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DARBOUX LIKE FUNCTIONS. OLD PROBLEMS AND NEW RESULTS

During the 14th Summer School on Real Functions Theory in Liptovsky Ján, Slovakia in 1996, Richard G. (Jerry) Gibson gave a talk concerning Darboux-like functions. Following his talk we wrote the survey article *Darboux-like functions* [GN], that was published in the *Real Analysis Exchange*. In that paper we collected not only known facts on the subject but also presented some open problems. In the last two years a considerable number of papers have been written that solve some of the open questions posed in our survey article. In this short abstract we would like to present those solutions and to list the problems that remain open. This paper is based on the talk given by Tomasz (Tomek) Natkaniec at the 15th Summer School on Real Functions Theory in Liptovsky Ján, Slovakia, in 1998.

Most of the new results are contained as a part of the Polish-American project *Set Theoretic Analysis* and has been described in K. Ciesielski's survey *Set Theoretic Real Analysis* [C]. You can find these results on the Set Theoretic Analysis web page:

http://www.math.wvu.edu/homepages/kcies/STA/STA.html

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1 Notations and definitions

We use the notations from our survey article [GN]. In particular, the questions in this paper are numbered the same as in the survey article.

Recall the main notations and definitions from the survey article.

D – f is a *Darboux function* if $f(C)$ is connected whenever C is connected in X;

Conn – f is a *connectivity function* if the graph of f restricted to C is connected in $X \times Y$ whenever $C \subset X$ is connected;

ACS – f is an *almost continuous function* in the sense of Stallings, if U is an open subset of $X \times Y$ containing the graph of f, then U contains the graph of a continuous function $g : X \to Y$;

Ext – f is an *extendable function* if there exists a connectivity function $g : X \times I \to Y$ such that $f(x) = g(x, 0)$ for all $x \in X$.

2 Borel measurable functions

It is well known that in the class of all functions from $\mathbb{R}$ into $\mathbb{R}$ the following chain of implications holds:

$$(\star) \quad \mathrm{Ext} \to \mathrm{ACS} \to \mathrm{Conn} \to \mathrm{D}$$

For Baire class 1 functions, $f : \mathbb{R} \to \mathbb{R}$, the properties defined above are equivalent, i.e.,

$$\mathrm{Ext} = \mathrm{ACS} = \mathrm{Conn} = \mathrm{D}.$$

However, for Borel measurable functions, Brown, Humke and Laczkovich proved that the implications $(\star)$ are not reversible except for possibly Ext $\to$ ACS [BHL]. Thus they posed the following question.

Question 3.21. *Does there exist a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in \mathrm{ACS} \setminus \mathrm{Ext}$?*

The answer to this question was given in the following theorem.

Theorem 1. (Ciesielski, Jastrzębski, 1998) [CJ]

$$(\mathrm{ACS} \setminus \mathrm{Ext}) \cap B_2 \neq \emptyset.$$

This answers also **Question 3.11**. (See the next section.)

J. Brown [JB] has considered whether the implications ($\star$) within the class of all functions with G_δ graphs are reversible. He noticed that in this class all implications ACS $\to$ Conn $\to$ D are not reversible. (Recall that every function in the class B_1 has a graph which is a G_δ subset of $\mathbb{R}^2$. Moreover, it is well-known that every function with a G_δ graph is Borel measurable. However, it can be of an arbitrary high Borel class. See e.g., [AK].)

The problem of whether the implication Ext $\to$ ACS in the class of all functions with a G_δ graph is reversible was posed by R. Gibson [RG].

Question 3.15. *Does there exist a function $f \in \mathrm{ACS} \setminus \mathrm{Ext}$ with a G_δ graph?*

This question was answered in the next theorem.

Theorem 2. (Ciesielski, Rosen, 1999) [CRos] There exists a Baire two function $f \in \mathrm{ACS} \setminus \mathrm{Ext}$ with a G_δ graph.

