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ON AN IMPROVEMENT OF THE HAKE THEOREM

Abstract

The well-known Hake Theorem asserts that if a function f is Denjoy* integrable then it is also Perron integrable, and the two integrals are equal. In [3] we introduced a very strong Perron integration $(\mathcal{P}_{1,1})$ and proved the corresponding Hake-type theorem, using the Vitali-Carathéodory Theorem. In this paper we give a new, less technical proof of this result, using essentially Lusin's Theorem.

1 Introduction

The well-known Hake Theorem asserts that if a function f is Denjoy* integrable then it is also Perron integrable, and the two integrals are equal. In fact these two integrals are equivalent (see the Hake-Alexandroff-Looman Theorem), and there are many definitions of Perron-type integrals that are equivalent to the Denjoy* integral. In [3, Corollary 5.9.1], we made a study of many (at least 108) of these equivalences. One of the strongest Perron type definition is that of Skljarenko, where the major and minor functions are AC^*G and continuous. Using the Tolstoff-Zahorski Theorem we showed that in addition, the major and minor functions have finite or infinite derivatives at each point, obtaining the $(\mathcal{P}_{1,1})$-integral. To prove that the $\mathcal{D}^*$-integrability implies the $(\mathcal{P}_{1,1})$-integrability (i.e., a Hake type theorem) we used essentially the Vitali-Carathéodory Theorem [11, p. 166]. In the present paper we give a different, less technical proof of this result, using essentially Lusin's Theorem [13, p. 72]. Both proofs use different techniques from that of Skljarenko. We conclude the paper with some comments and a question related to the subject.

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2 Preliminaries

For the definitions of VB, VB^*, AC, AC^*, AC^*G see [13]. We denote by $\mathcal{P}(E) = \{X : X \subseteq E\}$ whenever $E \subseteq \mathbb{R}$. By $\mathcal{O}(F; X)$ we mean the oscillation of the function F on the set X. We denote by $m(X)$ the Lebesgue measure of the Lebesgue measurable set X.

Definition 1. Let $F : [a, b] \mapsto \mathbb{R}$, and let P be a closed subset of [a,b], $c = \inf(P)$, $d = \sup(P)$. Let $F_P : [c, d] \to \mathbb{R}$ be defined as follows: $F_P(x) = F(x)$, $x \in P$ and F_P is linear on each $[c_k, d_k]$, where $\{(c_k, d_k)\}_{k\geq 1}$ are the intervals contiguous to P.

Definition 2. [3, p. 174]. Let $f : [a, b] \to \overline{\mathbb{R}}$. We define the following classes of major and minor functions and the corresponding $(\mathcal{P}_{1,1})$-integral:

- $\overline{\mathcal{M}}_1(f) = \{M : [a, b] \to \mathbb{R} : M(a) = 0,\ M \in AC^*G;\ M'(x)$ exists (finite or infinite); $f(x) \leq M'(x) \neq -\infty\}$;
- $\underline{\mathcal{M}}_1(f) = \{m : [a, b] \to \mathbb{R} : -m \in \overline{\mathcal{M}}_1(-f)\}$.
- If $\overline{\mathcal{M}}_1(f) \neq \emptyset$ then we denote by $\overline{I}_1(b)$ the lower bound of all $M(b)$, $M \in \overline{\mathcal{M}}_1(f)$. If $\underline{\mathcal{M}}_1(f) \neq \emptyset$ then we denote by $\underline{I}_1(b)$ the upper bound of all $m(b)$, $m \in \underline{\mathcal{M}}_1(f)$.
- f is said to have a $(\mathcal{P}_{1,1})$-integral on $[a, b]$, if $\overline{\mathcal{M}}_1(f) \times \underline{\mathcal{M}}_1(f) \neq \emptyset$ and $\overline{I}_1(b) = \underline{I}_1(b) = (\mathcal{P}_{1,1}) \int_a^b f(t)dt$.

Lemma A. [3, Lemma 5.8.2]. *Let $F, H, G : [a, b] \to \mathbb{R}$, and let $H(x) = \mathcal{O}(F; [a, b]) - \mathcal{O}(F; [x, b]) + \mathcal{O}(G; [a, x])$, $G = F + H$. If $F \in AC^*G$ on $[a, b]$ then: $H(a) = 0$ and $H(b) = 2 \cdot \mathcal{O}(F; [a, b])$; H is increasing and AC on $[a, b]$; $G \in AC^*G$ and $G(a) \leq G(x) \leq G(b)$ on $[a, b]$.*

Lemma B. [3, Theorem 2.11.1, (xviii)]. *Let $F : [a, b] \to \mathbb{R}$, and let P be a subset of $[a, b]$, $c = \inf(P)$ $d = \sup(P)$. $F \in AC$ on $\overline{P}$ if and only if $F_{\overline{P}} \in AC$ on $[c, d]$;*

Lemma C. *Let $F : [a, b] \to \mathbb{R}$, $F \in AC$, and let $V : [a, b] \to \mathbb{R}$, $V(x) = V(F; [a, x])$ (here $V(f; X)$ stands for the variation of F on the set X). Then $V \in AC$ on $[a, b]$.*

