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THE DYNKIN SYSTEM GENERATED BY THE LARGE BALLS OF $\mathbb{R}^n$

Abstract

We prove that in an at least three dimensional Euclidean space the Dynkin system generated by the family of all open balls with radii at least one (that is, the smallest collection containing the open balls with radii at least one and closed under complements and countable disjoint unions) does not contain all Borel sets. We also give a simple characterization of the sets of this Dynkin system.

A non-empty family $\mathcal{D}$ of subsets of a set X is said to be a *Dynkin system* provided $\mathcal{D}$ is closed under complements and countable disjoint unions. (The terms 'σ-class', 'concrete quantum logic' and 'q-σ-algebra' is also used for the same notion; the latter two in the study of quantum logic.)

In 1988 and 1995 Olejček [5, 6] proved that in $\mathbb{R}^2$ and in $\mathbb{R}^3$ the Dynkin system generated by all balls is equal to the Borel class. (Although it does not make much difference, by a ball we always mean here an open ball.) Recently Jackson and Mauldin [1] and independently Zelený [8] generalized this result for any finite dimensional Euclidean space (in fact, in [1] for any finite dimensional Banach space). In this note we show that in Euclidean spaces with dimension at least three it is not enough using large balls. More precisely we show

Theorem 1. *In $\mathbb{R}^n$ (for $n \geq 3$) the Dynkin system generated by the family of all (open) balls with radii at least 1 does not contain all Borel sets.*

The motivation of this result came from infinite dimension: According to two results about the positivity principle for large and small balls in a separable Hilbert space ([3] Theorem 3 and Theorem 4; the proof of the main statement is in [7]; see also in [4]) there is a strange analogy between the small balls in an

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infinite dimensional Hilbert space and the large balls in the finite dimensional one. The analogy turned out to be true enough to enable the solution of the following infinite dimensional question, which was asked 20 years ago: "Is it true that in any separable Banach space the Dynkin system generated by the family of all balls is equal to the Borel class?" Generalizing the proof of the above Theorem 1 it is proved in [2] that the answer is negative in an infinite dimensional separable Hilbert space. Actually, the proof in [2] carries over without any essential change to showing Theorem 1. The aim of this note is to show the simpler original proof of Theorem 1.

We shall also give a complete characterization of the sets of the Dynkin system generalized by the large balls of $\mathbb{R}^n$ ($n \geq 3$).

The idea of the proof of Theorem 1 is to show that the sets from the Dynkin system generated by the large balls, modulo a countable set, can be obtained by a relatively simply operation directly from large balls; in fact, as a union of simple building blocks.

Under a *generalized ball* in $\mathbb{R}^n$ we will understand either a ball, or a half-space or the whole space $\mathbb{R}^n$; by a *generalized sphere* we will mean a sphere or a hyper-plane. By the radius of an unbounded generalized ball or generalized sphere we will mean infinite.

A stereographic projection P maps the generalized spheres to (normal) spheres on a fixed sphere. We equip the set of generalized spheres by the Hausdorff distance of the mapped spheres; that is,

$$d(S,T) = \max\big(\sup_{x \in P(S)} d(x, P(T)), \sup_{y \in P(T)} d(y, P(S)) \big).$$

One can easily check the following:

Lemma 2. *Any monotone union or intersection of generalized balls with radii at least 1 is a generalized ball with radius at least 1. Moreover, if the union is not $\mathbb{R}^n$ then its boundary belongs to the closure of the set of the boundaries of the generalized balls from $\mathcal{B}$.*

If the intersection of different generalized spheres S, T contains more than one point then we shall say that S, T are *crossing*.

We will use the following obvious fact:

Lemma 3. *Let $S \neq T$ be crossing generalized spheres. Then there is $\varepsilon > 0$ such that any two generalized spheres S', T' with $d(S', S) < \varepsilon$ and $d(T', T) < \varepsilon$ are crossing.*

The following lemma will ensure that the exceptional set will be countable:

Lemma 4. *Suppose that $\mathcal{S}$ is a family of pairwise non-crossing generalized spheres in $\mathbb{R}^n$. Then the set M of those points that belong to at least two different generalized spheres from $\mathcal{S}$ is countable.*

