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STRUCTURE OF THE SET OF CONTINUOUS FUNCTIONS WITH LUZIN'S PROPERTY (N)

Abstract

We prove that the set of all continuous mappings of $[0,1]^n$ to R^n with Luzin's property (N) with respect to Lebesgue measure is a coanalytic non-Borel and first category subset of the space of all continuous mappings. Some generalizations, e.g. to cases of other Radon or Hausdorff measures are given.

1 Introduction

Let X and Y be measure spaces equipped with measures μ and ν, respectively, and let $f : X \to Y$ be a mapping. We say that f has (Luzin's) property (N) if

$$\nu(f(E)) = 0 \text{ whenever } \mu(E) = 0.$$

We shall investigate descriptive properties of the set of all mappings with property (N) in some topological spaces of continuous mappings of X to Y if X and Y are Hausdorff topological spaces.

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If Y is a metric space, $\mathcal{C}_b(X,Y)$ stands for the space of all bounded continuous mappings endowed with the supremum metric. On $\mathcal{C}_b(X,\mathbb{R}^n)$ we consider the norm induced by the Euclidean norm in $\mathbb{R}^n$. If (X,μ) and (Y,ν) are measure spaces, then $\mathcal{N}_b((X,\mu),(Y,\nu))$, or simply $\mathcal{N}_b(X,Y)$, stands for the set of all elements of $\mathcal{C}_b(X,Y)$ with Luzin's property (N).

The space $\mathcal{C}_b([a,b]^n,\mathbb{R}^n)$ will be also denoted by $\mathcal{C}([a,b]^n)$. The n-dimensional Lebesgue measure is denoted by λ_n. Unless specified otherwise we consider λ_n on $\mathbb{R}^n$ and $[a,b]^n$. We denote by $\mathcal{N}([a,b]^n)$ the set of all $f \in \mathcal{C}([a,b]^n)$ which have property (N).

We shall prove, in Theorem 3.1, that $\mathcal{N}([a,b])$ is a coanalytic non-Borel subset of $\mathcal{C}([a,b])$ and give several generalizations of this result.

Note that it was proved in [14] that the set of all functions f with the N^{-1}-property (i.e. $\lambda_1(f^{-1}(E))=0$ whenever $\lambda_1(E)=0$) is an $F_{\sigma\delta}$ first category subset of $\mathcal{C}([a,b])$.

The crucial observation for the proof of the coanalyticity of $\mathcal{N}([a,b])$ is the following fact due to N. Luzin ([7]).

If $f \in \mathcal{C}([a,b]) \setminus \mathcal{N}([a,b])$, then there exists a compact set $K \subset [a,b]$ such that $\lambda_1(K)=0$ and $\lambda_1(f(K))>0$.

This fact and its generalizations are easy consequences of the Choquet capacity theorem; this is the reason why we suppose that X is an analytic space in our generalizations of the result on coanalyticity of $\mathcal{N}([a,b])$.

We prove that $\mathcal{N}([a,b])$ is a non-Borel subset of $\mathcal{C}([a,b])$ by showing that it is even a complete coanalytic subset. (See Section 3 below for the definition and needed facts about $\mathcal{C}([a,b]$.) The proof of the complete coanalyticity of $\mathcal{N}([a,b])$ in $\mathcal{C}([a,b])$ is based on the well-known fact that "well-founded" trees (i.e. trees without infinite branches) form a complete coanalytic subset of the space of all trees on $\mathbb{N}$ and on a construction of a suitable embedding of the space of trees into $\mathcal{C}([a,b])$. The generalizations are proved by reduction to this simplest one-dimensional case.

We consider also the (topological) magnitude of the set of continuous mappings with property (N). The fact that $\mathcal{N}([a,b])$ is a subset of the first category in $\mathcal{C}([a,b])$ follows easily from the fact that each function from $\mathcal{N}([a,b])$ has a derivative (finite or infinite) at all points of an uncountable set (cf. [18], Chapter IX., Theorems 6.6. and 7.3.) and the well-known fact (cf. [1], Chapter 13, p. 143) that the set of all $f \in \mathcal{C}([a,b])$ which have a derivative (finite or infinite) at a point form a first category subset of $\mathcal{C}([a,b])$. We generalize the result on "category" of $\mathcal{N}([a,b])$ using the Banach-Mazur game.

2 Mappings with Property (N) Form a Co-Suslin set

We say that X is analytic if it is a metrizable space which is the image of a Polish (i.e. separable and completely metrizable) space P by a continuous surjection $\varphi : P \to X$.

In this section, X will be always an analytic space and P, φ ($\varphi : P \to X$) will be as above. We will also suppose that Y is a fixed Hausdorff topological space. Further, we suppose that X and Y are equipped with measures μ and ν, respectively, which have the following properties.

$$\mu \text{ and } \nu \text{ are completions of Borel measures.} \tag{1}$$

$$\mu(K) = \inf\{\mu(G) : K \subset G, G \text{ is open}\} \text{ if } K \subset X \text{ is compact.} \tag{2}$$

$$\nu(K) = \inf\{\nu(G) : K \subset G, G \text{ is open}\} < \infty \text{ if } K \subset Y \text{ is compact.} \tag{3}$$

Of course, if μ and ν are Radon measures (we consider only complete Radon measures), they satisfy the above conditions; we can use, as far as we know, any existing definition of a complete Radon measure on a topological space (cf. [19]).

Lemma 2.1. *Let (X, μ), (Y, ν) and $\varphi : P \to X$ be as above and let $f : X \to Y$ be a continuous mapping. Then f does not have property (N) iff there exists a compact set $K \subset P$ such that $\mu(\varphi(K)) = 0$ and $\nu(f(\varphi(K))) > 0$.*

PROOF. The sufficiency is obvious. To prove the necessity, we suppose that f does not have property (N). Then there exists a set $A \subset X$ such that $\mu(A) = 0$ and $\nu^*(f(A)) > 0$. According to (1), we may and will suppose that A is a Borel set. It is easy to verify that the outer measure ν^* is a capacity on Y. (We use here the definition of capacity given e.g. in [6], Definition 30.1.) Indeed, ν^* is clearly a regular outer measure and thus we can use 2.1.5 of [5]. Then also the function γ defined on $\exp(P)$ by the equation $\gamma(T) = \nu^*(f(\varphi(T)))$ is clearly a capacity (cf. [6], Example 2 of 30.B). Since $\gamma(\varphi^{-1}(A)) > 0$ and $\varphi^{-1}(A)$ is Borel, the Choquet capacity theorem ([6], Theorem 30.13) implies that there exists a compact set $K \subset \varphi^{-1}(A)$ such that $\gamma(K) = \nu(f(\varphi(K))) > 0$. Clearly $\mu(\varphi(K)) \leq \mu(A) = 0$. □

In the following we shall work with the hyperspace $\mathcal{K}$ of all compact subsets of P equipped with the Hausdorff metric. The induced topology on $\mathcal{K}$ coincides with the Vietoris topology ([6], 4.F, p. 24), in particular the set $\{K \in \mathcal{K} : K \subset U\}$ is open if $U \subset P$ is open.

Lemma 2.2. *The set $\{K \in \mathcal{K} : \mu(\varphi(K)) = 0\}$ is a G_δ subset of $\mathcal{K}$.*

PROOF. It is sufficient to prove that, for each $\alpha > 0$, the set $H_\alpha := \{K \in \mathcal{K} : \mu(\varphi(K)) < \alpha\}$ is open. Let $K_0 \in H_\alpha$ be fixed. Choose by (2) an open set $G \supset \varphi(K_0)$ such that $\mu(G) < \alpha$. Then $\{K \in \mathcal{K} : K \subset \varphi^{-1}(G)\}$ is an open subset of H_α containing K_0. □

Let us recall that a subset S of a topological space T is a Suslin set if it is the result of the Suslin operation (originally called the A-operation) applied to (a Suslin scheme of) closed subsets of T (cf. [6], Definition 25.4). If T is a Polish space, then $S \subset T$ is a Suslin set iff it is an analytic subset (i.e. an analytic subspace) of T ([6], p. 199). A subset of T is called co-Suslin if it is the complement of a Suslin set.

As we will investigate sets in several topological spaces of continuous mappings, we prove first the following general proposition.

Proposition 2.3. *Let X and Y be as above. Let $\mathcal{E}$ be a set of continuous mappings of X to Y equipped with a topology τ such that the following condition holds.*

(*) If $C \subset X$ is a compact set, $G \subset Y$ is an open set, $f \in \mathcal{E}$ and $f(C) \subset G$, then there exists an open set $V \supset C$ such that $\{g \in \mathcal{E} : g(V) \subset G\}$ is a τ-neighborhood of f.

