(y)) \cup F^{-1}(N'(y) \cap M_s(y_1)). \end{aligned}$$

Obviously, $F^{-1}(N'(y) \cap M_s(y))$ is open because $N'(y) \cap M_s(y) \subset M_s(y)$ and is an open neighbourhood of y. Now we set $N'(y) \cap M_s(y_1) = N''$. Then N'' is also an open set. If $y_1 \in N''$ then N'' is an open neighbourhood of y_1 for which $N'' \subset M_s(y_1)$. Hence $F^{-1}(N'')$ is open in this case. If $y_1 \notin N''$ and $Bd\, F^{-1}(N'') \neq \phi$, then this will imply the existence of at least one y_2-*l.s.d.* point x_2 in $Bd\, F^{-1}(N'') \subset X$ such that $y_2 \in M_s(y_1)$ by Corollary 2.5. But by Theorem 3.6, $x_2 = x$. Hence $Bd\, F^{-1}(N'') = \{x\}$. Therefore $Bd\, F^{-1}(N'') \subset F^{-1}(N'(y) \cap M_s(y))$. But $F^{-1}(N'(y) \cap M_s(y))$ is an open set. Hence x is an interior point of $F^{-1}(N'(y))$. Therefore $F^{-1}(N'(y))$ is an open set. This shows that $N(y) \subset M_s(y)$. But by construction $N(y) \supset M_s(y)$. Consequently $N(y) = M_s(y)$. Similarly it can be proved that $N(y) = M_s(y_1)$. Hence $M_s(y) = M_s(y_1)$, proving the theorem. □

Finally we have a theorem on the cardinality of the set of points of lower semi-discontinuity which stems from the following example.

Example 3.17. Let $F : (\mathbb{R}, \mathbb{U}) \to (\mathbb{R}, \tau)$ be a multifunction defined by

$$\begin{aligned} F(x) &= \{x+1\},\ x = 2,\ 3,\ 4,\ \ldots\ldots\ldots \\ &= \{1, 2\},\ x = 1 \\ &= \{x\},\ \textit{otherwise} \end{aligned}$$

where a base for τ is the set of all open intervals with rational end points and all of the singletons $\{x\}$ where $x = 0,\ \pm1,\ \pm2,\ \ldots\ldots$. It is easy to verify that the points of *l.s.d.* are $x = 0,\ \pm1,\ \pm2,\ \ldots\ldots$. It is also worthwhile to notice that $(\mathbb{R}, \tau)$ is locally connected and second countable.

Before going to the theorem, we first state (without proof) the following lemma.

Lemma 3.18. *Let $F : X \to Y$ be an open bijection. Then $F^{-1}(A)$ is connected in X if A is connected in Y.*

Theorem 3.19. *If $F : \mathbb{R} \to Y$ is an open bijection of the real line $\mathbb{R}$ into a locally connected space Y satisfying the second axiom of countability, then the set of points of lower semi-discontinuity of the multifunction F must be countable at most.*

Proof. Let $x \in \mathbb{R}_{ld}$. Then there exists an $y \in F(x)$ and an open neighbourhood $N(y)$ of y such that $x \in Bd\, F^{-1}(N(y))$. Since Y is locally connected, there exists a connected open neighbourhood $U(y)$ of y such that $U(y) \subset N(y)$. Then $x \in Bd\, F^{-1}(U(y))$. Let us consider the family Ω of open connected sets such that for any arbitrarily chosen $x \in \mathbb{R}_{ld}$ it will be possible to find at least one $U \in \Omega$ for at least one $y \in F(x)$ such that $y \in U$ and $x \in Bd\, F^{-1}(U)$. Since Y satisfies the second axiom of countability, there is a countable open base $\{O_n : n \in \mathbb{N}\}$ of Y. Hence it is possible to find an $O \in \{O_n : n \in \mathbb{N}\}$ such that $y \in O \subset U$. Hence as x runs over $\mathbb{R}_{ld}$ the corresponding set O runs over a subfamily of $\{O_n : n \in \mathbb{N}\}$. Since this subfamily of the sets O is countable, the corresponding sets $U \in \Omega$ also form a countable family. Let $\{U_n : n \in \mathbb{N}\}$ be the countable family of sets U so obtained. For any $x \in \mathbb{R}_{ld}$ it will be possible, obviously, to select an $U_n \in \{U_n : n \in \mathbb{N}\}$ for at least one $y \in F(x)$ such that $y \in U_n$ and $x \in Bd\, F^{-1}(U_n)$, by construction of the family $\{U_n : n \in \mathbb{N}\}$. So $\mathbb{R}_{ld} \subset \bigcup_{n \in \mathbb{N}} Bd\, F^{-1}(U_n)$. Since F is open and bijective and U_n is connected for each n, $F^{-1}(U_n)$ is also connected for each n and it will have at most two boundary points. Also $\{F^{-1}(U_n) : n \in \mathbb{N}\}$ is a countable family of connected sets. Hence $\bigcup_{n \in \mathbb{N}} Bd\, F^{-1}(U_n)$ is countable at most and so, $\mathbb{R}_{ld} \subset \bigcup_{n \in \mathbb{N}} Bd\, F^{-1}(U_n)$ is also at most countable. □

Acknowledgment. We are thankful to the referee for his valuable comments and suggestions for the improvement of the merit of the paper.

References

[1] J. Cao and W. B. Moors, *Quasicontinuous selections of upper continuous set-valued mappings*, Real Anal. Exchange, **31(1)** (2005), 1-7.

[2] E. Ekici, *On almost and weak forms of nearly continuous multifunctions*, Proc. Jangjeon Math. Soc., **9(2)** (2006), 109-120.

[3] S. R. Ghosh and H. Dasgupta, *Classification of points of continuities and discontinuities of a mapping in topological spaces*, Bull. Calcutta Math. Soc., **97(4)** (2005), 282-296.

[4] R. E. Smithson, *Multifunctions*, Nieuw Arch. Wiskd. (5), **XX** (1972), 31-53.

[5] R. Vaidyanathaswamy, *Set topology*, Chelsea Publishing Company, New York, 2nd Edition (1960).

[6] G. T. Whyburn, *Continuity of multifunctions*, Proc. Natl. Acad. Sci. USA, **54** (1965), 1494-1501.