Proof. For every $x \in M$ choose generalized spheres $S, T \in \mathcal{S}$ such that $S \neq T$ and $x \in S \cap T$; denote by U, V, W the three components of $\mathbb{R}^n \setminus (S \cup T)$, where the notation is such that $\partial U = S$, $\partial V = T$ and x is the only point of $\overline{U} \cap \overline{V}$, and choose points with rational coordinates $u \in U$, $v \in V$, and $w \in W$. We claim that x is uniquely determined by the triple $\langle u, v, w \rangle$; since the set of such triples is countable, this will finish the proof. To prove the claim, suppose, that from some $x' \in M$ we arrived to the same $\langle u, v, w \rangle$ using generalized spheres $S', T' \in \mathcal{S}$ and components U', V', W' of $\mathbb{R}^n \setminus (S' \cup T')$. Since the generalized spheres from $\mathcal{S}$ are non-crossing and since $U \cap U' \neq \emptyset$, we have that $U \subset U'$ or $U' \subset U$; similarly for V's and W's. Exchanging the role of x and x' and/or of U's and V's if necessary, there are only two cases to consider: (a) $U \supset U'$ and $V \supset V'$ and (b) $U \subset U'$, $V \supset V'$ and $W \subset W'$. In case (a) we recall that $\overline{U} \cap \overline{V}$ contains only x and that $x' \in \overline{U'} \cap \overline{V'}$, so $x = x'$. In case (b) we get from $W \subset W'$ that $U' \subset \overline{U \cup V}$; so since U' is open and connected, and $\overline{U} \cap \overline{V}$ is just one point, we infer that $U' = U$ and we are back in the already proved case (a). $\qquad\square$

   The points $x, y \in \mathbb{R}^n$ are said to be *separated* by a generalized sphere S if they lie in different components of $\mathbb{R}^n \setminus S$. For a class of generalized spheres $\mathcal{S}$ let $\sim_{\mathcal{S}}$ be the equivalence relation on $\mathbb{R}^n \setminus \bigcup_{S \in \mathcal{S}} S$ defined by

$$x \sim_{\mathcal{S}} y \qquad \leftrightarrow \qquad x, y \text{ are separated by no generalized spheres of } \mathcal{S}.$$

Lemma 5. *Suppose that $\mathcal{S}$ is a non-empty closed family of pairwise non-crossing generalized spheres with radii at least 1. Then each equivalence class of $\sim_{\mathcal{S}}$ is of the form $B_0 \setminus \bigcup_{i \geq 1} \overline{B}_i$, where B_0 is a generalized (open) ball; $B_1, B_2, \ldots \subset B$ is a (finite or infinite) sequence of pairwise disjoint generalized balls with radii at least 1; and $\partial B_i \in \mathcal{S} \cup \{\emptyset\}$ for all i.*

Proof. Let $y \in \mathbb{R}^n \setminus \bigcup_{S \in \mathcal{S}} S$. Let $\mathcal{B}$ contain $\mathbb{R}^n$ and all generalized balls with boundary in $\mathcal{S}$ and let $\mathcal{B}_1$ contain those of $\mathcal{B}$ that contain y. Since the generalized spheres from $\mathcal{S}$ are non-crossing, for every $B, B' \in \mathcal{B}$ we have that $B \cap B' = \emptyset$ or $B \subset B'$ or $B' \subset B$. According to Lemma 2 and using that $\mathcal{S}$ is closed and $\mathbb{R}^n \in \mathcal{B}_1$, Zorn lemma gives us a minimal generalized ball B_0 from $\mathcal{B}_1$. Let $\mathcal{B}_2$ be the family of those generalized balls $B \in \mathcal{B}$ for which $B \subset B_0$ and $y \notin B$. By Hausdorff maximal principle, each generalized ball from $\mathcal{B}_2$ is a subset of a maximal generalized ball from $\mathcal{B}_2$; these maximal balls are disjoint, hence there are only countably many of them and we may order them into a sequence $B_1, B_2, \ldots$. Clearly, any point which is not in $U = B_0 \setminus \bigcup_{i \geq 1} \overline{B}_i$ is

separated from y by an $S \in \mathcal{S}$. On the other hand, every point of B_0 that is separated from y belongs to some generalized ball from $\mathcal{B}_2$, hence to some B_k. □

A set K will be called a *K-set about a generalized sphere S* if either $K = S$ or it is a non-empty set of the form $B_0 \setminus \bigcup_{i \geq 1} \overline{B}_i$, where B_0 is a generalized (open) ball with radius at least 1; $B_1, B_2, \ldots \subset B$ is a (finite or infinite) sequence of pairwise disjoint generalized balls with radii at least 1; and $\partial C_i = S$ for an $i \geq 0$.

Lemma 5 showed that a non-empty closed family of pairwise non-crossing generalized spheres defines K-sets such that their union is $\mathbb{R}^n$ and the intersection of any two contains at most one point. As we will see later, any element of the Dynkin system generated by the large balls (in at least 3 dimension) can be built using these building blocks.