Then the set $\mathcal{N}_\mathcal{E}$ of all mappings $f \in \mathcal{E}$ with property (N) is a co-Suslin subset of $\mathcal{E}$.

PROOF. Let P, φ and $\mathcal{K}$ be as above. Consider the topological product space $\mathcal{E} \times \mathcal{K}$ and the projection π of this space onto $\mathcal{E}$. By Lemma 2.1 we have $\mathcal{E} \setminus \mathcal{N}_\mathcal{E} = \pi(M)$, where

$$M = \{(f, K) \in \mathcal{E} \times \mathcal{K} : \mu(\varphi(K)) = 0 \text{ and } \nu(f(\varphi(K))) > 0\}.$$

We prove first that M is an $F_{\sigma\delta}$ subset of $\mathcal{E} \times \mathcal{K}$.

Since $\mathcal{K}$ is a metric space, Lemma 2.2 easily implies that

$$M_1 := \{(f, K) \in \mathcal{E} \times \mathcal{K} : \mu(\varphi(K)) = 0\}$$

is an $F_{\sigma\delta}$ set. Thus it is sufficient to prove that the set

$$\begin{aligned} M_2 :=& \{(f, K) \in \mathcal{E} \times \mathcal{K} : \nu(f(\varphi(K))) > 0\} \\ =& \bigcup_{n=1}^{\infty} \{(f, K) \in \mathcal{E} \times \mathcal{K} : \nu(f(\varphi(K))) \geq 1/n\} \end{aligned}$$

is an F_σ set. So it is clearly sufficient to show that the set

$$S_n := \{(f, K) \in \mathcal{E} \times \mathcal{K} : \nu(f(\varphi(K))) < 1/n\}$$

is open. To this end, choose an arbitrary $(f_0, K_0) \in S_n$. Choose by (3) an open set $G \supset f_0(\varphi(K_0))$ such that $\nu(G) < \frac{1}{n}$ and an open set $V \supset C := \varphi(K_0)$ by the condition (*). Let $Z := \{K \in \mathcal{K} : K \subset \varphi^{-1}(V)\}$. Then $T := \{g \in \mathcal{E} : g(V) \subset G\} \times Z$ is a neighborhood of (f_0, K_0) and clearly $T \subset S_n$. Since M is an $F_{\sigma\delta}$ (and therefore a Suslin) subset of $\mathcal{E} \times \mathcal{K}$ and $\mathcal{K}$ is a Polish space (cf. [6], p. 26), we obtain (cf. [16], Theorem 2.6.6) that $\mathcal{E} \setminus \mathcal{N}_{\mathcal{E}} = \pi(M)$ is a Suslin subset of $\mathcal{E}$ and we are done. □

As a consequence of Proposition 2.3 we easily obtain the following theorem which deals with three more concrete topological spaces $\mathcal{E}$ of continuous mappings.

Theorem 2.4. *Let X be an analytic space and let Y be a Hausdorff topological space. Let $\mathcal{E}$ be one of the topological spaces of continuous mappings of X to Y described in one of the following three cases:*

(i) Y is a metric space and $\mathcal{E} = C_b(X, Y)$ is the space of all bounded continuous mappings $f : X \to Y$ equipped with the supremum metric.

(ii) X is locally compact and $\mathcal{E} = \mathcal{C}(X, Y)$ is the set of all continuous mappings $f : X \to Y$ equipped with the compact-open topology; i.e. with the topology with a subbase

$$\{\{f \in \mathcal{E} : f(K) \subset G\} : K \subset X \text{ compact}, \ G \subset Y \text{ open } \}.$$

(iii) $\mathcal{E} = \mathcal{C}(X, Y)$ is equipped with the "closed-open" topology with a subbase formed by the sets of the form

$$\{f \in \mathcal{E} : f(F) \subset G\},$$

where $F \subset X$ is closed and $G \subset Y$ is open.

Let μ and ν be Radon measures on X and Y, respectively, or more generally, measures which satisfy conditions (1), (2), (3) from the beginning of this section. Then the set $\mathcal{N}_{\mathcal{E}}$ of all $f \in \mathcal{E}$ which have Luzin's property (N) is a co-Suslin subset of $\mathcal{E}$.

Proof. By Proposition 2.3 it is sufficient to check that $\mathcal{E}$ satisfies the condition $(*)$ in all three cases. To this end suppose that a compact set $C \subset X$, an open set $G \subset Y$ and $f \in \mathcal{E}$ such that $f(C) \subset G$ are given. Assign to each $x \in C$ an open neighborhood U_x such that

(a) $\operatorname{diam} f(U_x) < \frac{\varepsilon}{2}$, where $\varepsilon := \operatorname{dist}(f(C), Y \setminus G)$ in the case (i),

(b) $\overline{U_x} \subset f^{-1}(G)$ and $\overline{U_x}$ is compact in the case (ii) and

(c) $\overline{U_x} \subset f^{-1}(G)$ in the case (iii).

Since C is compact, we can find a finite set $F \subset C$ such that $V := \bigcup_{x\in F} U_x \supset C$. Since clearly

$$f \in \{g \in \mathcal{E} : g(\overline{V}) \subset G\} \subset E := \{g \in \mathcal{E} : g(V) \subset G\},$$

we see that E is a neighborhood of f in the cases (ii) and (iii). In the case (i) the $\frac{\varepsilon}{2}$-neighborhood of f is clearly a subset of E. Thus the condition (*) is satisfied and we are done. □

Corollary 2.5. *$\mathcal{N}([a,b]^n, \mathbb{R}^k)$ is a coanalytic subset of $\mathcal{C}([a,b]^n, \mathbb{R}^k)$ for all $n, k \in \mathbb{N}$.*

3 The Set of Functions with Luzin's Property (N) Need Not be Borel

We consider the set $\mathcal{N} = \mathcal{N}([0,1])$ of continuous functions with Luzin's property (N) as a subset of the space $\mathcal{C} = \mathcal{C}([0,1])$. By Corollary 2.5 we know that $\mathcal{N}$ is a coanalytic subset of $\mathcal{C}$. We are going to prove that $\mathcal{N}$ is complete coanalytic. Let us recall that C is a complete coanalytic subset of a Polish space P if C is coanalytic in P and given any coanalytic set $D \subset \mathbb{N}^{\mathbb{N}}$, there is a continuous map $f : \mathbb{N}^{\mathbb{N}} \to P$ such that $D = f^{-1}(C)$. Let us remark that the definition is not identical with Definition 22.9 in [6]. However, we may easily notice that these definitions are equivalent realizing that every zero-dimensional Polish space is homeomorphic to some closed subspace of $\mathbb{N}^{\mathbb{N}}$ ([6], Theorem 7.8).

We also use the two following facts. The first follows easily from the definition of complete coanalytic sets. The other uses also the well-known facts that there is a coanalytic subset of $\mathbb{N}^{\mathbb{N}}$ which is not Suslin, that every Borel subset of a metric space is Suslin and that continuous preimages of Suslin sets are Suslin.

(A) Let C be a Polish space and N be a coanalytic subset of C. Further, let L be a complete coanalytic subset of a Polish space T, $F : T \to C$ be continuous and let $F^{-1}(N) = L$. Then N is complete coanalytic in C.

(B) Every complete coanalytic subset of a Polish space P is not Suslin and thus not Borel in P.

Theorem 3.1. *The set $\mathcal{N}$ is complete coanalytic and so it is not Borel nor Suslin in $\mathcal{C}$.*

We do first some auxiliary considerations and constructions. We shall construct a suitable continuous (even homeomorphic) mapping to $\mathcal{C}$ of the compact metric space $\mathcal{T}$ of trees on $\mathbb{N}$ with the root $\emptyset$, i.e. the space of sets $T \subset \mathbf{S} = \bigcup_{k=1}^{\infty} \mathbb{N}^k \cup \{\emptyset\}$ such that $(n_1, \dots, n_{k+1}) \in T$ implies that $(n_1, \dots, n_k) \in T$ and $\emptyset \in T$, endowed with the compact metrizable topology induced by the topology of pointwise convergence of the characteristic functions $\chi_T : \mathbf{S} \to \{0, 1\}$ of T's from $\mathcal{T}$. As we already mentioned, we are going to use the following fact.

(C) The set $\mathcal{L}$ of "well founded" trees, i.e. the trees which have finite branches only, is complete coanalytic in $\mathcal{T}$ (cf. [6], Theorem 27.1).

First of all we construct one particular function $s : [0, 1] \to [0, 1]$ which will be the crucial tool to built up suitable $f^T \in \mathcal{C}$ for every $T \in \mathcal{T}$ at the end. Let D be a closed nowhere dense subset of $[0, 1]$ which has positive Lebesgue measure. The starting point of the construction of s is the function s_0 defined by $s_0(x) = \dfrac{1}{\lambda_1(D)} \displaystyle\int_0^x \chi_D(y) dy$, where χ_D is the characteristic function of D. Notice that s_0 is Lipschitz (with constant $\frac{1}{\lambda_1(D)}$), $s_0(0) = 0, s_0(1) = 1$ and $s_0([0, 1] \setminus D)$ is dense in $[0, 1]$.

