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VARIATIONS ON A THEOREM ON RINGS OF CONTINUOUS FUNCTIONS

Abstract

We find variations of the classical theorem that any nonzero real ring homomorphism, u, of $C(X)$, for a compact Hausdorff space, X, is fixed. We let X be a locally compact Hausdorff space and we let u be defined on certain subrings of $C(X) >$ We also vary the hypothesis on u in other ways.

1 Introduction

For a topological space X, let $C(X)$ denote the ring of all continuous real valued functions on X, and let $C^*(X)$ denote the subring of all bounded functions in $C(X)$. If X is locally compact Hausdorff, let $C^{**}(X)$ denote the family $\{f \in C(X) : f - 1 \text{ vanishes at infinity}\}$. Hence $C^{**}(X) \subset C^*(X)$. (Note: we say that g vanishes at infinity [GJ, 7FG] if the set $\{x \in X : |g(x)| \geq 1/n\}$ is compact for all integers $n > 0$. We say that p is the limit of g at infinity if $g - p$ vanishes at infinity.)

We say that a real ring homomorphism u of a subring S of $C(X)$ is fixed [GJ, 10.5], if for some $x_0 \in X$, $f(x_0) = u(f)$ for all $f \in S$. A completely regular Hausdorff space X is said to be realcompact [GJ, 10.5], if every nonzero real ring homomorphism on $C(X)$ is fixed. Perhaps the best known result on realcompact spaces is [K, 7S], [GJ, 8.2].

Proposition A. *Every compact Hausdorff space X is realcompact.*

In Section 4 we will discover a proof of Proposition A that does not require the use of ideals. Unfortunately a locally compact Hausdorff space need not be realcompact. The standard counterexample [GJ, 5.12] is the space X of

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countable ordinal numbers under the order topology. If however we define a binary operation $*$ from $\mathbb{R}^2$ to $\mathbb{R}$ by

$$a * b = a - ab$$

we can achieve something that resembles realcompactness in a limited way, for locally compact spaces X. In Theorem 2.1 we let u be a nonzero real function preserving $*$,

$$u(f - fg) = u(f) - u(f)u(g)\,,$$

on a subring S of $C(X)$ containing $C^{**}(X)$ such that u is nonconstant on $C^{**}(X)$, and we prove that u is fixed. Note that our theorem requires that u be nonconstant on $C^{**}(X)$, and that u preserve only the one operation $*$, rather than the two, addition and multiplication. Moreover, note that the subring S need not be $C(X)$ or $C^*(X)$, provided $C^{**}(X) \subset S$.

In Theorem 3.1 we let u be a nonzero real valued function on a subring S of $C(Y)$ containing $C^*(Y)$, where Y is any topological space. We let u preserve $*$ and prove that u must be a ring homomorphism on S.

This differs from Theorem 2.1 in that we have all the bounded continuous functions at our disposal in S. Note there is no restriction on the space Y. If for example, Y is discrete, then Theorem 3.1 is just a statement about real valued functions on the set Y. In Section 3 we also offer some applications to realcompact spaces.

In Section 4 we let X be compact Hausdorff and regard $C(X)$ to be a real vector space. We let L be a linear subspace of $C(X)$ containing 1, and let u be a nonzero real linear functional on L with $u \geq 0$ and $u(1) = 1$. In Theorem 4.1 we prove that under certain hypotheses u must be fixed. We conclude with some examples arising from Theorem 4.1.

2 The Locally Compact Hausdorff Case

We begin with the following assertion.

Theorem 2.1. *Let X be a locally compact Hausdorff space, and let S be a subring of $C(X)$ containing $C^{**}(X)$. Let u be a real valued function on S that is not constant on $C^{**}(X)$ such that*

$$u(f * g) = u(f) * u(g) \qquad (f, g \in S)\,.$$

Then u is a real fixed ring homomorphism on S. Moreover, u extends to a unique real ring homomorphism on $C(X)$.

We will prove fairly easily that u preserves multiplication. Most of the work will be in the remainder of the proof, where we will use the one-point compactification to establish that u preserves addition.

