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A VARIATIONAL INTEGRAL FOR BANACH–VALUED FUNCTIONS

Abstract

It is shown that for Banach–space–valued functions the variational Henstock integral is equivalent to the Henstock integral if and only if the range space is of a finite dimension. The same is true for the equivalence of the variational McShane integral and the McShane integral.

There are various ways to define the Henstock integral. The original Henstock–Kurzweil definition is based on generalized Riemann sums (the H-integral). For real–valued functions this integral is known to be equivalent to the variational Henstock integral (the V-integral, see [10]) and to the Denjoy–Perron integral (the D_*-integral, see [7]). The first of this equivalence is a corollary of the so–called Saks–Henstock Lemma (see Lemma 1 below).

Here we are considering Henstock type integral for Banach–space–valued functions. It was noticed by S. S. Cao in [2] that for such functions Saks–Henstock Lemma might fail to be true. Because of that for some spaces the V-integral is not equivalent to the H-integral. It is natural to ask what is a characterization of those Banach spaces for which such equivalence holds.

We are showing here that for Banach–space–valued functions the V-integral is equivalent to the H-integral if and only if the range space is of a finite dimension. At the same time for any Banach space the V-integral is equivalent to the Denjoy–Bochner integral.

Similar problems are considered for the variational McShane integral.

First we recall some notations and definitions. We denote by X a Banach space with the norm $\|\cdot\|$, by $\mathbb{R}$ the real line, by $[a, b]$ a closed interval on the line, and by $|E|$ Lebesgue measure of a set E.

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Let $\mathcal{I}$ be a collection of all closed intervals that are contained in $[a, b]$. A collection T of pairs $(\Delta_k, \xi_k) \in \mathcal{I} \times [a, b]$, $i = 1, \ldots, n$, is called a *partition* of the interval $[a, b]$ if the intervals Δ_i and Δ_j are non-overlapping for $i \neq j$, and $\bigcup_{k=1}^{n} \Delta_k = [a, b]$.

Let $\delta : [a, b] \longrightarrow (0, \infty)$ be a positive function defined on $[a, b]$. A partition T of $[a, b]$ is called *Henstock δ-fine* if every pair $(\Delta, \xi) \in T$ satisfies

$$\xi \in \Delta \subset \big(\xi - \delta(\xi), \xi + \delta(\xi)\big).$$

Definition 1. ([6]). A function $f : [a, b] \longrightarrow X$ is called *Henstock integrable (H-integrable)* on a closed interval $[a, b]$ with integral value $I \in X$ if for every $\varepsilon > 0$ there exists a positive function $\delta : [a, b] \longrightarrow (0, \infty)$ such that for every Henstock δ-fine partition T of $[a, b]$

$$\Big\| \sum_{T} f(\xi_k)|\Delta_k| - I \Big\| < \varepsilon.$$

We denote $I = (H) \int_a^b f \, dt$.

A partition T of $[a, b]$ is called *McShane δ-fine* (δ being a positive function on $[a, b]$) if every pair $(\Delta, \xi) \in T$ satisfies

$$\Delta \subset \big(\xi - \delta(\xi), \xi + \delta(\xi)\big).$$

Definition 2. ([9]). A function $f : [a, b] \longrightarrow X$ is called *McShane integrable (M-integrable)* on a closed interval $[a, b]$ with integral value $I \in X$ if for every $\varepsilon > 0$ there exists a positive function $\delta : [a, b] \longrightarrow (0, \infty)$ such that for every McShane δ-fine partition T of $[a, b]$

$$\Big\| \sum_{T} f(\xi_k)|\Delta_k| - I \Big\| < \varepsilon.$$

We denote $I = (M) \int_a^b f \, dt$.

It is clear that M-integrability implies H-integrability. Since H-integrability and M-integrability on $[a, b]$ imply integrability on any subinterval $\Delta \subset [a, b]$, we can define *the indefinite H-integral* and *the indefinite M-integral* by putting $F(\Delta) = (H) \int_\Delta f \, dt$ $\big(F(\Delta) = (M) \int_\Delta f \, dt\big)$.