We shall get s by modifications of s_0 done separately on a disjoint sequence of closed intervals in $[0, 1] \setminus D$ which we choose first. Since $s_0([0, 1] \setminus D)$ is dense in $[0, 1]$, we can choose (pairwise distinct) $y_n(k) \in [0, 1]$ and pairwise disjoint closed intervals $J_n^i(k) \subset [0, 1] \setminus D$ for $n \in \mathbb{N}$, $k \in \{1, \dots, 2^{n-1}\}$ and $i = 0, 1$, such that

$$\left| y_n(k) - \frac{2k-1}{2^n} \right| < \frac{1}{2^{n+1}} \text{ and } s_0(J_n^0(k)) = s_0(J_n^1(k)) = \{y_n(k)\}.$$

Let $L_n^i(k)$ be the closed left half, and $R_n^i(k)$ the closed right half, of $J_n^i(k)$. We define s by s_0 on $[0, 1] \setminus \bigcup \{J_n^i(k); n \in \mathbb{N}, k = 1, \dots, 2^{n-1}, i = 0, 1\}$. We define s to be an affine function on $L_n^i(k)$, and also on $R_n^i(k)$, such that

$$s(x) = \begin{cases} y_n(k) & \text{if } x \text{ is an endpoint of } J_n^i(k) \\ \frac{2k-2}{2^n} & \text{if } x \text{ is the midpoint of } J_n^0(k) \\ \frac{2k}{2^n} & \text{if } x \text{ is the midpoint of } J_n^1(k). \end{cases}$$

We see immediately that $s(\bigcup \mathcal{L}_n) = [0, 1]$, where $\mathcal{L}_n = \{L_n^i(k); i = 0, 1, k = 1, \dots, 2^{n-1}\}$ for every $n \in \mathbb{N}$. Also $s(0) = 0$ and $s(1) = 1$. Now s is a continuous function on $[0, 1]$ due to the fact that it is the uniform limit of the continuous functions s_n defined such that s_n equals s on $\bigcup_{m=1}^{n} \bigcup \mathcal{L}_m$ and s_n equals s_0 otherwise. By the definition of s, we have that $\sup_{x\in[0,1]} |s(x) - s_n(x)| \leq \frac{3}{2^{n+1}}$. Also s has Luzin's property (N) because s_0 is Lipschitz and s is Lipschitz on each of the countably many intervals $J_n^i(k)$. Put $L = \bigcup \bigcup_{n=1}^{\infty} \mathcal{L}_n$. The Lebesgue measure $\lambda(L)$ of L is less than $\frac{1}{2}$ because L consists of the left halves of $J_n^i(k)$ only.

We have proved the following lemma. It describes all properties of s which will be used below.

Lemma 3.2. *There are a continuous function* $s : [0,1] \to [0,1]$ *with Luzin's property (N) and finite families* $\mathcal{L}_n$ *of closed subintervals of* $[0,1]$ *such that*

(i) $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n$ *is a disjoint family*;

(ii) $q = \lambda(\bigcup \bigcup_{n=1}^{\infty} \mathcal{L}_n) < 1$;

(iii) $s(0) = 0, s(1) = 1$;

(iv) s *is affine on elements of* $\mathcal{L}_n$;

(v) $|s(\max I) - s(\min I)| \leq \frac{3}{2^{n+1}} \leq \frac{3}{4}$ *for* $I \in \mathcal{L}_n$*;*

(vi) $s(\bigcup \mathcal{L}_n) = [0,1]$ *for every* $n \in \mathbb{N}$.

Now we modify s by some similarity transformations.

Definition 3.3. *For* $a, b, c, d \in [0,1], a < b$, *we put*

$$s_{ab}^{cd}(x) = c + (d-c)\ s\left(\frac{x-a}{b-a}\right)$$

and $\Delta_{ab}^{cd}(x) = s_{ab}^{cd}(x) - [c + (d-c)\frac{x-a}{b-a}]$ *for* $x \in [a,b]$.

Notice that $s_{ab}^{cd}(a) = c, s_{ab}^{cd}(b) = d, \|\Delta_{ab}^{cd}\| \leq |d-c|$, $\Delta_{ab}^{cd}(a) = \Delta_{ab}^{cd}(b) = 0$ and $\Delta_{ab}^{cd}(x) \neq 0$ for some $x \in (a,b)$.

Let $\mathcal{L}_n(a,b)$ be the family obtained from $\mathcal{L}_n$ using the natural similarity transformation of $[0,1]$ onto $[a,b]$. For a fixed sequence $(n_1,\dots,n_k) \in \mathbf{S}$ we shall define a disjoint family $\mathcal{I}_{n_1,\dots,n_k}$ of closed intervals. Then we put $F_{n_1,\dots,n_k} = \bigcup \mathcal{I}_{n_1,\dots,n_k}$ and define a function $h_{n_1,\dots,n_k}$ on $F_{n_1,\dots,n_k}$. First of all, we put $\mathcal{I}_\emptyset = \{[0,1]\}$, $F_\emptyset = [0,1]$ and $h_\emptyset = s$.

Let $h_{n_1,\ldots,n_k}$ and $\mathcal{I}_{n_1,\ldots,n_k}$ be already defined. We define $\mathcal{I}_{n_1,\ldots,n_k,n_{k+1}}$ to be the family $\bigcup\{\mathcal{L}_{n_{k+1}}(A,B);[A,B]\in\mathcal{I}_{n_1,\ldots,n_k}\}$. For $x\in[a,b],[a,b]\in\mathcal{I}_{n_1,\ldots,n_k,n_{k+1}}$, using the notation $c=h_{n_1,\ldots,n_k}(a),d=h_{n_1,\ldots,n_k}(b)$, we put

$$h_{n_1,\ldots,n_k,n_{k+1}}(x)=h_{n_1,\ldots,n_k}(x)+\Delta_{ab}^{cd}(x)=s_{ab}^{cd}(x).$$

The last equality follows by the linearity of $h_{n_1,\ldots,n_k}$ on $[a,b]$. We observe that

$$\|\Delta_{ab}^{cd}\|\leq\frac{3}{2^{n_{k+1}+1}}|h_{n_1,\ldots,n_k}(A)-h_{n_1,\ldots,n_k}(B)|$$

by (v) of Lemma 3.2, with $[A,B]\in\mathcal{I}_{n_1,\ldots,n_k}$ and $[a,b]\in\mathcal{L}_{n_{k+1}}([A,B])$. We note that our construction yields the following lemma.

Lemma 3.4. *There are disjoint families $\{F_{n_1,\ldots,n_k};(n_1,\ldots,n_k)\in\mathbb{N}^k\}$, $k=0,1,\ldots,$ of compact subsets of $[0,1]$ and continuous functions $h_{n_1,\ldots,n_k}$ on $F_{n_1,\ldots,n_k}$ such that the following hold. (We use here that $h_{n_1,\ldots,n_0}\equiv h_\emptyset$ and $F_{n_1,\ldots,n_0}\equiv F_\emptyset$.)*

(a) $F_\emptyset=[0,1]$, $h_\emptyset=s$.

(b) $\lambda(F_{n_1,\ldots,n_k})\leq q^k$ *for some* $q<1$.

(c) $F_{n_1,\ldots,n_{k+1}}\subset F_{n_1,\ldots,n_k}$.

(d) $h_{n_1,\ldots,n_k}$ *has Luzin's property (N).*

(e) $h_{n_1,\ldots,n_k}(F_{n_1,\ldots,n_k})=[0,1]$.

(f) $h_{n_1,\ldots,n_k,n_{k+1}}(x)=h_{n_1,\ldots,n_k}(x)$ *for* $x\in\partial F_{n_1,\ldots,n_{k+1}}$.

(g) $|h_{n_1,\ldots,n_k,n_{k+1}}(x)-h_{n_1,\ldots,n_k}(x)|\leq\frac{3^{k+1}}{2^{n_1+\cdots+n_{k+1}+k+1}}\leq(\frac{3}{4})^{k+1}$ *for* $x\in F_{n_1,\ldots,n_{k+1}}$.

PROOF. The points (a) to (f) are obvious by the above construction.