PROOF OF THEOREM 2.1. We have by hypothesis $u(0) = u(0 - 0^2) = u(0) - u(0)^2$ and hence $u(0) = 0$. Also $0 = u(0) = u(1 - 1^2) = u(1) - u(1)^2$; so $u(1)$ is either 0 or 1. But $u(1) = 0$ is impossible because otherwise

$$u(1 - f) = u(1 - 1f) = u(1) - u(1)u(f) = 0$$

for any $f \in S$, and consequently u would vanish on S and $C^{**}(X)$, contrary to hypothesis. Hence $u(1) = 1$. Now for any $f \in S$

$$u(1 - f) = u(1 - 1f) = u(1) - u(1)u(f) = 1 - u(f)\,.$$

For any $f, g \in S$,

$$u(fg) = u\big(f - f(1 - g)\big) = u(f) - u(f)u(1 - g) =$$

$$= u(f) - u(f)\big(1 - u(g)\big) = u(f)u(g)\,.$$

For $f, g \in S$ we have established

(i) $u(fg) = u(f)u(g)$,

(ii) $u(1 - f) = 1 - u(f)$.

Let $\{p\} \cup X$ be the one point compactification of X where p is the point at infinity. Let D denote the family of all continuous functions f from $\{p\}\cup X$ into $[0, 1]$ with $f(p) = 1$. Essentially $D \subset C^{**}(X)$. Let $1 - D$ denote $\{1 - f : f \in D\}$. Now $D\cup(1-D)$ is closed under multiplication ($f, g \in D\cup(1-D)$ implies $fg \in D \cup (1 - D)$), and is closed under complementation ($f \in D \cup (1 - D)$ implies $1 - f \in D \cup (1 - D)$). Moreover (i) and (ii) hold for functions in $D \cup (1 - D)$.

Let $f \in D$. By (ii), $u(1-f) = 1-u(f)$. But $f^{\frac{1}{2}} \in D$ and $(1-f)^{\frac{1}{2}} \in 1-D$. Then by (i), $u(f) = u(f^{\frac{1}{2}})^2 \geq 0$ and $u(1 - f) = u\big((1 - f)^{\frac{1}{2}}\big)^2 \geq 0$. Clearly u maps $D \cup (1 - D)$ into $[0, 1]$. It follows from [C, Theorem 1] that there is an $x_0 \in \{p\} \cup X$ such that $f(x_0) = u(f)$ for all $f \in D \cup (1 - D)$. Our next task is to prove that $x_0 \neq p$.

Let $h \in C^{**}(X)$. Put $h(p) = 1$. Let k be a function in D (with $k(p) = 1$) that vanishes on the closed set $\{x \in X : h(x) \leq 1/2\}$ such that $k \leq 1/h$ on the support of k. Then $hk \in D$ and $u(hk) = u(h)u(k)$. Now if $x_0 = p$, then

$$1 = (hk)(p) = u(hk) = u(h)u(k) \quad \text{and} \quad 1 = k(p) = u(k)\,.$$

Hence $u(h) = 1$, contrary to the hypothesis that u is not constant on $C^{**}(X)$. We have established that $x_0 \in X$.

Let $f \in S$ with $f(x_0) > 0$. Let f_1 be a function in $1 - D$ such that f_1 vanishes on the closed set $\{x \in X : f(x) \leq f(x_0)/2\}$, $f_1(x_0) > 0$, and $f_1 \leq 1/f$ on the support of f_1. Then $ff_1 \in 1 - D$. So by (i),

$$f(x_0)f_1(x_0) = (ff_1)(x_0) = u(ff_1) = u(f)u(f_1) = u(f)f_1(x_0)\,,$$

and (because $f_1(x_0) \neq 0$) $u(f) = f(x_0)$. Let $f_0 \in S$ with $f_0(x_0) \leq 0$. By applying (ii) and this argument to $1 - f_0$ we see that

$$1 - f_0(x_0) = u(1 - f_0) = 1 - u(f_0) \quad \text{and} \quad u(f_0) = f_0(x_0)\,.$$

Thus $g(x_0) = u(g)$ for all $g \in S$. Hence $(g_1 + g_2) = u(g_1) + u(g_2)$ for $g_1 \in S,\ g_2 \in S\,$.