The next problem is connected with the <u>Cesáro-Vietoris function</u> $\varphi : \mathbf{I} \to \mathbf{I}$ defined by the formula:

$$\varphi(x) = \overline{\lim}_{n\to\infty} \frac{a_1 + \ldots + a_n}{n}$$

where a_i are given by the unique nonterminating binary expansion of the number $x = (0.a_1 a_2 \ldots)$.

Claims.

- $\varphi \in B_2$.
- *(Vietoris, 1921).* $\varphi \in \mathrm{Conn}$.
- *(Brown, 1975).* $\varphi \in \mathrm{ACS}$.

Question 3.22. *Does the Cesáro-Vietoris function belong to* Ext*?*
Open.

3 Cantor intermediate values properties

Recall the following definitions.

PR – f has a *perfect road* if for every $x \in \mathbb{R}$, there exists a perfect set P having x as a bilateral limit point such that $f\restriction P$ is continuous at x;

CIVP – *Cantor Intermediate Value Property*: $f \in \mathrm{CIVP}$ if for all $p, q \in \mathbb{R}$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$;

SCIVP – *Strong Cantor Intermediate Value Property*: $f \in \mathrm{SCIVP}$ if for all $p, q \in \mathbb{R}$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$ and $f{\restriction}C$ is continuous.

PC – f is *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset $W \subset U$ such that $x \in W$ and $f(\mathrm{bd}\,(W)) \subset V$, where $\mathrm{bd}\,(W)$ denotes the boundary of W.

For Baire class 1 functions, $f\colon \mathbb{R} \to \mathbb{R}$, the functions defined above are equivalent, i.e.,

$$\mathrm{Ext} = \mathrm{ACS} = \mathrm{Conn} = \mathrm{D} = \mathrm{SCIVP} = \mathrm{CIVP} = \mathrm{PR} = \mathrm{PC}.$$

For arbitrary function, $f\colon \mathbb{R} \to \mathbb{R}$, we have only the following implications

$$\begin{tikzcd}
 & \mathrm{ACS} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[dr] & & & & \mathrm{PC} \\
 & \mathrm{SCIVP} \arrow[r] & \mathrm{CIVP} \arrow[r] & \mathrm{PR} \arrow[ur] &
\end{tikzcd}$$

Chart 1

The following theorems and questions are related to Chart 1.

Theorem 3. (K. Banaszewski, Natkaniec, 1996) [BN] *Assume CH. Then* $(\mathrm{ACS} \cap \mathrm{CIVP}) \setminus \mathrm{Ext} \neq \emptyset$.

Question. *Can the above be proved in ZFC?*

Theorem 4. (Ciesielski, 1997) [KC] *Yes.*

Question 3.11. *Does there exist* $f \in (\mathrm{ACS} \cap \mathrm{SCIVP}) \setminus \mathrm{Ext}$?

Theorem 5. (Rosen, 1997) [HR] *Yes, under CH.*

Theorem 6. (Ciesielski, Rosłanowski, 1998) [CR] *Every* $g \in \mathrm{Ext}$ *with a dense graph satisfies the following condition:*

SSCIVP – *for all* $a < b$ *and for every perfect set* K *between* $g(a)$ *and* $g(b)$ *there exists a perfect set* $C \subset (a, b)$ *such that* $g(C) \subset K$ *and* $g|C$ *is continuous strictly increasing.*[1]

(Note that there exists an $f \in (\mathrm{ACS} \cap \mathrm{SSCIVP}) \setminus \mathrm{Ext}$. This example has been constructed recently by K. Ciesielski.[2])

The following corollary follows immediately from the above theorem, as well as from Theorem 1.[3]

Corollary 7. [CR], [CJ] $(\mathrm{ACS} \cap \mathrm{SCIVP}) \setminus \mathrm{Ext} \neq \emptyset$.

4 Additive functions

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is additive if it satisfies the Cauchy equation: $f(x + y) = f(x) + f(y)$ for each $x, y \in \mathbb{R}$.