The estimate (g) can be verified using Lemma 3.2 (v) inductively. Let $x\in F_{n_1,\ldots,n_{k+1}}$. Then, there is only one interval $[a_{k+1},b_{k+1}]\in\mathcal{I}_{n_1,\ldots,n_{k+1}}$ with $x\in[a_{k+1},b_{k+1}]$. Let $[a_l,b_l]\in\mathcal{I}_{n_1,\ldots,n_l}, l=0,\ldots,k+1$ be such that $[a_0,b_0]=[0,1]\supset[a_1,b_1]\supset\cdots\supset[a_{k+1},b_{k+1}]$. We estimate $\Delta_{a_lb_l}^{c_ld_l}(x)$ for $x\in[a_l,b_l]$ inductively for $l=0,1,\ldots,k+1$. Here $c_l=h_{n_1,\ldots,n_l}(a_l),d_l=h_{n_1,\ldots,n_l}(b_l)$ and we use the fact that $\|\Delta_{a_lb_l}^{c_ld_l}\|\leq|c_l-d_l|$ and $|c_l-d_l|\leq\frac{3}{2^{n_l+1}}|c_{l-1}-d_{l-1}|$. It follows from Lemma 3.2 (v), the observation preceding Lemma 3.4 and the fact that $\|\Delta_{a_{l-1}b_{l-1}}^{c_{l-1}d_{l-1}}\|\leq|c_{l-1}-d_{l-1}|$. □

We still need one more observation for the proof of Theorem 3.1.

Lemma 3.5. *Let $F_k, k \in \mathbb{N}$, be a nonincreasing sequence of compact spaces, $f_k, k \in \mathbb{N}$, be a sequence of continuous maps of F_1 to a metric space which converges uniformly to f. Then*

$$\bigcap_{k=1}^{\infty} f_k(F_k) \subset f(\bigcap_{k=1}^{\infty} F_k).$$

PROOF. Let $y \notin f(\bigcap_{k=1}^{\infty} F_k)$. Since f is continuous and since the F_k's are compact, the set $f(\bigcap_{k=1}^{\infty} F_k)$ is compact and there is a $\delta > 0$ such that the closed ball $\overline{B(y,\delta)}$ does not intersect $f(\bigcap_{k=1}^{\infty} F_k)$. By continuity of f there is an open $G \supset \bigcap_{k=1}^{\infty} F_k$ such that $f(G) \cap \overline{B(y,\delta)} = \emptyset$. The compactness and monotonicity of the sequence $F_k, k \in \mathbb{N}$, ensures the existence of some $k_0 \in \mathbb{N}$ with $F_{k_0} \subset G$. Thus $f(F_k) \cap \overline{B(y,\delta)} = \emptyset$ for $k \geq k_0$. For $k \geq k_0$ large enough we have $\|f_k - f\| < \delta$ and so $y \notin f_k(F_k)$. So $y \notin \bigcap_{k=1}^{\infty} f_k(F_k)$ and the lemma is proved. □

PROOF OF THEOREM 3.1. Let $F_{n_1,\dots,n_k}$ and $h_{n_1,\dots,n_k}$ be as in Lemma 3.4. We define, for a fixed $T \in \mathcal{T}$, a function f_k^T on $[0,1]$ so that $f_k^T(x) = h_{n_1,\dots,n_k}(x)$ for $x \in F_{n_1,\dots,n_k}$ if $(n_1,\dots,n_k) \in T$ and by $f_k^T = f_{k-1}^T$ otherwise. Due to (a) and the fact that each $h_{n_1,\dots,n_k}$ is defined on $F_{n_1,\dots,n_k} \subset [0,1]$ and $\{F_{n_1,\dots,n_k}; (n_1,\dots,n_k) \in \mathbb{N}^k\}$, $k = 0,1,\dots$ are disjoint families, the function is well defined on the interval $[0,1]$. Notice that, by (a), (f) and (g) for fixed k, each f_k^T is continuous. Define $f^T = \lim_{k\to\infty} f_k^T$. By (g), the limit is uniform. Hence $f^T \in \mathcal{C}$.

If $T \in \mathcal{L}$, i.e. $T \in \mathcal{T}$ and there are only finite branches in T, and if $x \in [0,1]$, there is an $(n_1,\dots,n_k) \in \mathbb{N}^k$ such that $f^T(x) = h_{n_1,\dots,n_k}(x)$. Since all these countably many functions $h_{n_1,\dots,n_k}$ have Luzin's property (N) by (d), f^T has Luzin's property (N), too.

If $T \in \mathcal{P}$, i.e. if there is some sequence $(n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $(n_1, n_2, \dots, n_k) \in T$ for every $k \in \mathbb{N}$, then $f_k^T(F_{n_1,\dots,n_k}) = [0,1]$ by (e) and the definition of f_k^T.

As the sets $F_{n_1,\dots,n_k}$ are compact, f_k^T are continuous functions which uniformly converge, we get, using (c) and (e) above, that $f^T(\bigcap F_{n_1,\dots,n_k}) = [0,1]$ by Lemma 3.5. Since $\lambda(F_{n_1,\dots,n_k}) \leq q^k$ by (b), we have that $\lambda(\bigcap_{k=1}^{\infty} F_{n_1,\dots,n_k}) = 0$ and so f^T does not have Luzin's property (N).

Now we show that the map is continuous. Suppose that two trees $T_1, T_2 \in \mathcal{T}$ have the same intersection with $\{1,\dots,N_0\}^{k_0}$. Let $x \in [0,1]$ be arbitrary. Let $S(x) = \{(n_1,\dots,n_k); x \in F_{n_1,\dots,n_k}\}$. We notice that there is a unique, finite or infinite, sequence $n_1(x), n_2(x), \dots$ such that the set $S(x)$ is the set of the finite initial subsequences of $n_1(x), n_2(x), \dots$ by the facts that $\{F_{n_1,\dots,n_k}; (n_1,\dots,n_k) \in \mathbb{N}^k\}$ is disjoint and $F_{n_1,\dots,n_{k+1}} \subset F_{n_1,\dots,n_k}$ for every $k \in \mathbb{N}$.

If $T_1 \cap S(x) = T_2 \cap S(x)$, then $f^{T_1}(x) = f^{T_2}(x)$. If $T_1 \cap S(x) \neq T_2 \cap S(x)$, then either there is $k_1 \leq k_0$ such that $n_{k_1}(x) \geq N_0$ and $(n_1(x),\dots,n_{k_1-1}(x)) \in T_1 \cap T_2$, or $(n_1(x),\dots,n_{k_0}(x)) \in T_1 \cap T_2$ and there is $k_1 > k_0$ such that $(n_1(x),\dots,n_{k_1}(x)) \in (T_2 \setminus T_1) \cup (T_1 \setminus T_2)$. So, either $|f^{T_1}(x) - f^{T_2}(x)| \leq \sum_{k=0}^{\infty} \frac{3}{2^{N_0+1}} (\frac{3}{4})^k$, or $|f^{T_1}(x) - f^{T_2}(x)| \leq \sum_{k=k_0}^{\infty} (\frac{3}{4})^k$ by (g). So, for any positive ε, $|f^{T_1}(x) - f^{T_2}(x)| < \varepsilon$ if k_0 and N_0 are sufficiently large and the map defined by $F(T) = f^T$ is continuous on $\mathcal{T}$. Hence, we have that $F : \mathcal{T} \to \mathcal{C}$ is a continuous mapping such that $F^{-1}(\mathcal{N}) = \mathcal{L}$. As $\mathcal{L}$ is complete coanalytic in $\mathcal{T}$ by (C), we conclude that $\mathcal{N}$ is complete coanalytic in $\mathcal{C}$ by (A). Thus $\mathcal{N}$ is not Borel in $\mathcal{C}$ by (B). □

Remark 3.6. We may easily notice that the map $T \in \mathcal{T} \mapsto f^T \in \mathcal{C}$ from the previous proof is a homeomorphic embedding. As we proved that it is continuous, it is enough to show that it is one-to-one.

Notice first that $h_{n_1,\dots,n_{k+1}}(x) \neq h_{n_1,\dots,n_k}(x)$ for some $x \in F_{n_1,\dots,n_{k+1}} \setminus \bigcup_{n\in\mathbb{N}} F_{n_1,\dots,n_{k+1},n}$, where the functions $h_{n_1,\dots,n_k}$ and the sets $F_{n_1,\dots,n_k}$ are those constructed in Lemma 3.4. Indeed, given any $[a,b] \in \mathcal{I}_{n_1,\dots,n_{k+1}}$, $h_{n_1,\dots,n_k}$ is affine on $[a,b]$ and the corresponding function Δ_{ab}^{cd} is not monotone on $[a,b]$.

The injectivity is now obvious from our construction. If $(n_1,\dots,n_k)$ belongs to one tree T and not to the other tree T', then the functions $f^T, f^{T'}$ differ at some point of $F_{n_1,\dots,n_k} \setminus \bigcup_{n\in\mathbb{N}} F_{n_1,\dots,n_k,n}$.

Now we extend the result of Theorem 3.1 to several more general situations. Before coming to more interesting cases, we make an easy observation.