Now u extends to an obvious ring homomorphism of $C(X)$ to $\mathbb{R}$. If v is any other ring homomorphism of $C(X)$ to $\mathbb{R}$ that extends u, the argument in the preceding paragraph shows that $g(x_0) = v(g)$ for all $g \in C(X)$. □

Corollary 2.2. *Let S and X be as in Theorem 2.1. Let W be any space, and let U be a mapping from S into $C(W)$ such that $U(C^{**}(X))(w)$ is not a singleton set for any $w \in W$, and $U(f * g) = U(f) * U(g) \quad (f, g \in S)$. Then there is a continuous function t from W into X such that $U(f)(w) = f(t(w)) \quad (f \in S,\ w \in W)$.*

This is proved by an argument like the solution to [K, problem 7S, pp. 245-6].

A real valued mapping preserving operation $*$ on an arbitrary commutative ring with identity need not be a ring homomorphism. For example, consider the mapping on the ring of integers taking even integers to 0 and odd integers to 1. This inspires the following corollary.

Corollary 2.3. *Let Y be a compact space and let $p \in Y$. Let S be the family $\{f \in C(Y) : f(p)$ is an integer$\}$. Let u be a nonzero mapping from S into $\mathbb{R}$ such that*

$$u(f * g) = u(f) * u(g) \qquad (f, g \in S)\,.$$

Then either u is fixed, or $u(f) = 0$ when $f(p)$ is even and $u(f) = 1$ when $f(p)$ is odd.

Proof. First note that $C^{**}(X) \subset S$. Our proof then proceeds just like the proof of Theorem 2.1 until (i) and (ii) are proved, D and $1-D$ are defined, and x_0 is found. If $x_0 \neq p$, the conclusion follows just as in the proof of Theorem

2.1. It suffices then to let $x_0 = p$. As before, $u(1) = 1$ and $u(0) = 0$. Now $u(-1)^2 = u(1) = 1$ by (i); so $u(-1) = 1$ or -1.

Case 1. $u(-1) = -1$. Here $u(2) = u\big(1-(-1)\big) = 1 - u(-1) = 2$ by (ii). By (i), $u(-2) = u(-1)u(2) = -2$. So $u(3) = u\big(1-(-2)\big) = 1-u(-2) = 3$. Similarly, $u(-3) = -3$ and $u(4) = 4$. Clearly an induction argument will establish that $u(n) = n$ for any integer, positive, negative or zero.

Let $f \in S$ with $f(p) = 1$. Let $f_0 \in D$ such that f_0 vanishes on the closed set $\{x \in X : f(x) \leq 1/2\}$ and $f_0 \leq 1/f$ on the support of f_0. Then $ff_0 \in D$ and $f_0 \in D$; so by (i) $f(p) = 1 = (ff_0)(p) = u(ff_0) = u(f)u(f_0) = u(f)$. Now let $g \in S$ and $g(p) = n$ where n is a nonzero integer. We have $n^{-1}g \in S$, $(n^{-1}g)(p) = 1$, and by the preceding paragraph, $u(n^{-1}g) = 1$. Finally, $u(g) = u(n)u(n^{-1}g) = n$ by (i). If $h \in S$ and $h(p) = 0$, then $1 = u(1-h) = 1-u(h)$ because $(1-h)(p) = 1$. So $u(h) = 0$. Thus in Case 1, u is fixed.

Case 2. $u(-1) = 1$. Here $u(2) = u\big(1-(-1)\big) = 1-u(-1) = 0$. Thus if $f \in S$ and $f(p)$ is even, then $f/2 \in S$ and by (i), $u(f) = u(2)u(f/2) = 0$. If $g(p)$ is odd, then $(1-g)(p) = 1-g(p)$ is even and hence $0 = u(1-g) = 1-u(g)$ and $u(g) = 1$. □

Note that of all the mappings u possible in Corollary 2.3, the two associated with the point p have countable range and all the others have uncountable range. Now let V be a locally compact Hausdorff, noncompact space and let S_V denote the family $\{f \in C(V)$: the limit of f at infinity is an integer$\}$. Arguments similar to the solution to [K, problem 7S(e), p. 246] prove that the algebraic structure $(S_V, *)$ determines the space V in the following sense. If V_1 and V_2 are such spaces and if $(S_{V_1}, *)$ and $(S_{V_2}, *)$ are isomorphic, then V_1 and V_2 are homomorphic. Of course the ring S_V also determines V, but again the one operation $*$ suffices.