Let $\Phi : \mathcal{I} \times [a, b] \longrightarrow X$ be an interval- -point function. *The Henstock* and *the McShane variations* of Φ are defined as

$$\mathrm{V}_H(\Phi) = \inf_{\delta} \sup_{T} \sum_{T} \big\| \Phi(\xi_k, \Delta_k) \big\|$$

(sup is taken over all Henstock δ–fine partitions T and inf is taken over all positive functions δ on $[a,b]$), and

$$\mathrm{V}_M(\Phi) = \inf_{\delta} \sup_{T} \sum_{T} \big\|\Phi(\xi_k, \Delta_k\big\|$$

(sup is taken over all McShane δ–fine partitions T and inf is taken over all positive functions δ on $[a,b]$).

Note that any interval function $\Phi : \mathcal{I} \longrightarrow X$ can be considered as an interval–point function dependent only on the first argument.

The following two definitions of variational integrals are natural extensions of the definitions for the real-valued case (see [10]).

Functions $\Phi_1, \Phi_2 : \mathcal{I} \times [a,b] \longrightarrow X$ are said to be *Henstock variationally equivalent* if $\mathrm{V}_H(\Phi_1 - \Phi_2) = 0$.

Definition 3. A function $f : [a,b] \longrightarrow X$ is called *Henstock variationally integrable (V-integrable)* on $[a,b]$ if there exists an additive interval function $F : \mathcal{I} \longrightarrow X$ such that the interval–point function $f(t)|\Delta|$ and $F(\Delta)$ are Henstock variationally equivalent, $F(\Delta)$ being *the indefinite V-integral* of f.

Functions $\Phi_1, \Phi_2 : \mathcal{I} \times [a,b] \longrightarrow X$ are said to be *McShane variationally equivalent* if $\mathrm{V}_M(\Phi_1 - \Phi_2) = 0$.

Definition 4. A function $f : [a,b] \longrightarrow X$ is called *McShane variationally integrable (MV-integrable)* on $[a,b]$ if there exists an additive interval function $F : \mathcal{I} \longrightarrow X$ such that the interval–point function $f(t)|\Delta|$ and $F(\Delta)$ are McShane variationally equivalent, $F(\Delta)$ being *the indefinite MV-integral* of f.

Definition 5. ([8]). A function $F : \mathcal{I} \longrightarrow X$ is said to be *AC-function* on a set $E \subset [a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection of non-overlapping closed intervals $\{\Delta_i\}_{i=1}^n$ with the end points belonging to E and with $\sum_{i=1}^n |\Delta_i| < \delta$, we have $\sum_{i=1}^n \|F(\Delta_i)\| < \varepsilon$.

Definition 6. ([8]). A function $f : [a,b] \longrightarrow X$ is said to be *Bochner integrable (B-integrable)* on $[a,b]$, if there exists a function $F : \mathcal{I} \longrightarrow X$ that is AC on $[a,b]$ and such that it is differentiable a. e. and $F'(t) = f(t)$ a. e. on $[a,b]$, $F(\Delta)$ being *the indefinite B-integral* of f.

Definition 7. ([1]). A function $F : \mathcal{I} \longrightarrow X$ is said to be *AC^*-function* on a set $E \subset [a,b]$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every collection of non-overlapping closed intervals $\{\Delta_i\}_{i=1}^n$ with one of the end points belonging to E and with $\sum_{i=1}^n |\Delta_i| < \delta$, we have $\sum_{i=1}^n \|F(\Delta_i)\| < \varepsilon$.

It is clear that for $E = [a,b]$ the class of AC^*-functions coincides with the class of AC-functions.