Corollary 3.7. *The set $\mathcal{N}([0,1]^n, \mathbb{R}^n)$ is a complete coanalytic subset of $\mathcal{C}([0,1]^n, \mathbb{R}^n)$. In particular, it is not Suslin in $\mathcal{C}([0,1]^n, \mathbb{R}^n)$.*

Proof. By Theorem 2.4 the set $\mathcal{N}([0,1]^n, \mathbb{R}^n)$ is coanalytic. For each $g \in \mathcal{C}([0,1])$ put $T(g) = f$, where $f(x_1, x_2, \dots, x_n) = (g(x_1), x_2, \dots, x_n)$. Obviously, T is a homeomorphic embedding of $\mathcal{C}([0,1])$ into $\mathcal{C}([0,1]^n, \mathbb{R}^n)$ and $T(g) \in \mathcal{N}([0,1]^n, \mathbb{R}^n)$ if and only if $g \in \mathcal{N}([0,1])$ by the Fubini theorem. Hence, by Theorem 3.1 and the fact (A), $\mathcal{N}([0,1]^n, \mathbb{R}^n)$ is complete coanalytic. According to the fact (B), $\mathcal{N}([0,1]^n, \mathbb{R}^n)$ is not Suslin in $\mathcal{C}([0,1]^n, \mathbb{R}^n)$. □

The next lemma will be used to transfer the study of general situations to simpler ones. In Theorems 3.15 and 3.17 below we study $\mathcal{N}_b((X,\mu), \mathbb{R}^n)$ for several cases of (X,μ) and $n \in \mathbb{N}$. We reduce those cases using Lemma 3.8 to the study of $\mathcal{N}_b([0,1]^n, \mathbb{R}^n)$.

Lemma 3.8. *Let (I,ν) and (X,μ) be analytic spaces equipped with Radon measures such that there exists a continuous mapping h of X onto I satisfying*

(i) for every $A \subset I$ with $\nu(A)=0$, there exists $B \subset X$ such that $h(B)=A$ and $\mu(B)=0$;

(ii) $\nu(h(B))=0$ whenever $\mu(B)=0, B \subset X$.

Let $n \in \mathbb{N}$. If the set $\mathcal{N}_b(I,\mathbb{R}^n)$ is not Suslin in $\mathcal{C}_b(I,\mathbb{R}^n)$, then the set $\mathcal{N}_b(X,\mathbb{R}^n)$ is not Suslin in $\mathcal{C}_b(X,\mathbb{R}^n)$.

PROOF. We define a mapping T of $\mathcal{C}_b(I,\mathbb{R}^n)$ into $\mathcal{C}_b(X,\mathbb{R}^n)$ by

$$T : f \mapsto f \circ h.$$

The mapping T is clearly continuous and, according to (ii), $T(f) \in \mathcal{N}_b(X,\mathbb{R}^n)$ whenever $f \in \mathcal{N}_b(I,\mathbb{R}^n)$. Now we show that $T^{-1}(g) \subset \mathcal{N}_b(I,\mathbb{R}^n)$ whenever $g \in \mathcal{N}_b(X,\mathbb{R}^n)$. Suppose to the contrary that there exist $g \in \mathcal{N}_b(X,\mathbb{R}^n)$ and $f \in \mathcal{C}_b(I,\mathbb{R}^n) \setminus \mathcal{N}_b(I,\mathbb{R}^n)$ with $T(f)=g$. Then there exists a Borel set $A \subset I$ such that $\nu(A)=0$ and $\lambda_n(f(A))>0$. Due to (i) there exists $B \subset X$ such that $\mu(B)=0$ and $h(B)=A$. Thus we have $\lambda_n(g(B))>0$, a contradiction. Now we can conclude that $\mathcal{N}_b(X,\mathbb{R}^n)$ is not Suslin since continuous preimages of Suslin sets are Suslin and $\mathcal{N}_b(I,\mathbb{R}^n)=T^{-1}(\mathcal{N}_b(X,\mathbb{R}^n))$. □

In applications of the preceding lemma, we use the following easy observation.

Remark 3.9. Let $n \in \mathbb{N}$ and M be an analytic space with a Radon measure μ. Suppose that μ has a non-zero continuous part ν. It is easy to see that $\mathcal{N}_b((M,\mu),\mathbb{R}^n)=\mathcal{N}_b((M,\nu),\mathbb{R}^n)$.

When looking for the map h of Lemma 3.8, we use the following well-known properties of the distribution function of a non-atomic probability measure concentrated on a compact set $F \subset \mathbb{R}$.

Lemma 3.10. *Let μ be a non-zero non-atomic Radon measure on $\mathbb{R}$ and F be a compact subset of $\mathbb{R}$ with $\mu(F)>0$. Put $h(x)=\frac{1}{\mu(F)}\mu((-\infty,x)\cap F)$ for $x \in \mathbb{R}$. Then h is continuous and*

(i) $h(\mathbb{R})=h(F)=[0,1]$,

(ii) $\mu(h^{-1}(A)\cap F)=0$ whenever $\lambda_1(A)=0$,

(iii) $\lambda_1(h(B))=0$ whenever $\mu(B)=0$,

(iv) $\lambda_1(h(\mathbb{R}\setminus F))=0$.

PROOF. It is well-known (cf. [17], pp. 163–164) that h is non-decreasing, continuous (since μ is a non-atomic measure), and $\lambda_1|_{[0,1]}$ is the image of $\mu|_F$ under the function $h|_F$. Therefore h satisfies conditions (i) and (ii). Let $K = \operatorname{supp}(\mu|_F)$. Obviously, $K \subset F$ and (iv) follows from the fact that $h(\mathbb{R}\setminus F)$ is a subset of the countable set $h(\mathbb{R} \setminus K)$. To prove (iii), we may suppose that B is Borel and notice that h is injective on $K \setminus L$, where L is the set of left endpoints of intervals contiguous to K. Thus $\lambda_1(h(B)) = \mu|_F(h^{-1}(h(B)) \cap K) \leq \mu|_F((B \cup L) \cap K) = 0$. □

Lemma 3.11. *Let $n \in \mathbb{N}$ and X be an analytic space with a non-zero non-atomic Radon measure μ. Then there exist a compact set $K \subset X$ homeomorphic to the Cantor set with $\mu(K) > 0$ and a continuous mapping ψ of K onto $[0,1]^n$ with property (N) and property (N^{-1}), i.e. $\mu(\psi^{-1}(A)) = 0$ whenever $\lambda_n(A) = 0$.*

PROOF. Let $\mathbb{I}r = [0,1] \setminus \mathbb{Q}$. Fix a non-empty zero-dimensional compact set $D \subset \mathbb{I}r$ such that $\lambda_1(D \cap V) > 0$ for every open subset V of $\mathbb{I}r$ intersecting D. Put $\tilde{D} = \prod_{j=1}^{n} D$.

For every $x \in X$, the set $\{r > 0;\ \mu(\partial B(x,r)) = 0\}$ is dense in $(0,\infty)$ since μ is a Radon measure. Thus there exists a countable open basis $\mathcal{B}$ of the topology of X consisting of open balls with μ-zero boundaries. Put $\tilde{X} = X \setminus \bigcup\{\partial B;\ B \in \mathcal{B}\}$. The set $\tilde{X}$ is zero-dimensional and $\mu(\tilde{X}) = \mu(X)$. Now we can find a non-empty compact set $L \subset \tilde{X}$ such that $\mu(L \cap G) > 0$ for every open set $G \subset X$ intersecting L. The set L is zero-dimensional. Take a countable set $C \subset L$ dense in L and put $H = L \setminus C$. The sets $\mathbb{I}r^n$ and H are homeomorphic to $\mathbb{I}r$ (see [20], Theorem 1.2.5). The Oxtoby theorem from [9] says that, for a topological space Z with a Radon measure ν, there exists a homeomorphism ρ of Z onto $\mathbb{I}r$ with $\rho(\nu) = \lambda_1|_{\mathbb{I}r}$ if and only if

- Z is homeomorphic to $\mathbb{I}r$,
- $\nu(Z) = 1$,
- $\nu(V) > 0$ for every non-empty open subset of Z,
- ν is non-atomic.