Lemma 6. *Suppose that in $\mathbb{R}^n$, where $n \geq 3$, K and K' are K-sets about crossing generalized spheres S and S', respectively. Then $K \cap K'$ is uncountable.*

Proof. The boundary of a K-set about a generalized sphere S is clearly always of the form $S \cup S_1 \cup S_2 \cup \ldots$, where $S, S_1, S_2, \ldots$ is a (finite or infinite) sequence of pairwise non-crossing generalized spheres with radii at least 1. For K and K' this way we get $\partial K = S \cup S_1 \cup S_2 \cup \ldots$ and $\partial K' = S' \cup S'_1 \cup S'_2 \cup \ldots$.

The intersection of the crossing generalized spheres S and S' is a generalized sphere in an $(n-1)$-dimensional affine subspace, so (since $n - 1 \geq 2$) it is an uncountable set. Thus, by Lemma 4, there must be a point $x \in S \cap S'$ such that none of $S_1, S_2, \ldots, S'_1, S'_2, \ldots$ contains x. Since $S_1, S_2, \ldots, S'_1, S'_2, \ldots$ have radii at least 1, a small neighborhood of x is disjoint to $S_1, S_2, \ldots, S'_1, S'_2, \ldots$. Then in all cases it is easy to see that this neighborhood contains uncountable points of $K \cap K'$. □

We say that the class $\mathcal{S}$ of generalized spheres *separates E modulo M* if each generalized sphere of $\mathcal{S}$ and each equivalence class of $\sim_{\mathcal{S}}$ is contained in $E \cup M$ or $E^c \cup M$. The generalized sphere A is *essential for separating E modulo M* if for every $x \in A \setminus M$ there exists arbitrarily close $y \in M^c$ such that $y \in E$ if $x \in E^c$ and $y \in E^c$ if $x \in E$.

Lemma 7. *If in $\mathbb{R}^n$ (where $n \geq 3$), for $i = 1, 2$, the closed family of pairwise non-crossing generalized spheres $\mathcal{S}_i$ separates E_i modulo M_i; $A_i \in \mathcal{S}_i$ is essential for separating E_i modulo a countable set M_i; and A_1, A_2 are crossing then $E_1 \cap E_2$ is non-empty.*

Proof. By Lemma 5 each equivalence class of $\sim_{\mathcal{S}_i}$ is a K-set. Since $\mathcal{S}_i$ separates E_i modulo M_i and $A_i \in \mathcal{S}_i$ is essential for separating E_i modulo M_i,

we have K-set $K_i \subset E_i \cup M$ about generalized sphere arbitrarily close to A_i. By Lemma 3 and Lemma 6, this implies that $E_1 \cap E_2$ is non-empty. $\square$

Lemma 8. *For every set E from the Dynkin system generated by the balls of $\mathbb{R}^n$ (for $n \geq 3$) with radii ≥ 1 one may choose a countable set M and a closed (in the metric d) family $\mathcal{S}$ of pairwise non-crossing generalized spheres such that $\mathcal{S}$ separates E modulo M and each generalized sphere of $\mathcal{S}$ is essential for separating E modulo M.*

PROOF. If E is a ball, it suffices to take $\mathcal{S} = \{\partial E\}$ and $M = \emptyset$. If the statement of the lemma holds for E with some M and $\mathcal{S}$, then it also holds for E^c with the same M and $\mathcal{S}$.

Hence we just have to find the required set M and family $\mathcal{S}$ assuming that $E = \bigcup_{k=1}^{\infty} E_k$, the sets E_k are disjoint, and there are countable sets M_k and closed families $\mathcal{S}_k$ of pairwise non-crossing generalized spheres such that for each k, $\mathcal{S}_k$ separates E_k modulo M_k and each generalized sphere of $\mathcal{S}_k$ is essential for separating E_k modulo M_k.

Define $\mathcal{S}'$ as the family of those generalized spheres that belong to at least one of the $\mathcal{S}_k$ and $\mathcal{S}''$ as the closure of $\mathcal{S}'$ (in the metric d). By Lemma 7, $\mathcal{S}'$ is a family of pairwise non-crossing generalized spheres; and by Lemma 2 and Lemma 3, so is $\mathcal{S}''$. Let M_0 be the set of those points that belong to at least two different generalized spheres from $\mathcal{S}''$. By Lemma 4, M_0 is countable, and so $M = \bigcup_{k=0}^{\infty} M_k$ is countable.