Definition 8. ([1]). A function $F:\mathcal{I}\longrightarrow X$ is said to be *ACG^*-function* on a set $E\subset[a,b]$ if E can be represented as a union of a sequence of sets such that F is AC^*-function on each of them.

Definition 9. ([1]). A function $f:[a,b]\longrightarrow X$ is said to be *Denjoy–Bochner integrable (D_*B-integrable)* on $[a,b]$, if there exists an ACG^*-function $F:\mathcal{I}\longrightarrow X$ such that it is differentiable a. e. and $F'(t)=f(t)$ a. e. on $[a,b]$, $F(\Delta)$ being *the indefinite D_*B-integral* of f.

The Definitions 5 and 7 — 9 are extensions of the respective definitions for the real–valued case (see [11]).

The following proposition is a direct corollary of the definitions.

Proposition 1. *If $f:[a,b]\longrightarrow X$ is V-integrable (MV-integrable) on $[a,b]$ then it is also H-integrable (M-integrable)on $[a,b]$ and the indefinite integrals coincide.*

The following assertion is known as Saks–Henstock Lemma for real–valued functions and is easily extended to the case of vector–valued functions with range spaces being spaces of finite dimensions.

Lemma 1. ([3]). *Let X be a Banach space of a finite dimension. If a function $f:[a,b]\longrightarrow X$ is H-integrable (M-integrable) with the indefinite integral $F:\mathcal{I}\longrightarrow X$ then for every $\varepsilon>0$ there exists a function $\delta:[a,b]\longrightarrow(0,\infty)$ such that for every Henstock (McShane) δ-fine partition T of $[a,b]$*

$$\sum_{T}\left\|f(\xi_k)|\Delta_k|-F(\Delta_k)\right\|<\varepsilon.$$

S. S. Cao (see [3]) introduced a definition of HL-integrability of a function $f:[a,b]\longrightarrow X$ which is a restriction of H-integrability by the requirement that the assertion of Lemma 1 is valid for f. It is proved in [13] that the HL-integral is equivalent to the D_*B-integral. (The same equivalence was stated in [1], but there was some gap in the proof which was overcome in [13].) Since V-integrability of a function $f:[a,b]\longrightarrow X$ is obviously equivalent to the assertion of Saks–Henstock Lemma we get

Theorem 1. *Let X be a Banach space. For functions taking values in X the V-integral is equivalent to the D_*B-integral.*

Analogous fact for the MV-integral is the following one.

Theorem 2. *Let X be a Banach space. For functions taking values in X the MV-integral is equivalent to the B-integral.*

The proof is the same as above. It is enough to use [4] instead of [13].

Proposition 1 and Theorems 1 and 2 imply

Proposition 2. *If a function $f : [a, b] \longrightarrow X$ is D_*B-integrable (B-integrable) it is also H-integrable (M-integrable) and the indefinite integrals coincide.*

Now we consider the relation between the H-integral (the M-integral) and the V-integral (the MV-integral). Our aim in the rest of the paper is to prove the following theorem.

Theorem 3. *Consider functions on $[a, b]$ taking values in a fixed Banach space X. Then the V-integral (the MV-integral) is equivalent to the H-integral (the M-integral) on this class of functions if and only if X is of a finite dimension.*

Proof. The sufficiency follows easily from Lemma 1. The proof of the necessity is based on a geometric idea (see [12]) which in turn follows from the construction by A. Dvoretzky and C. A. Rogers used in [5] to show that in every infinite–dimensional Banach space there exists a series that is unconditionally but not absolutely convergent.

Lemma 2. ([5]). *Let B be a body in $\mathbb{R}^n$ which is convex and has the origin as a center and let r be an integer with $1 \le r \le n$. Then there exist r vectors $A_1, A_2, \dots, A_r \in \mathbb{R}^n$ on the boundary of B such that if $\lambda_1, \lambda_2, \dots, \lambda_r$ are any r real numbers then*

$$\sum_{i=1}^{r} \lambda_i A_i \in \lambda B, \text{ where } \lambda^2 = \left(2 + \frac{r(r-1)}{n}\right) \sum_{i=1}^{r} \lambda_i^2.$$

(λB is the set $\{\lambda x,\ x \in B\}$.)