3 Arbitrary Spaces and Realcompact Spaces

We first establish the following assertion about arbitrary spaces.

Theorem 3.1. *Let Y be any topological space and let S be any subring of $C(Y)$ that contains $C^*(Y)$. Let u be a nonzero real valued function on S such that $u(f*g) = u(f)*u(g) \quad (f, g \in S)$. Then u is a real ring homomorphism on S.*

We begin the proof with a lemma.

Lemma. *Let Y be any space and let S be a subring of $C(Y)$ such that $C^*(Y) \subset S$. Let $f \in S$. Then* $\max(0, f) \in S$ *and* $\min(0, f) \in S$.

Proof. Let $g = \min\big(1, \max(0,f)\big)$. Then $g \in C^*(Y)$ and hence $g \in S$. Observe that fg vanishes on the set $\{y \in Y : f(y) \le 0\}$ and fg coincides with f on the set $\{y \in Y : f(y) \ge 1\}$. It follows that $f - fg = \min(0,f) + h$ where $h \in C^*(Y) \subset S$. But $f - fg \in S$; so $\min(0,f) \in S$. Finally, $\max(0,f) = f - \min(0,f) \in S$. □

Proof. [Proof of Theorem 3.1] Just as in the proof of Theorem 2.1 we prove that $u(0) = 0$, $u(1) = 1$ and prove that (i) and (ii) hold. We claim that $u(2) \neq 0$; for otherwise, $1 = u(1) = u(2)u(1/2) = 0$. But by (i), $1 = u(1) = u(-1)^2$ and $u(-1) = 1$ or -1. We claim that $u(-1) \neq 1$; for otherwise $u(2) = u\big(1-(-1)\big) = 1 - u(-1) = 0$ by (ii). Hence $u(-1) = -1$. If $g \in S$, then by (i), $u(-g) = u(-1)u(g) = -u(g)$.

In what follows, k_1 and k_2 are in S and $k_1 \ge 0$. Then $(1+k_1)^{-1} \in C^*(Y) \subset S$ and $(1+k_1)^{-1}(-k_2) \in S$. By (i), (ii) and $u(-k_2) = -u(k_2)$,

$$u\big(1+k_1-(1+k_1)(1+k_1)^{-1}(-k_2)\big) = u(1+k_1) - u(-k_2) = u(1+k_1) + u(k_2)\,.$$

Moreover $u\big(1-(-k_1)\big) = 1 - u(-k_1) = 1 + u(k_1)$. Combining,

$$u(1+k_1+k_2) = 1 + u(k_1) + u(k_2)\,. \tag{1}$$

Again by (ii), $u(1+k_1+k_2) = 1 - u(-k_1-k_2) = 1 + u(k_1+k_2)$. Thus

$$u(1+k_1+k_2) = 1 + u(k_1+k_2)\,. \tag{2}$$

From (1) and (2) we obtain $1 + u(k_1) + u(k_2) = 1 + u(k_1+k_2)$, or

$$u(k_1+k_2) = u(k_1) + u(k_2)\,. \tag{3}$$

Now let $f, g \in S$. Say $f_+ = \max(0,f)$, $g_+ = \max(0,g)$, $f_- = -\min(0,f)$, $g_- = -\min(0,g)$. Then f_+, g_+, f_-, g_- are all nonnegative, $f = f_+ - f_-$, $g = g_+ - g_-$, and by the Lemma, f_+, g_+, f_-, g_- are all in S. By the preceding paragraph,

$$\begin{aligned} u(f+g) &= u(f_+ + g_+ - f_- - g_-) \\ &= u(f_+ + g_+) + u(-f_- - g_-) \\ &= u(f_+ + g_+) - u(f_- + g_-) \\ &= u(f_+) + u(g_+) - u(f_-) - u(g_-)\,. \end{aligned}$$

Likewise

$$\begin{aligned} u(f) + u(g) &= u(f_+ - f_-) + u(g_+ - g_-) \\ &= u(f_+) + u(-f_-) + u(g_+) + u(g_-) \\ &= u(f_+) + u(g_+) - u(f_-) - u(-g_-)\,. \end{aligned}$$

By comparing, we obtain $u(f+g) = u(f) + u(g)$. □

This corollary is almost immediate.