PROOF. This follows by definitions. □

Lemma D. [3, Lemma 5.8.3]. *Let $\{r_k\}_k$ be a sequence of positive numbers such that $\sum_{k=1}^{\infty} r_k = r < +\infty$. Let $F_k : [a, b] \to [0, r_k]$ such that F_k is increasing and AC on $[a, b]$. Let $F : [a, b] \to \mathbb{R}$, $F(x) = \sum_{k=1}^{\infty} F_k(x)$. Then: $F(a) = 0$ and $F(b) < r$; F is increasing and AC on $[a, b]$.*

3 A Hake Type Theorem

Lemma 1. *Let $F : [a,b] \to \mathbb{R}$, $F \in AC^*G$ on $[a,b]$. Let P be a closed subset of $[a,b]$ such that $F \in AC^*$ on P. Then for $\epsilon > 0$, there is a $\delta > 0$ such that for every open set E with $m(E) < \delta$, there exists a function $F^E : [a,b] \to [0,+\infty)$ having the following properties:*

a) $F^E(a) = 0$, $F^E(b) < \epsilon$*;*

b) F^E *is* AC *and increasing on* $[a,b]$*;*

c) $\underline{D}(F + F^E)(x) \geq 0$ *for all* $x \in P \cap E \neq \emptyset$*;*

d) For some E *we have that* $\underline{D}(F + F^E)$ *is bounded below on* P*.*

Proof. Let $Q = \{a,b\} \cup P$, $V : [a,b] \to \mathbb{R}$, $V(x) = V(F_Q; [a,x])$. By Lemma B and Lemma C, the function V belongs to AC on $[a,b]$. Let $\{(a_i, bi)\}_{i=1}^{\infty}$ (we suppose the infinite case, otherwise the situation is easy) be the intervals contiguous to Q. Let $H_i : [a,b] \to [0,+\infty)$,

$$H_i(x) = \begin{cases} 0 & \text{if } x \in [a, a_i] \\ \mathcal{O}\big(F; [a_i, b_i]\big) - \mathcal{O}\big(F; [x, b_i]\big) + \mathcal{O}\big(F; [a_i, x]\big) & \text{if } x \in [a_i, b_i] \\ 2 \cdot \mathcal{O}\big(F; [a_i, b_i]\big) & \text{if } x \in [b_i, b] \end{cases}$$

By Lemma A, each H_i is AC and increasing on $[a,b]$. Since $V(b) + \sum_{i=1}^{\infty} H_i(b) \neq +\infty$ (because F is VB^* on P, so on Q), by Lemma D, the function $G : [a,b] \to \mathbb{R}$, $G(x) = V(x) + \sum_{i=1}^{\infty} H_i(x)$ is increasing and AC on $[a,b]$. Hence G' exists and is finite *a.e.* on $[a,b]$, $G'(x) \geq 0$ *a.e.* on $[a,b]$, G' is Lebesgue integrable on $[a,b]$, and $G(x) = (\mathcal{L}) \int_a^x G'(t)\,dt$ (see [11, p. 255] or [3, p. 62]). For $\epsilon > 0$, let $\delta > 0$ be such that $\int_S G'(t)\,dt < \epsilon$, whenever S is a Lebesgue measurable subset of $[a,b]$ with $m(S) < \delta$ (see [11, p. 148]. Let E be an open set with $m(E) < \delta$, $E \cap P \neq \emptyset$, and let $F^E : [a,b] \to \mathbb{R}$, $F^E(x) = (\mathcal{L}) \int_{[a,x] \cap E} G'(t)\,dt$. Clearly we have a) and b).

c) Fix some $x_o \in E \cap P$, and let $x > x_o$ such that $[x_o, x] \subset E \cap [a,b]$. Then

$$F^E(t) - F^E(x_o) = (\mathcal{L}) \int_{[x_o,t]} G'(z)\,dz\,, \quad (\forall)\ t \in [x_o, x]\,.$$

Since $(F^E)'(t) = G'(t)$ *a.e.* on $[x_o, x]$, and $F^E, G \in AC$ on $[a,b]$, it follows that $F^E - G$ is constant on $[x_o, x]$. For $x \in P$ we have

$$\big(F + F^E\big)(x) - \big(F + F^E\big)(x_o) = F(x) - F(x_o) + G(x) - G(x_o) \geq$$

$$\geq F(x) - F(x_o) + V(x) - V(x_o) \geq 0\,.$$

If $x \in (a_i, b_i)$ then from above, $\big(F + F^E\big)(a_i) - \big(F + F^E\big)(x_o) \geq 0$ and

$$\big(F + F^E\big)(x) - \big(F + F^E\big)(a_i) = F(x) - F(a_i) + G(x) - G(a_i) \geq$$
$$\geq F(x) - F(a_i) + H_i(x) - H_i(a_i) \geq 0$$

(see the last part of Lemma A). Thus $\big(F + F^E\big)(x) - \big(F + F^E\big)(x_o) \geq 0$. Similarly, we obtain that $\big(F + F^E\big)(x_o) - \big(F + F^E\big)(x) \geq 0$ for $x < x_o$ and $[x, x_o] \subset E \cap [a, b]$ (in the computations we shall use b_i instead of a_i). Therefore $\underline{D}\big(F + F^E\big)(x_o) \geq 0$.