We claim that each generalized sphere $T \in \mathcal{S}''$ is contained in $E \cup M$ or $E^c \cup M$. Indeed, suppose to the contrary that $x \in (T \cap E) \setminus M$ and $y \in (T \setminus E) \setminus M$ and choose k such that $x \in E_k$. Since $x \in E_k \setminus M_k$, $y \in E_k^c \setminus M_k$ and $\mathcal{S}_k$ separates E_k modulo M_k, we get that $T \notin \mathcal{S}_k$. Since $x \notin M_0$, x is not contained by any generalized sphere from $\mathcal{S}'' \setminus \{T\} \supset \mathcal{S}_k$, so x is contained in an equivalence class of $\sim_{\mathcal{S}_k}$. Using that $\mathcal{S}_k$ separates E_k modulo M_k we get that this equivalence class is in $E_k \cup M_k$, so y is not in this equivalence class, which means that x, y are separated by a generalized sphere $S \in \mathcal{S}_k \subset \mathcal{S}''$. But then $S, T \in \mathcal{S}''$ are crossing, which is a contradiction.

We prove that every $S \in \mathcal{S}''$ such that $S \cap (E^c \setminus M) \neq \emptyset$ is essential for separating E modulo M: The preceding paragraph shows that then $S \setminus M \subset E^c$. Moreover, for every $\varepsilon > 0$ we find $T \in \mathcal{S}_k$ such that $d(T, S) < \varepsilon/2$ and, using that T is essential for separating E_k modulo M_k, for any $x \in S \setminus M \subset E^c \subset E_k^c$ there exists arbitrarily close $y \in E_k \subset E$.

Finally, we define $\mathcal{S}$ as the family of those generalized spheres of $\mathcal{S}''$ that are essential for separating E modulo M. Thus $\mathcal{S}$ is clearly a closed subset of $\mathcal{S}''$, so it remains to prove that each equivalence class of $\sim_{\mathcal{S}}$ is contained in either $E \cup M$ or in $E^c \cup M$. Let $x \in E \setminus (M \cup \bigcup_{S \in \mathcal{S}} S)$ and $y \in E^c \setminus (M \cup \bigcup_{S \in \mathcal{S}} S)$.

We need to show that $x \not\sim_{\mathcal{S}} y$. If $S \cap (E^c \setminus M) \neq \emptyset$ for some $S \in \mathcal{S}''$ that separates x and y, then (by the preceding paragraph) this S belongs to $\mathcal{S}$, and we are done. We may therefore assume that $S \setminus M \subset E$ for every $S \in \mathcal{S}''$ that separates x and y. By Lemma 5, the equivalence class of $\sim_{\mathcal{S}''}$ that contains y is of the form $G = B_0 \setminus \bigcup_{i \geq 1} \overline{B}_i$, where B_0 is a generalized (open) ball; $B_1, B_2, \ldots \subset B$ is a (finite or infinite) sequence of pairwise disjoint generalized balls; and $\partial B_i \in \mathcal{S}'' \cup \{\emptyset\}$ for all i. Since each $\mathcal{S}_k$ separates E_k modulo M_k, the class $\mathcal{S}''$ separates E modulo M, so x cannot be in G. It cannot be on the boundary of G either, so x, y are separated by one of the generalized spheres $\partial B_i \in \mathcal{S}''$. Then $\partial B_i \subset E \cup M$ but $G \subset E^c \cup M$, so ∂B_i is essential for separating E modulo M, hence it is in $\mathcal{S}$, which implies that $x \not\sim_{\mathcal{S}} y$. $\square$

The following result is clearly stronger than Theorem 1, which we wanted to prove:

Theorem 9. *Every set E of the Dynkin system generated by the balls of $\mathbb{R}^n$ ($n \geq 3$) with radii ≥ 1 has diameter at least 2.*

PROOF. Using Lemma 8 and Lemma 5, up to some changes on a countable set, E is the union of K-sets. On the other hand, clearly all K-sets have diameter at least 2, even if we remove a countable set of points. $\square$

Remark 10. Lemma 8 is not true in 2-dimension: Let F_1 and F_2 be the union of the closed discs with radii 1 and with centers of the form $(2i, 2j)$ and $(2i + 1, 2j + 1)$, respectively. Then $E = F_1^c \cup F_2^c$ is a counter-example.

The author does not know weather Theorem 9 (or at least Theorem 1) holds in 2-dimension.

In the remaining part of this note we give a complete characterization of the Dynkin system generated by the large balls of $\mathbb{R}^n$ for $n \geq 3$. Some of the details of the proof will be omitted.