Question 5.2. *Does there exist a discontinuous additive* ACS *function whose graph is small in the sense of measure or category?*

Theorem 8. (Ciesielski, 1997) [KC]

- *If* $\mathbb{R}$ *is not a union of less than continuum meager sets then there exists an additive discontinuous* ACS *function* f *with the graph of measure zero.*
- *If* $\mathbb{R}$ *is not a union of less than continuum sets of measure zero then there exists an additive discontinuous* ACS *function* f *with a meager graph.*
- *Under CH there exists an additive discontinuous* ACS *function* f *with the graph which is meager and of measure zero.*[4]

Question 5.2 remains open in ZFC.

Question 5.5. *Does there exist* $f \in \mathrm{Add} \cap \mathrm{Conn} \setminus \mathrm{ACS}$?

[1] "*strictly increasing*" in this statement can be replaced by "*strictly decreasing*"

[2] Unpublished.

[3] This is a consequence of the fact that every Borel measurable Darboux function has the SCIVP.

[4] J. Pawlikowski has recently noticed that the same example exists also under the Martin's axiom.

Theorem 9. (Ciesielski, Rosłanowski, 1998) [CR] *Yes, under CH.*

Open in ZFC.

Question. *Does there exist a discontinuous function $f\colon \mathbb{R}^n \to \mathbb{R}$, $f \in \mathrm{Add} \cap \mathrm{Ext}$?*

Theorem 10. (Ciesielski, Jastrzębski, 1998) [CJ] *Yes, for $n = 1$. No for $n > 1$.*

5 Darboux like functions and quasi-continuity

In general Darboux like functions and quasi-continuous functions are not related. However, under certain conditions quasi-continuous functions are extendable. One of these conditions is that the function have a graph whose closure is bilaterally dense in itself. (It means that the right and left cluster sets of f at any point x of the domain of f coincide.)

QC $-f : X \to Y$ is a quasi-continuous function if and only if for each $p \in X$ the following condition holds: for every open set $U \subset X$ with $p \in U$ and open set $V \subset Y$ with $f(p) \in V$ there exist a non-empty open set $W \subset U$ such that $f(W) \subset V$.

Theorem 11. (Rosen, 1998) [HR1] *If f is Darboux, quasi-continuous and has a graph whose closure is bilaterally dense in itself, then f is extendable and $D(f)$ is f-negligible.*

(Here $D(f)$ denotes the set of all points at which f is discontinuous.)

Recently, Francis Jordan [FJ] improved this results by showing that we need not assume that the function is a Darboux function.

6 Functions of $n > 1$ variables

For $n > 1$, the implications in Chart 1 are no longer valid. In fact, we have the followig diagram.

$$\begin{tikzcd} & & \mathrm{ACS} \\ \mathrm{Ext} \arrow[r] & \mathrm{Conn} = \mathrm{PC} \arrow[ur] \arrow[dr] & \\ & & \mathrm{D} \end{tikzcd}$$

Question 8.3. *Is the inclusion* Ext $\subset$ Conn *proper for* $f\colon \mathbb{R}^n \to \mathbb{R}$*, if* $n > 1$*?*

Theorem 12. (Ciesielski, Natkaniec, Wojciechowski, 1998) [CNW] *Every connectivity function* $f\colon \mathbb{R}^n \to \mathbb{R}$*,* $n > 1$ *can be extended to a connectivity function of* $n+1$ *variables.*

Thus for functions of $n > 1$ variables we have the equalities

$$\mathrm{Ext} = \mathrm{Conn} = \mathrm{PC}.$$

7 Compositions

This section contains the new results concerning compositions of Darboux like functions.

Question 9.3. *Is every Darboux function the composition of two (finitely many)* ACS *(or* Conn*) functions?*

Theorem 13. (Natkaniec, 1991) [TN] *Assume CH. Every* $f \in \mathrm{D}$ *with dense level sets is the composition of two* ACS *functions.*

Question. *Can the above be proved in ZFC?*
Open.

Theorem 14. (Kellum, 1998) [KK] *There exists an* $f \in \mathrm{Conn}$ *such that* f *is not the composition of finitely many* ACS *functions.*

Theorem 15. (Ciesielski, Kellum, 1998) [CK] *There exists an* $f \in \mathrm{D}$ *such that* f *is not the composition of finitely many* Conn *functions.*

Question 9.1. *If* $f, g \in \mathrm{Ext}$*, is* $g \circ f \in \mathrm{Ext}$*?*
Open.