Corollary 3.2. *Let Y be a realcompact space, let u be a real valued function on $C(Y)$ with $u(1) = 1$ satisfying $u(f * g) = u(f) * u(g) \quad \big(f, g \in C(Y)\big)$. Then u is a fixed ring homomorphism of $C(Y)$ into $\mathbb{R}$.*

Proof. That u is a ring homomorphism on $C(Y)$ follows from Theorem 3.1. Because Y is realcompact, u is fixed [GJ, 10.5]. □

Corollary 3.3. *Let X be realcompact. Let W be any space, and let U be a mapping from $C(X)$ into $C(W)$ such that $U(1) = 1$ and*

$$U(f * g) = U(f) * U(g) \quad \big(f, g \in C(X)\big).$$

Then there is a continuous function t from W into X such that

$$U(f)(w) = f\big(t(w)\big) \quad \big(f \in C(X),\, w \in W\big).$$

The proof is similar to the solution of [K, problem 7S(e), p. 246]; so we leave it. We note the algebraic structure $\big(C(V), *\big)$ determines a realcompact space V in the following way. If V_1 and V_2 are realcompact spaces and if $\big(C(V_1), *\big)$ and $\big(C(V_2), *\big)$ are isomorphic, then V_1 and V_2 are homomorphic. Of course the ring $C(V)$ also determines V, but again the one operation $*$ suffices.

4 Compact Hausdorff Spaces

Here we prove the following theorem.

Theorem 4.1. *Let X be a compact Hausdorff space and regard $C(X)$ to be a real vector space. Let L be a linear subspace of $C(X)$ containing 1, and let u be a nonzero real linear functional on L with $u \geq 0$ and $u(1) = 1$. Let*

$$K = \big\{f \in L : f^2 \in L \text{ and } u(f^2) = (u(f))^2\big\},$$

and let A be the smallest uniformly closed subalgebra of $C(X)$ containing K. Then there is an $x_0 \in X$ such that $u(f) = f(x_0)$ for all $f \in L \cap A$. Moreover, if K separates points in X, then $u(f) = f(x_0)$ for all $f \in L$.

Proof. Let $f_1, f_2, \ldots, f_n$ be finitely many functions in K. Let

$$F = \big(f_1 - u(f_1)\big)^2 + \big(f_2 - u(f_2)\big)^2 + \ldots + \big(f_n - u(f_n)\big)^2.$$

Now

$$u\Big(\big(f_i - u(f_i)\big)^2\Big) = u\Big(f_i^2 - 2u(f_i)f_i + u(f_i)^2\Big)$$
$$= u(f_i)^2 - 2u(f_i)^2 + u(f_i)^2 = 0$$

by the linearity of u, and it follows that $u(F) = 0$. But $F \geq 0$ on X.

We claim that F vanishes at some point in X. Assume not. By compactness, there is a number $d > 0$ such that $F \geq d$ on X. Because $u \geq 0$, we have $u(F) \geq u(d) = d > 0 = u(F)$, which is impossible. So there is some point $x_1 \in X$ with $F(x_1) = 0$. But then $f_1(x_1) = u(f_1)$, $f_2(x_1) = u(f_2), \ldots, f_n(x_1) = u(f_n)$. It follows then that for each $f \in K$, there is a nonvoid closed set $X_f \subset X$ with $f(x) = u(f)$ for $x \in X_f$. The family X_f $(f \in K)$ has the finite intersection property; so by compactness there is a point $x_0 \in \cap_{f \in K} X_f$. Thus $f(x_0) = u(f)$ for all $f \in K$. It remains to prove that $g(x_0) = u(g)$ for all $g \in L \cap A$. (Observe that we now have a proof of Proposition A without ideals.)