Lemma 3. *Let X be a Banach space of the infinite dimension. Then for every natural number r there exist unit vectors $x_1, x_2, \dots, x_r \in X$ such that for every numbers $\theta_1, \theta_2, \dots, \theta_r$ with $|\theta_i| \le 1$, $1 \le i \le r$,*

$$\left\| \sum_{i=1}^{r} \theta_i x_i \right\|^2 \le 3r.$$

Proof. Since X is of the infinite dimension, for any n there exist linear independent vectors $z_1, z_2, \dots, z_n \in X$. Take $n = r(r-1)$. Consider the set of vectors $z = \sum_{i=1}^{n} \mu_i z_i$, where μ_i are numbers such that $\|z\| \le 1$. In Euclidean space with the norm generated by vectors $(\mu_1, \mu_2, \dots, \mu_n)$ they form a convex body B having the origin as a center. According to Lemma 2 there exist vectors $x_1, x_2, \ldots, x_r$ on the boundary of a set B with the following property: for every numbers $\theta_1, \theta_2, \ldots, \theta_r$ with $|\theta_i| \leq 1$, $1 \leq i \leq r$,

$$\sum_{i=1}^{r} \theta_i x_i \in \theta B, \text{ where } \theta^2 = 3\sum_{i=1}^{r} \theta_i^2 \leq 3r. \tag{1}$$

Since for all $i = 1, 2, \ldots, r$ vectors x_i belong to the boundary of B they have unit norm in X. Since $\|z\| \leq 1$ for all $z \in B$ it follows from (1) that

$$\Big\|\sum_{i=1}^{r} \theta_i x_i\Big\|^2 \leq 3r.$$

□

Now we can complete the proof of the necessity in Theorem 3. For simplicity we suppose $[a, b] = [0, 1]$. Let C be the Cantor ternary set, (a_i^r, b_i^r), $r \geq 0$, $1 \leq i \leq 2^r$, being the intervals of rank r contiguous to C (we have $b_i^r - a_i^r = 3^{-r-1}$) and d_i^r being the middle points of the intervals (a_i^r, b_i^r).

Assume that X is of the infinite dimension. According to Lemma 3, for every r we may construct vectors $x_1^r, x_2^r, \ldots, x_{2^r}^r \in X$ such that

$$\|x_i^r\| = \frac{1}{2^r},\ 1 \leq i \leq 2^r,$$

and for every numbers $\theta_1^r, \theta_2^r, \ldots, \theta_{2^r}^r$, with $|\theta_i^r| \leq 1, 1 \leq i \leq 2^r$,

$$\Big\|\sum_{i=1}^{2^r} \theta_i^r x_i^r\Big\|^2 \leq \frac{3}{2^r}.$$

Define the function $f : [0, 1] \longrightarrow X$ in the following way

$$f(t) = \begin{cases} 0, \text{ if } t \in C \text{ or } t = d_i^r, & r \geq 0, 1 \leq i \leq 2^r, \\ 2 \cdot 3^r x_i^r, \text{ if } t \in (a_i^r, d_i^r), & r \geq 0, 1 \leq i \leq 2^r, \\ -2 \cdot 3^r x_i^r, \text{ if } t \in (d_i^r, b_i^r), & r \geq 0, 1 \leq i \leq 2^r. \end{cases}$$

It is proved in [12] that this function is M-integrable (and consequently H-integrable) and its indefinite integral $F(\Delta)$ is not an ACG^*-function. Hence in view of the Proposition 2 function f is not D_*B-integrable (and is not B-integrable) and therefore Theorems 1 and 2 imply that f is not V-integrable and is not MV-integrable. This completes the proof. □

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