It is not difficult to verify that this theorem gives that there exist a homeomorphism τ_1 of H onto $\mathbb{I}r$ such that $\tau_1(\frac{1}{\mu(H)}\mu|_H) = \lambda_1|_{\mathbb{I}r}$ and a homeomorphism τ_2 of $\mathbb{I}r$ onto $\mathbb{I}r^n$ such that $\tau_2(\lambda_1|_{\mathbb{I}r}) = \lambda_n|_{\mathbb{I}r^n}$. Put $\tau = \tau_2 \circ \tau_1$. The mapping τ is a homeomorphism of H onto $\mathbb{I}r^n$ such that $\tau(\frac{1}{\mu(H)}\mu|_H) = \lambda_n|_{\mathbb{I}r^n}$. Define $g : \mathbb{R} \to [0,1]$ by $g(x) = \frac{1}{\lambda_1(D)}\lambda_1((-\infty, x) \cap D)$. Using Fubini's theorem and Lemma 3.10, it is not difficult to show that the mapping $G : \tilde{D} \to [0,1]^n$ defined by

$$G(x_1, \ldots, x_n) = (g(x_1), \ldots, g(x_n)),\ (x_1, \ldots, x_n) \in \tilde{D},$$

maps $\tilde{D}$ onto $[0,1]^n$, has property (N) and property (N^{-1}). The desired compact K and the mapping ψ can be defined by $K = \tau^{-1}(\tilde{D})$ and $\psi = G \circ \tau|K$. Since the zero-dimensional compact set $\tilde{D}$ satisfies $\lambda_n(\tilde{D} \cap V) > 0$ for every open set $V \subset \mathbb{R}^n$ intersecting $\tilde{D}$, $\tilde{D}$ is dense in itself and therefore $\tilde{D}$ is homeomorphic to the Cantor set (see [20] Theorem 1.3.1). Thus the set K is homeomorphic to the Cantor set. □

We need the following notation. If A and B are elements of $\mathbb{R}^n$, then the closed segment with endpoints A and B is denoted by $[A, B]$.

Lemma 3.12. *Let P be a metric space. Let $n \in \mathbb{N}$ and G, G_0, G_1 be nonempty open subsets of P such that $\overline{G_0} \cup \overline{G_1} \subset G$, $\overline{G_0} \cap \overline{G_1} = \emptyset$. Let A, A_0, A_1 be points in $\mathbb{R}^n$. Then there exists a continuous mapping T of $D = \overline{G} \setminus (G_0 \cup G_1)$ to $\mathbb{R}^n$ such that*

(i) $T(D) \subset [A, A_0] \cup [A, A_1]$,

(ii) $T(\partial G) \subset \{A\}$,

(iii) $T(\partial G_i) \subset \{A_i\}$, $i = 0, 1$.

PROOF. The assertion follows immediately from the fact that $[A, A_0] \cup [A, A_1]$ is homeomorphic to an interval $[-K, K]$, $K \geq 0$, and from the Tietze theorem. □

The set of all finite (possibly empty) sequences of 0's and 1's will be denoted by $\mathbf{Y}$.

The construction of h, which is presented below, was influenced by [8] (where the idea of construction is attributed to L. Cesari, [2]).

Lemma 3.13. *Let μ be a non-zero non-atomic Radon measure on an analytic space X. Let $n \in \mathbb{N}$, $n \geq 2$. Then there exists a continuous mapping h of X onto $[0,1]^n$ such that*

(i) for every $A \subset [0,1]^n$ with $\lambda_n(A) = 0$, there exists $B \subset X$ such that $h(B) = A$ and $\mu(B) = 0$;

(ii) $\lambda_n(h(B)) = 0$ whenever $\mu(B) = 0$, $B \subset X$.

PROOF. According to Lemma 3.11 there exist a compact set $K \subset X$ homeomorphic to the Cantor set with $\mu(K) > 0$ and a continuous mapping ψ of

K onto $[0,1]^n$ satisfying property (N) and property (N^{-1}). Now the desired mapping h can be constructed as an arbitrary continuous extension of ψ defined on X with values in $[0,1]^n$ and having property (N), i.e. property (ii). Such a mapping h clearly satisfies property (i) automatically.

Since K is homeomorphic to the Cantor set we can easily find a scheme

$$\{G_{s_1,\dots,s_k};\ k \in \mathbb{N} \cup \{0\},\ (s_1,\dots,s_k) \in \mathbf{Y}\}$$

of open sets satisfying

(S1) $G_\emptyset = X$,

(S2) $K \subset \bigcup\{G_{s_1,\dots,s_k};\ (s_1,\dots,s_k) \in \mathbf{Y}\}$ for every $k \in \mathbb{N} \cup \{0\}$,

(S3) $\overline{G_{s_1,\dots,s_k,0}} \cup \overline{G_{s_1,\dots,s_k,1}} \subset G_{s_1,\dots,s_k}$,

(S4) $\overline{G_{s_1,\dots,s_k,0}} \cap \overline{G_{s_1,\dots,s_k,1}} = \emptyset$,

(S5) $G_{s_1,\dots,s_k} \cap K \neq \emptyset$ for every $k \in \mathbb{N} \cup \{0\}$, $(s_1,\dots,s_k) \in \mathbf{Y}$,

(S6) $\lim_{k\to+\infty} \operatorname{diam} G_{\nu_1,\dots,\nu_k} = 0$ for every $\nu \in \{0,1\}^{\mathbb{N}}$.

We pick an arbitrary point $x_{s_1,\dots,s_k}$ in $G_{s_1,\dots,s_k} \cap K$ for every $(s_1,\dots,s_k) \in \mathbf{Y}$. Fix $(s_1,\dots,s_k) \in \mathbf{Y}$. Lemma 3.12 shows that there exists a continuous mapping $f_{s_1,\dots,s_k}$ defined on $D_{s_1,\dots,s_k} = \overline{G_{s_1,\dots,s_k}} \setminus (G_{s_1,\dots,s_k,0} \cup G_{s_1,\dots,s_k,1})$ with values in $\mathbb{R}^n$ such that

(P1) $f_{s_1,\dots,s_k}(D_{s_1,\dots,s_k}) \subset [\psi(x_{s_1,\dots,s_k}), \psi(x_{s_1,\dots,s_k,0})] \cup [\psi(x_{s_1,\dots,s_k}), \psi(x_{s_1,\dots,s_k,1})]$,

(P2) $f_{s_1,\dots,s_k}(\partial G_{s_1,\dots,s_k}) \subset \{\psi(x_{s_1,\dots,s_k})\}$,

(P3) $f_{s_1,\dots,s_k}(\partial G_{s_1,\dots,s_k,i}) \subset \{\psi(x_{s_1,\dots,s_k,i})\},\ i = 0,1$.

We define a mapping $h : X \to [0,1]^n$ by

$$h(x) = \begin{cases} f_{s_1,\dots,s_k}(x), & x \in D_{s_1,\dots,s_k},\ k \in \mathbb{N} \cup \{0\},\ (s_1,\dots,s_k) \in \mathbf{Y}, \\ \psi(x), & x \in K. \end{cases}$$

The definition is correct. Indeed, $K \cap D_{s_1,\dots,s_k} = \emptyset$ for every $(s_1,\dots,s_k) \in \mathbf{Y}$ and if $a \in D_{u_1,\dots,u_l} \cap D_{v_1,\dots,v_m}$ for $l \geq m$, then $(u_1,\dots,u_l) = (v_1,\dots,v_m)$ or $l = m+1$ and $(u_1,\dots,u_l) = (v_1,\dots,v_m,i)$ for some $i \in \{0,1\}$. In the second case (P2) and (P3) give $f_{u_1,\dots,u_l}(a) = f_{v_1,\dots,v_m}(a)$.

The mapping h is continuous at each point of $X \setminus K$ by the continuity of each map $f_{s_1,\dots,s_k}$, (P2), (P3), and by the fact that for every $a \in X \setminus K$ there exist $(v_1, \dots, v_k) \in \mathbf{Y}$ and $i \in \{0, 1\}$ such that $a \in \operatorname{int}(D_{v_1,\dots,v_k} \cup D_{v_1,\dots,v_k,i})$. The continuity at points of K follows from the fact that ψ is continuous on K and from the condition (S6) and (P1).

Observe that

$$h(X \setminus K) \subset \bigcup \{[\psi(x_{s_1,\dots,s_k}), \psi(x_{s_1,\dots,s_k,i})];\ k \in \mathbb{N} \cup \{0\}, \\ (s_1, \dots, s_k) \in \mathbf{Y},\ i \in \{0,1\}\}.$$

This gives that $\lambda_n(h(X \setminus K)) = 0$. (Here we have used the assumption $n \geq 2$ which ensures that each segment is λ_n-null.) Thus h has property (N) and we are done. □

Remark 3.14. The previous lemma answers positively the following question (posed by P. Holický and L. Zajíček at 26th Winter School on Abstract Analysis). Does there exist a continuous mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ with property (N) such that $\lambda_2(f(\mathbb{R}^3)) > 0$?

Now we are prepared to generalize Theorem 3.1 and its Corollary 3.7.