d) Since $F \in AC^*G$ on $[a, b]$, it follows that $F'(x)$ exists and is finite *a.e.* on P. But F' is also Lebesgue measurable on P. Thus by Lusin's Theorem (see [13, p. 72]), there exists a closed subset P_o of P, such that $(F')_{|P_o}$ is continuous and $m(P \setminus P_o) < \delta/2$. Hence there exists an open set E such that $(P \setminus P_o) \subset E$ and $m(E) < \delta$. It follows that $(F')_{|P\setminus E}$ is bounded, and by b) and c) we have that $\underline{D}(F + F^E)$ is bounded below on P. □

Theorem 1. [3, Corollary 5.8.1]. *Let $F : [a, b] \to \mathbb{R}$, $F \in AC^*G$ on $[a, b]$, and let $r > 0$. Then there exists $H : [a, b] \to \mathbb{R}$ such that:*

(i) $H(a) = 0$ and $H(b) < r$;

(ii) H is increasing and AC on $[a, b]$;

(iii) $G = F + H$ is AC^*G and $\underline{D}G(x) \neq -\infty$ on $[a, b]$.

Proof. Since $F \in AC^*G$ on $[a, b]$, it follows that there exists a sequence $\{E_n\}_n$ of closed subsets of $[a, b]$ that cover $[a, b]$, such that F is AC^* on each E_n. By Lemma 1, for each positive integer n, there exists $h_n : [a, b] \to \big[0, \frac{r}{2^{n+1}}\big)$, such that $\underline{D}(F + h_n)(x) \neq -\infty$ for $x \in E_n$, and h_n is AC and increasing on $[a, b]$. Let $H : [a, b] \to \mathbb{R}$, $H(x) = \sum_{n=1}^{\infty} h_n(x)$. Clearly we have (i), and by Lemma D, we also have (ii) and (iii). □

Theorem 2. [3, Theorem 5.8.1]. *Let $f : [a, b] \to \overline{\mathbb{R}}$. If f is $\mathcal{D}^*$- integrable on $[a, b]$ then f is $(\mathcal{P}_{1,1})$-integrable on $[a, b]$ and the two integrals are equal.*

Proof. The proof follows by Theorem 1 and the Tolstoff-Zahorski Theorem (see for example [3, Theorem 2.14.6]). □

Remark 1. Special cases of Theorem 2 and Theorem 3 have been studied by many authors, from different points of view, using the constructive theory of Denjoy ([11, Ch. XVI, §8]), or the descriptive theory of Denjoy ([13], [14], [9], [3]), or the Kurzweil-Henstock theory ([4], [5], [15]), or combining the descriptive theory of Denjoy with the Kurzweil-Henstock theory ([10]).

4 Some Comments and Remarks

In [2, Corollary 1, (i), (vii)] and [3, Corollary 2.27.1, (i), (vii)] we have proved, without using the Kurzweil-Henstock theory, the following result (two years later, B. Bongiorno, L. Di Piazza, and V. Skvortsov gave another proof, using the Kurzweil-Henstock theory [1, Theorem 4]):

Theorem A. *A function* $F : [a,b] \to \mathbb{R}$ *is* AC^*G *on* $[a,b]$ *if and only if* F *is* Y_{D^o} *on* $[a,b]$.

Remark 2. Condition Y_{D^o} has been introduced by Jarnik and Kurzweil, without naming it, in [6, (3.16) on p. 655]; later it has been used by several authors, but with different names: the strong Lusin condition, short SLC in [8, p. 557], well-behaved in [7, p. 124], AC_* in [12, p. 115], Y_{D_o} in [2, p. 503], [3, p. 89], absolute continuity of the variational measure with respect to the Lebesgue measure in [1], etc.

From Theorem A we easily obtain the following theorem:

Theorem B. [3, Theorem 5.12.1, (i), (iv)]. $f : [a,b] \to \mathbb{R}$ *is* $\mathcal{D}^*$*-integrable on* $[a,b]$ *if and only if* F *is* (KH)*-integrable (Kurzweil-Henstock integrable) on* $[a,b]$ *and the two integrals are equal.*

Using Theorem B $\big($or only Theorem A and the facts that $F \in Y_{D^o}$ on $[a,b]$, and $F'(x) = f(x)$ *a.e.* on $[a,b]$ for $F(x) = (KH)\int_a^x f(t)\,dt$, $x \in [a,b]\big)$ together with Theorem 2, we obtain:

Theorem 3. *If* $f \in (KH)$*-integrable on* $[a,b]$ *then* f *is* $(\mathcal{P}_{1,1})$*-integrable on* $[a,b]$ *and the two integrals are equal.*

Question. In [15, Theorem 2], using the Kurzweil-Henstock theory, Skvortsov constructed some continuous major and minor functions M, m for a KH-integrable function. Is it possible to show that M and m are also major respectively minor functions for the $(\mathcal{P}_{1,1})$-integral?

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