Theorem 11. *For $n \geq 3$ a set $E \subset \mathbb{R}^n$ is in the Dynkin $\mathcal{D}$ system generated by the balls of $\mathbb{R}^n$ with radii ≥ 1 if and only if*

(1) *E is a Borel set, and*

(2) *there is a collection $\mathcal{K}$ of K-sets such that any two of them have at most one common point and the symmetric difference of $\cup \mathcal{K}$ and E is countable.*

PROOF. If E is in the Dynkin system then (1) is obvious; Lemma 8 and Lemma 5 imply (2).

For the other direction first note that any one point set is in the Dynkin system $\mathcal{D}$. (Indeed, using that $\mathcal{D}$ is clearly also closed under monotone union

and monotone intersection and contains all the large closed balls as well, one can first generate the union of a large open ball and a point of its boundary, then the union of the complement of this set and the open ball is the complement of the point.) Thus, if we modify a set of $\mathcal{D}$ on a countable set, the set we get is still in $\mathcal{D}$. Therefore we can assume that $E = \cup\mathcal{K}$. Moreover, for generating the Dynkin system, we can also allow pairwise "almost" disjoint countable union as well, where by almost disjoint we mean that the intersection is a countable set.

Let $\mathcal{K}_1$ contain those K-sets of $\mathcal{K}$ that are generalized spheres and let $\mathcal{K}_2$ contain the others. Since $\mathcal{K}_2$ contains disjoint open sets, it is countable. On the other hand, each K-set is obviously in the Dynkin system $\mathcal{D}$, so we get that $\cup\mathcal{K}_2 \in \mathcal{D}$. Using that $\mathcal{K}$ contains pairwise almost disjoint K-sets, $\mathcal{K}_1$ contains generalized spheres and $\mathcal{K}_2$ contains open sets, we deduce that $\cup\mathcal{K}_1$ and $\cup\mathcal{K}_2$ are disjoint. Thus $\cup\mathcal{K}_1$ is a Borel set and we need to prove that it is also in $\mathcal{D}$.

Let $\mathcal{B}$ be the class of all generalized balls with boundary in $\mathcal{K}_1$. Since $\mathcal{K}_1$ is a class of pairwise non-crossing spheres, for any $B_1, B_2 \in \mathcal{B}$ we have either $B_1 \cap B_2 = \emptyset$ or $B_1 \subset B_2$ or $B_2 \subset B_1$. Using this property, the following relation is an equivalent relation on $\mathcal{B}$: $B_1 \sim B_2$ if they are contained in each other and whenever $B \in \mathcal{B}$ meets one of them it also meets the other. Using that all balls from $\mathcal{B}$ have radii at least 1, one can check that the set of equivalence classes is countable: $\mathcal{B}_1, \mathcal{B}_2, \ldots$. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots$ be the corresponding classes of generalized spheres from $\mathcal{K}$. Since $\cup\mathcal{C}_1, \cup\mathcal{C}_2, \ldots$ are almost disjoint it is enough to prove that each of them (say $\cup\mathcal{C}_1$) is in the Dynkin system. The set $\cup\mathcal{C}_1$ is Borel since it is, up to a countable set, the intersection of the Borel set $\cup\mathcal{K}$ and a closed set of the form $\overline{B_1} \setminus B_2$, where $B_2 \subset B_1$ are generalized balls.

It is easy to extend $\mathcal{C}_1$ to a class $\mathcal{C} \supset \mathcal{C}_1$ of pairwise non-crossing generalized spheres with radii at least 1 such that they cover the whole $\mathbb{R}^n$ except a ball $B(O, 1)$ with radius 1. By Lemma 4 every point is covered once except a countable set. Let e be a line through O that avoid all these exceptional points. Let $f \subset e$ be the closed half-line that has distance 1 from O and let $F = (\cup\mathcal{C}_1) \cap f$. Then $\Psi(S) = S \cap f$ is a bijection between the generalized spheres of $\mathcal{C}$ and the points of f. Clearly $\Psi(\cup\mathcal{C}_1) = F$ and F is a Borel subset of f. On the other hand, in a half-line the Dynkin system generated by the subintervals obviously contains all Borel sets. The pre-image (Ψ^{-1}) of a subinterval is of the form $B_1 \setminus \overline{B_2}$ $(B_1 \supset B_2 \in \mathcal{C})$, so it is clearly in the Dynkin system $\mathcal{D}$ generated by the large balls. The same way as we can generate F from the subintervals of f we can generate $\cup\mathcal{C}_1$ from the corresponding sets of the form $B_1 \setminus \overline{B_2}$, so $\cup\mathcal{C}_1 \in \mathcal{D}$. $\square$

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