Theorem 3.15. *Let $n \in \mathbb{N}$, $n \geq 2$, X be an analytic space, and μ be a Radon measure on X with a non-zero continuous part. Then $\mathcal{N}_b(X, \mathbb{R}^n)$ is a co-Suslin subset of $\mathcal{C}_b(X, \mathbb{R}^n)$ which is not Suslin. In particular, $\mathcal{N}_b(X, \mathbb{R}^n)$ is not Borel.*

PROOF. Theorem 2.4 gives that $\mathcal{N}_b(X, \mathbb{R}^n)$ is co-Suslin. According to Remark 3.9 we may assume that μ is a non-zero non-atomic Radon measure. Using Lemma 3.13, we obtain a mapping h of X onto $[0,1]^n$ satisfying the assumptions of Lemma 3.8. Therefore $\mathcal{N}_b(X, \mathbb{R}^n)$ is not Suslin since $\mathcal{N}([0,1]^n, \mathbb{R}^n)$ is not Suslin by Corollary 3.7. □

Remark 3.16. The assumption $n \geq 2$ in Theorem 3.15 cannot be omitted. Indeed, the set $\mathcal{N}_b(\mathbb{R}^2, \mathbb{R})$ contains only constant functions (Each non-constant function maps some segment onto a non-degenerate interval.) and therefore $\mathcal{N}_b(\mathbb{R}^2, \mathbb{R})$ is a closed subset of $\mathcal{C}_b(\mathbb{R}^2, \mathbb{R})$. Nevertheless, in Theorem 3.17, we prove that $\mathcal{N}_b(X, \mathbb{R})$ is not Suslin under some additional requirements on X and μ.

Theorem 3.17. *Let X be an analytic space with a measure μ satisfying one of the following conditions*

(i) $X \subset \mathbb{R}$ and μ is a Radon measure with a non-zero continuous part;

(ii) μ is a non-zero locally finite 1-dimensional Hausdorff measure;

(iii) X is a zero-dimensional space and μ is a Radon measure with a non-zero continuous part.

Then $\mathcal{N}_b(X, \mathbb{R})$ is a co-Suslin subset of $\mathcal{C}_b(X, \mathbb{R})$ which is not Suslin. In particular, $\mathcal{N}_b(X, \mathbb{R})$ is not Borel.

PROOF. Theorem 2.4 gives that $\mathcal{N}_b(X, \mathbb{R}^n)$ is co-Suslin in all cases above. Indeed, in the cases (i) and (iii) the considered measures are Radon. In the case (ii) we use separability of X and the fact that every analytic space is a Radon space ([19], Chap. II, Theorem 10) to prove that μ is a Radon measure.

Now we prove that $\mathcal{N}_b(X, \mathbb{R})$ is not Suslin in each of cases (i)–(iii) separately.

(i) There exists a compact set $F \subset X$ with $\mu(F) > 0$. We define a measure τ on $\mathbb{R}$ by $\tau(B) = \mu(B \cap F)$ for every Borel subset B of $\mathbb{R}$. Let σ be a completion of the measure τ. The measure σ is a Radon measure on $\mathbb{R}$ and Lemma 3.10 ensures the existence of a continuous mapping $h : \mathbb{R} \to [0, 1]$ satisfying (i) – (iv) in Lemma 3.10, where μ is replaced by σ. The restricted mapping $h|X$ of X onto $[0, 1]$ satisfies the assumptions of Lemma 3.8, where $(I, \nu) = ([0, 1], \lambda_1)$. Now Lemma 3.8 and Theorem 3.1 give that $\mathcal{N}_b(X, \mathbb{R})$ is not Suslin in $\mathcal{C}_b(X, \mathbb{R})$.

(ii) According to Lemma 3.11 there exist a compact set $K \subset X$ and a continuous mapping ψ of K onto $[0, 1]$ with property (N) and property (N^{-1}). There exists a continuous extension $h : X \to [0, 1]$, which is locally Lipschitz on $X \setminus K$. Indeed, this follows from Dugundji's construction of Φ from [4], Theorem 5.1. Notice that λ_U's used in the construction (defined on page 355) are locally Lipschitz. Lipschitz mappings map sets with zero 1-dimensional Hausdorff measure onto sets with zero 1-dimensional Hausdorff measure (cf. [15], p. 53). We can easily deduce that h has the same property since X is separable. We can conclude that h satisfies the assumptions of Lemma 3.8 with $(I, \nu) = ([0, 1], \lambda_1)$. Now Theorem 3.1 gives the conclusion.

(iii) The assertion follows from the case (i) above and from the fact that each zero-dimensional analytic space can be homeomorphically embedded into $\mathbb{R}$ (see [6], Theorem 7.8). □

4 The Category

In this section by a simplex we mean a closed (geometrical) n-simplex in $\mathbb{R}^n$. We suppose that n and a simplex $S \subset \mathbb{R}^n$ are fixed. By a figure we mean a finite disjoint union of n-simplexes. If a geometrical complex K is a subdivision of S, we denote by $\nu(K)$ the maximum of diameters of n-simplexes from K and by V_K the set of all vertices of simplexes from K. If a mapping $p : V_K \to \mathbb{R}^n$ is given, we denote by $f_p : S \to \mathbb{R}^n$ the unique (simplicial) extension of p which is affine on each n-simplex from K. The set of all mappings of the form f_p

(i.e. of "piecewise affine mappings of S to $\mathbb{R}^n$") will be denoted by $\mathcal{A}$. The Lebesgue measure of a set $A \subset \mathbb{R}^n$ will be denoted by $|A|$ in this section. We need several simple lemmas.

Lemma 4.1. *Let $f \in \mathcal{C}(S, \mathbb{R}^n)$ and $\varepsilon > 0$ be given. Then there exists a $\varphi \in \mathcal{A}$ such that $\|f - \varphi\| < \varepsilon$ and $|\varphi(S)| > 0$.*

Proof. We can clearly choose a subdivision K of S such that $\operatorname{diam} f(T) < \frac{\varepsilon}{2}$ for every n-simplex $T \in K$ and a mapping $p : V_K \to \mathbb{R}^n$ such that $\|f(v) - p(v)\| < \frac{\varepsilon}{2}$ for every $v \in V_K$ and the set $\{p(v) : v \in T_0\}$ is affinely independent for some n-simplex $T_0 \in K$. It is easy to see that $\varphi := f_p$ satisfies the assertion of the lemma. □

Lemma 4.2. *Let $\varphi \in \mathcal{A}$, a figure $F \subset S$, $c > 0$ and $\varepsilon > 0$ be given so that $|\varphi(F)| > c$. Then there exist $g \in \mathcal{A}$ and a figure $F^* \subset F$ such that $\|\varphi - g\| < \varepsilon$, $|F^*| < \varepsilon$ and $|g(F^*)| > c$.*

Proof. Consider a complex K^* which is a subdivision of S and $\nu(K^*) = \delta$. Let $\{S_1, \dots, S_k\}$ be the set of all n-simplexes $S^* \in K^*$ such that $S^* \subset F$ and φ is affine on S^*. It is easy to see that $\sum_{i=1}^k |\varphi(S_i)| > c$ and $\operatorname{diam} \varphi(S_i) < \varepsilon, i = 1, \dots k$, if δ is sufficiently small. Fix a K^* which corresponds to such a δ. Now consider a subdivision K of K^*. If $\nu(K)$ is sufficiently small, we can clearly choose pairwise disjoint n-simplexes $T_1, \dots, T_k$ from K such that $T_i \subset S_i$ and $\sum_{i=1}^k |T_i| < \varepsilon$. Let V_K be the set of all vertices of simplexes from K. Choose a $p : V_K \to \mathbb{R}^n$ such that

(i) $p(v) = \varphi(v)$ if v is vertex of no simplex $T_i, i = 1, \dots, k$, and

(ii) $p(V_{T_i}) = \varphi(V_{S_i}), i = 1, \dots, k$, where V_{T_i} and V_{S_i} are the sets of all vertices of T_i and S_i, respectively.

It is easy to verify that the corresponding simplicial mapping $g := f_p$ and $F^* = T_1 \cup \dots \cup T_k$ satisfy the assertion of the lemma. □

Lemma 4.3. *Let $g \in \mathcal{A}, c > 0$ and a figure $F \subset S$ be given so that $|g(F)| > c$. Then there exists $\delta > 0$ such that $|f(F)| > c$ whenever $f \in \mathcal{C}(S, \mathbb{R}^n)$ and $\|f - g\| < \delta$.*

Proof. We can clearly find n-simplexes $T_1, \dots, T_m$ such that $F = T_1 \cup \dots \cup T_m$, the sets $\operatorname{int} T_i, i = 1, \dots m$, are pairwise disjoint and g is affine on each T_i. Let $1 \le i \le m$ be fixed; put $\Omega = \operatorname{int} T_i$. If $|g(\Omega)| > 0$, then the topological degree $d(g, \Omega, y) = \operatorname{sgn} J_g(g^{-1}(y)) \neq 0$ for each $y \in \operatorname{int} g(T_i) = g(\Omega)$ (cf. [3], Definition 2.1). Let now $\delta > 0$, $f \in \mathcal{C}(S, \mathbb{R}^n)$ such that $\|f - g\| < \delta$ and $y \in \operatorname{int} g(\Omega)$ such that $\operatorname{dist}(y, g(\partial\Omega)) > \delta$ be given. Then Theorem 3.1 (d5) of [3] implies that $d(f,\Omega,y) \neq 0$ and therefore $y \in f(T_i)$ by Theorem 3.1,(d4) of [3]. Now it is easy to see that the assertion of the lemma holds for any sufficiently small δ. □

The main tool of this section is the Banach–Mazur game. It is the following infinite game between two players.