The next result seems to be connected with the above problem. (Note that every B_2 function can be expressed as the composition of two B_1 functions.[5])

Question. *Is every* $f \in \mathrm{D}B_2$ *the composition of two derivatives (*$\mathrm{D}B_1$ *functions)?*

No. See Kellum's example in [KK]. So the folowing problems seems to be **open**.

Question. (Ciesielski) Has the compositions of two derivatives ($\mathrm{D}B_1$ or Ext) functions from $\mathbb{I}$ into $\mathbb{I}$ a fix point?

(Recall that every connectivity function from $\mathbb{I}$ into $\mathbb{I}$ (so also every derivative, every $\mathrm{D}B_1$ and Ext function) has fixed points.)

Question. *Characterize compositions of of two derivatives (*$\mathrm{D}B_1$ *functions).*

[5] This fact was proved for us by S. Solecki.

8 Uniform limits

Let $\overline{\mathcal{F}}$ denote the class of uniform limits of sequences of functions from $\mathcal{F}$.

Question 9.13. *Does there exists* $f \in \mathrm{ACS} \cap \mathrm{PR} \setminus \overline{\mathrm{Ext}}$*?*

The answer is affirmative. Both Rosen's function from [HR] (constructed under CH) and ZFC Ciesielski-Jastrzębski's example from [CJ] are in the class $\mathrm{ACS} \cap \mathrm{SCIVP}$ but are not in $\overline{\mathrm{Ext}}$.

Question 9.14. *Characterize the uniform limits of sequences of* Ext *functions (*ACS *functions or* Conn *functions).*

This problem is still **open**. However, the following conjecture was formulated by K. Kellum during the last Miniconference in Real Analysis (March 1999) at Auburn University.

Conjecture. *Let* $f \colon \mathbb{R} \to \mathbb{R}$ *be bounded. Then* $f \in \overline{\mathrm{ACS}}$ *iff*
- $f \in \mathrm{U}$ $(= \overline{\mathrm{D}})$*;*
- f *is away-almost continuous:*

A function $f \colon \mathbb{R} \to \mathbb{R}$ is *away almost continuous* if for every closed set $B \subset \mathbb{R}^2$ such that $\sup\{\mathrm{dist}(f(x), B_x) \colon\ x \in \mathbb{R}\} > 0$ there exists a continuous function $g \colon \mathbb{R} \to \mathbb{R}$ which is disjoint with B.

9 Universal summands

It is easy to see that there is no $g \in B_\alpha$ such that $B_\alpha + g \subset \mathrm{D}$. Similarly, there is no Borel measurable function g such that $f + g \in \mathrm{D}$ for all Borel functions f.

Question. (Ceder, 1965) [JC] *Does there exist a Borel measurable* g *such that* $B_\alpha + g \subset \mathrm{D}$*?*

Theorem 16. (Natkaniec, Recław, 1997) [NR] *For each* $\alpha < \omega_1$ *there is a Borel function with* $B_\alpha + g \subset \mathrm{ACS}$.

Question. *Does there exist* $g \in B_{\alpha+1}$ *such that* $B_\alpha + g \in \mathrm{D}$ *(*$B_\alpha + g \in \mathrm{ACS}$ *or* $B_\alpha + g \in \mathrm{Ext}$*)?*

Theorem 17. (Solecki, 1998) [SS] *For each* $\alpha < \omega_1$ *there is a* $g \in B_\alpha$ *such that* $\bigcup_{\beta<\alpha} B_\beta + g \in \mathrm{Ext}$.