Let $V = \big\{f \in L : f(x_0) = u(f)\big\}$. Then $K \subset V$ and indeed if $f \in K$ then $f \in V$ and $f^2 \in V$. Moreover, if $f \in K$ and a is a real number, then

$$u\Big(a\big(f - u(f)\big)^2\Big) = au\Big(f^2 - 2u(f)f + u(f)^2\Big)$$
$$= a\Big(u(f)^2 - 2u(f)^2 + u(f)^2\Big) = 0\,.$$

Select $g \in L \cap A$ and let c be a number $> g(x_0)$. The set $P = \{x \in X : g(x) \geq c\}$ is a (closed) compact subset of X and $x_0 \notin P$. Take $y \in P$. Then g separates x_0 and y, and because $g \in A$ there exists a positive number a and $f \in K$ such that $a\big(f(y) - f(x_0)\big)^2 > g(y) - c$. Because P is compact, there are finitely many positive numbers $a_1, a_2, \ldots, a_n$ and functions $f_1, f_2, \ldots, f_n \in K$ such that

$$a_1\big(f_1 - u(f_1)\big)^2 + a_2\big(f_2 - u(f_2)\big)^2 + \ldots + a_n\big(f_n - u(f_n)\big)^2 > g - c$$

on P. Put

$$G = c + \sum_{i=1}^{n} a_i\big(f_i - u(f_i)\big)^2 \in V\,.$$

Then $G \geq g$ on X and $u(G) = c$. Because $u \geq 0$ we have $c = u(G) \geq u(g)$. But c $(> g(x_0))$ was arbitrary; so $g(x_0) \geq u(g)$. By the same kind of argument $-g(x_0) \geq u(-g) = -u(g)$; so finally $g(x_0) = u(g)$.

If K separates points in X, then $A = C(X)$ by the Stone-Weierstrass Theorem [GJ, (16.4)], and $g(x_0) = u(g)$ for all $g \in L$. □

(Observe that we used compactness on three occasions in this proof; once to show that F vanishes at some point, once to obtain x_0, and finally to obtain the function G.)

We turn to Borel measures.

Corollary 4.2. *Let X be a compact Hausdorff space and let m be a nonnegative regular Borel measure on X with $m(X) = 1$. Let*

$$K = \left\{ f \in C(X) \,:\, \left(\int f\, dm \right)^2 = \int f^2\, dm \right\}$$

and let K separate points in X. Then there is a singleton subset of X whose complement has measure zero.

We omit the proof. Note that if x_0 is a point satisfying $f(x_0) = \int f\, dm$ for $f \in C(X)$, and if $h \in C(X)$ such that $0 \le h \le 1$ on X, $h(x_0) = 0$ and $h = 1$ on a compact set P not containing x_0, we see that $m(P) = 0$.

Example 1. Let L be the linear space spanned by the functions 1, $\cos(nx)$ $(n = 1, 2, 3, \ldots)$ on $[0, \pi]$. Let u be a nonnegative linear functional on L with $u(1) = 1$. Let

$$u\big(\cos(2x)\big) = 2\big(u(\cos x)\big)^2 - 1\,.$$

Then u is fixed on L.

Proof. We have

$$u\big(\cos(2x)\big) = u(2\cos^2 x - 1) = 2u\big(\cos^2 x\big) - 1 = 2\big(u(\cos x)\big)^2 - 1\,,$$

and hence $u\big(\cos^2 x\big) = \big(u(\cos x)\big)^2$. The rest follows from Theorem 4.1. □

Example 2. Let W be the compact square $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$. Let u be a nonnegative linear functional on $C(W)$ with $u(1) = 1$. Let $g(x, y) = x$, $h(x, y) = y$ on W. Let $\big(u(g)\big)^2 = u(g^2)$, $\big(u(h)\big)^2 = u(h^2)$. Then u is fixed on $C(W)$.

We omit the proof.

Example 3. Let u be a nonnegative linear functional on $C[0, \pi]$ with $u(1) = 1$, $u(g^2) = \big(u(g)\big)^2$, $u(h^2) = \big(u(h)\big)^2$ where $g(x) = \sin x$, $h(x) = \sin(x + .01)$ on $[0, \pi]$. Then u is fixed on $C[0, \pi]$.

We omit the proof.

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