Let P be a metric space and let $M \subset P$ be given. In the first step the first player chooses an open ball $B(f_1,\varepsilon_1)$. In the second step the second player chooses an open ball $B(g_1,\delta_1) \subset B(f_1,\varepsilon_1)$, in the third step the first player chooses an open ball $B(f_2,\varepsilon_2) \subset B(g_1,\delta_1)$ and so on. If

$$\bigcap_{i=1}^{\infty} B(f_i,\varepsilon_i) \cap M = \emptyset,$$

then the second player wins. In the opposite case the first player wins.

We shall need the following result essentially due to Banach.

(BM) The second player has a winning strategy in the Banach–Mazur game if and only if M is of the first category in P.

A proof of this theorem can be found in [10] for the case $P = (0,1)$; the proof for the general case (cf. [11]) is essentially the same.

Proposition 4.4. *The set $\mathcal{N}_b(S,\mathbb{R}^n)$ of all mappings from $\mathcal{C}(S,\mathbb{R}^n)$ having property (N) is a first category subset of $\mathcal{C}(S,\mathbb{R}^n)$.*

PROOF. By (BM) it is sufficient to find a winning strategy for the second player of the Banach-Mazur game in the space $\mathcal{C}(S,\mathbb{R}^n)$ corresponding to the set $M := \mathcal{N}_b(S,\mathbb{R}^n)$.

We are now going to describe such a strategy. Suppose that the first player chose the ball $U_1 = B(f_1,\varepsilon_1)$ in his first move. Then the second player will choose a function $g_1 \in U_1 \cap \mathcal{A}$ such that $c := |g_1(S)| > 0$ in his first move; it exists by Lemma 4.1. Further, he will put $F_1 := S$ and by Lemma 4.3 choose a number $\delta_1 > 0$ such that $V_1 := B(g_1,\delta_1) \subset U_1$ and $|f(F_1)| > c$ for each $f \in V_1$.

In its m-th move, the second player will construct not only a ball $V_m = B(g_m,\delta_m)$ but also a figure F_m such that

$$|F_m| \leq \frac{1}{m}|S|, \tag{4}$$

$$|f(F_m)| > c \text{ whenever } f \in V_m, \text{ and} \tag{5}$$

$$F_m \subset F_{m-1} \text{ if } m > 1. \tag{6}$$

The conditions (4),(5) and (6) are clearly satisfied for $m = 1$.

Suppose now that $k > 1$ is given and $U_m = B(f_m, \varepsilon_m)$ are constructed for all $1 \leq m \leq k$ and $V_m = B(g_m, \delta_m), F_m$ are constructed for all $1 \leq m < k$ so that

$$U_1 \supset V_1 \supset \cdots \supset V_{k-1} \supset U_k$$

and conditions (4),(5) and (6) hold for each $1 \leq m \leq k$. Our aim is to construct $V_k = B(g_k, \delta_k)$ and a figure F_k such that $V_k \subset U_k$ and such that conditions (4),(5) and (6) hold for $m = k$. To this end, we choose an arbitrary $\varphi \in U_k \cap \mathcal{A}$ by Lemma 4.1. Since (5) holds for $k-1$, we have $|\varphi(F_{k-1})| > c$. Now we use Lemma 4.2 with $F = F_{k-1}$ and $\varepsilon > 0$ so small that $B(\varphi, \varepsilon) \subset U_k$ and $\varepsilon \leq \frac{1}{k}|S|$. We obtain a function $g = g_k \in \mathcal{A} \cap U_k$ and a figure $F^* = F_k \subset F_{k-1}$ such that $|g_k(F_k)| > c$ and $|F_k| < \frac{1}{k}|S|$. Further we use Lemma 4.3 with $g = g_k, F = F_k$ and obtain $\delta > 0$ such that $|f(F_k)| > c$ whenever $f \in B(g_k, \delta)$. We put $V_k := B(g_k, \delta_k)$, where $0 < \delta_k \leq \delta$ is chosen so small that $V_k \subset U_k$.

Thus the strategy for the second player is now well defined. To prove that it is a winning strategy, suppose that $f \in \bigcap_{k=1}^{\infty} V_k$ is given. Since conditions (4),(5) and (6) hold for every k, we obtain by Lemma 3.5 that $|f(\bigcap F_k)| \geq |\bigcap g_k(F_k)| \geq c > 0$ and therefore $f \notin \mathcal{N}_b(S, \mathbb{R}^n)$. □

Now we can prove the main result of this section.

Theorem 4.5. *Let $m \leq n$ be natural numbers and let X be a metric space equipped with a non-atomic Radon measure μ such that there exists an open set $G \subset X$ such that G is homeomorphic to $\mathbb{R}^n$ and $G \subset \operatorname{supp} \mu$. Then $\mathcal{N}_b(X, \mathbb{R}^m)$ is a first category subset of the space $C_b(X, \mathbb{R}^m)$.*

Proof. Using the Oxtoby-Ulam theorem (cf. [12]) we easily see that there exists a closed n-simplex $S \subset \mathbb{R}^n$ and a homeomorphism h of S onto an $F \subset G$ such that $h(\lambda_n) = \mu|_F$.

Suppose first that $m = n$. Then Proposition 4.4 implies that the set $\mathcal{N}_b(F, \mathbb{R}^n)$ of all mappings from $\mathcal{C}(F, \mathbb{R}^n)$ with property (N) is of the first category in $\mathcal{C}(F, \mathbb{R}^n)$. Now consider the restriction mapping $\mathcal{R} : \mathcal{C}_b(X, \mathbb{R}^n) \to \mathcal{C}(F, \mathbb{R}^n)$, $\mathcal{R}(f) = f|F$. Since $\mathcal{R}$ is continuous, linear and surjective, the open mapping theorem implies that $\mathcal{R}$ is an open mapping. Thus it is easy to see that $\mathcal{R}^{-1}(\mathcal{N}_b(F, \mathbb{R}^n))$ is of the first category in $\mathcal{C}_b(X, \mathbb{R}^n)$. Since clearly $\mathcal{N}_b(X, \mathbb{R}^n) \subset \mathcal{R}^{-1}(\mathcal{N}_b(F, \mathbb{R}^n))$, the case $m = n$ is settled.

In the case $m < n$ it is sufficient to observe that the canonical "projection" mapping $P : \mathcal{C}_b(X, \mathbb{R}^n) \to \mathcal{C}_b(X, \mathbb{R}^m)$ is a continuous and open surjection and therefore the P-image of the residual set $\mathcal{C}_b(X, \mathbb{R}^n) \setminus \mathcal{N}_b(X, \mathbb{R}^n)$ is residual as well (see [13], Lemma 4.25). By Fubini's theorem, this image is clearly disjoint with $\mathcal{N}_b(X, \mathbb{R}^m)$ and we are done. □

Remark 4.6. (a) Some analogues of Theorem 4.5 in spaces of unbounded continuous mappings also hold. For example, it is easy to prove that the set of all continuous mappings of $\mathbb{R}^n$ to $\mathbb{R}^n$ with Luzin's property (N) is of the first category in the space of all continuous mappings of $\mathbb{R}^n$ to $\mathbb{R}^n$ with the compact-open topology.

(b) In some cases, it is easy to see that the analogue of Theorem 4.5 does not hold. For example, $\mathcal{N}_b(X,Y)$ is residual in the space $\mathcal{C}_b(X,Y)$ in the following cases. (We suppose that X, Y are metric spaces equipped with Radon measures μ, ν, respectively.)

(i) X is a compact null-dimensional metric space and ν is non-atomic.

(ii) X is a compact subset of $\mathbb{R}^n$, $Y = \mathbb{R}^m$, $m > n$ and $\mu = \lambda_n, \nu = \lambda_m$ are Lebesgue measures.

In fact, in these cases it is not difficult to prove that

$$Z := \{f \in \mathcal{C}_b(X,Y) : \nu(f(X)) = 0\}$$

is a dense G_δ subset of $\mathcal{C}_b(X,Y)$.

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