For an ordinal $\alpha < \omega_1$ define the cardinal

$$u(\alpha) = \min(\{|\mathcal{F}| \colon\ \mathcal{F} \subset B_\alpha \ \&\ (\forall\, g \in B_\alpha)\ (\mathcal{F} + g \not\subset \mathrm{D})\} \cup \{\mathfrak{c}^+\})$$

Clearly, $u(0) = \mathfrak{c}^+$. Moreover, $u(1) = \omega$ [PP] and $\omega < u(\alpha) < \mathfrak{c}$ for $\alpha > 1$ (an easy corollary from [JC]). Thus we have the following

Theorem 18. *Under CH, $u(\alpha) = \omega_1$ for all $\alpha > 1$.*

Question. *Can the above be proved in ZFC?*
Open.[6]

10 Products

Theorem 19. *For every $f : \mathbb{R} \to \mathbb{R}$ the following conditions are equivalent:*
- *f is a product of two* D *functions;*
- *f is a product of two* Conn *functions;*
- *f is a product of two* ACS *functions;*
- *f possesses the (JC) property: f has a zero in each subinterval in which it changes sign.*

Question 9.31. *Is every $f : \mathbb{R} \to \mathbb{R}$ with (JC) property the product of two* Ext *functions?*
Open.

References

[BN] K. Banaszewski, T. Natkaniec, *Sierpiński-Zygmund functions that have the Cantor intermediate value property*, Real Anal. Exchange, to appear.

[JB] J. Brown, *Measurable Darboux functions*, Real Anal. Exchange **15** (1989–90), 65–66.

[BHL] J. Brown, P. Humke, M. Laczkovich, *Measurable Darboux functions*, Proc. of Amer. Math. Soc. **102** No. 3 (1988), 603-610.

[JC] J. Ceder, *On representing functions by Darboux functions*, Acta Sci. Math. (Szeged) **26** (1965), 283–288.

[C] K. Ciesielski, *Set Theoretic Real Analysis*, J. Appl. Anal. **3**(2) (1997), 143–190.

[KC] K. Ciesielski, *Some additive Darboux-like functions*, J. Appl. Anal. **4** (1998), 43–51.

[6] In fact, this is a reformulation of Ceder's problem from [JC].

[CJ] K. Ciesielski, J. Jastrzębski, *Darboux-like functions within the classes of Baire one, Baire two, and additive functions*, Topol. Appl., to appear.

[CK] K. Ciesielski, K. Kellum, *Compositions of Darboux and connectivity functions*, Real Anal. Exchange **24**(1) (1998–99), to appear.

[CNW] K. Ciesielski, T. Natkaniec, J. Wojciechowski, *Extending connectivity functions on* $\mathbb{R}^n$, preprint.

[CRos] K. Ciesielski, H. Rosen, *Two examples concerning extendable and almost continuous functions*, preprint.

[CR] K. Ciesielski, A. Rosłanowski, *Two examples concerning almost continuous functions*, Topol. Appl., to appear.

[RG] R. Gibson, *Darboux functions with a perfect road*, Real Anal. Exchange **15** (1989–90), 582–591.

[GN] R. Gibson, T. Natkaniec, *Darboux like functions*, Real Anal. Exchange **22**(2) (1996–97), 492–533.

[FJ] F. Jordan, *Quasicontinuous functions with a little symmetry are extendable*, Real Anal. Exchange, submitted.

[AK] A. S. Kechris, *Classical Descriptive Set Theory*, Springer Verlag, Berlin 1995.

[KK] K. Kellum, *Compositions of Darboux-like functions*, Real Anal. Exchange, **23**(1) (1997–98), 211–217.

[TN] T. Natkaniec, *On compositions and products of almost continuous functions*, Fund. Math. **139** (1991), 59–74.

[NR] T. Natkaniec, I. Recław, *Universal summands for families of measurable functions*, Acta Sci. Math. (Szeged) **64** (1998), 463–471.

[PP] H. W. Pu and H. H. Pu, *On representations of Baire functions in a given family as sums of Baire Darboux functions with a common summand*, Časopis Pěst. Mat. **112** (1987), 320–326.

[HR] H. Rosen, *An almost continuous nonextendable function*, Real Anal. Exchange **23**(2) (1997–98), 567–570.

[HR1] H. Rosen, *Darboux Quasicontinuous functions*, Real Anal. Exchange **23**(2) (1997-98), 631-640.

[SS] S. Solecki, *Making Borel functions like continuous*, preprint.